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10 JAN 2019

UNIVERZITET CRNE GORE

-Centar za doktorske studije-

-Senat-

U skladu sa članom 41 Pravila doktorskih studija, u prilogu akta dostavljamo predlog Odluke Vijeća o imenovanju komisije za ocjenu doktorske disertacije Aladina Crnkića pod nazivom "Kolektivno ponašanje i samoorganizacija u mrežama povezanih oscilatora" radi davanja saglasnosti.

Prilog:

-D2 obrazac sa neophodnim priložima

DEKAN

Prof. dr Predrag Miranović



ISPUNJENOST USLOVA DOKTORANDA

OPŠTI PODACI O DOKTORANDU			
Titula, ime, ime roditelja, prezime	MSc Aladin, Alija, Crnkić		
Fakultet	Prirodno-matematički fakultet		
Studijski program	Matematika		
Broj indeksa	1/13		
NAZIV DOKTORSKE DISERTACIJE			
Na službenom jeziku	Kolektivno ponašanje i samoorganizacija u mrežama povezanih oscilatora		
Na engleskom jeziku	Collective behavior and self-organization in networks of coupled oscillators		
Naučna oblast	Matematika i Fizika		
MENTOR/MENTORI			
Prvi mentor	Prof. dr Vladimir Jaćimović	Prirodno-matematički fakultet, Univerzitet Crne Gore, Crna Gora	Matematika
Drugi mentor	(Titula, ime i prezime)	(Ustanova i država)	(Naučna oblast)
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Doc. Dr Marijan Marković	PMF UCG, CG	Matematika	
Datum značajni za ocjenu doktorske disertacije			
Sjednica Senata na kojoj je data saglasnost na ocjenu teme i kandidata	16.10.2017.		
Dostavljanja doktorske disertacije organizacionoj jedinici i saglasnost mentora	10. 12. 2018.		
Sjednica Vijeća organizacione jedinice na kojoj je dat prijedlog za imenovanje komisija za pregled i ocjenu doktorske disertacije	21. 12. 2018.		
ISPUNJENOST USLOVA DOKTORANDA			
U skladu sa članom 38 pravila doktorskih studija kandidat je dio sopstvenih istraživanja vezanih za doktorsku disertaciju publikovao u časopisu sa (SCI/SCIE) liste kao prvi autor.			
Spisak radova doktoranda iz oblasti doktorskih studija koje je publikovao u časopisima sa (upisati odgovarajuću listu)			

Spisak radova iz oblasti doktorskih studija koje je kandidat publikovao u časopisima sa SCI/SCIE liste:

[1] Crnkić, A., Jaćimović, V.: Swarms on the 3-sphere with adaptive synapses: Hebbian and anti-Hebbian learning rule. Systems & Control Letters, Vol. 122, pp. 32--38 (2018), DOI: 10.1016/j.sysconle.2018.10.004

Link na rad: <https://www.sciencedirect.com/science/article/pii/S0167691118301786>

[2] Crnkić, A., Jaćimović, V.: Data clustering based on quantum synchronization. Natural Computing (2018), DOI: 10.1007/s11047-018-9720-z

Link na rad: <https://link.springer.com/article/10.1007/s11047-018-9720-z>

[3] Crnkić, A., Jaćimović, V.: Consensus and balancing on the three-sphere. Journal of Global Optimization (2018), DOI: 10.1007/s10898-018-0723-1

Link na rad: <https://link.springer.com/article/10.1007/s10898-018-0723-1>

[4] Jaćimović, V., Crnkić, A.: Low-dimensional dynamics in non-Abelian Kuramoto model on the 3-sphere. Chaos: An Interdisciplinary Journal of Nonlinear Science, Vol. 28, No. 8, pp. 083105 (2018), DOI: 10.1063/1.5029485

Link na rad: <https://aip.scitation.org/doi/abs/10.1063/1.5029485>

[5] Jaćimović, V., Crnkić, A.: Modelling mean fields in networks of coupled oscillators. Journal of Geometry and Physics, Vol. 124, pp. 241--248 (2018), DOI: 10.1016/j.geomphys.2017.11.009

Link na rad: <https://www.sciencedirect.com/science/article/pii/S0393044017302899>

[6] Jaćimović, V., Crnkić, A.: Collective motions of globally coupled oscillators and some probability distributions on circle. Physics Letters A, Vol. 381, No. 24, pp. 1989--1994 (2017), DOI: 10.1016/j.physleta.2017.04.024

Link na rad: <https://www.sciencedirect.com/science/article/abs/pii/S0375960117304012>

[7] Jaćimović, V., Crnkić, A.: Characterizing complex networks through statistics of Möbius transformations. Physica D: Nonlinear Phenomena, Vol. 345, pp. 56--61 (2017), DOI: 10.1016/j.physd.2016.12.007

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Međunarodna konferencija (indeksirana u bazi SCOPUS):

[8] Crnkić, A., Jaćimović, V.: Exploring complex networks by detecting collective dynamics of Kuramoto oscillators. VIII International Conference on Optimization and Applications (OPTIMA-2017), CEUR Workshop Proceedings, Petrovac, Montenegro (2017).

Link na rad: <http://ceur-ws.org/Vol-1987/paper22.pdf>

Obrazloženje mentora o korišćenju doktorske disertacije u publikovanim radovima

Kandidat je radove publikovane u naučnim žurnalima djelimično ili integralno prenio u disertaciju u prevodu sa engleskog jezika.

Sadržaj Poglavlja 2 doktorske teze odgovara članku [6]. Sadržaj Poglavlja 3 je publikovan u člancima [5,7,8]. Sadržaj Poglavlja 4 je preuzet iz članaka [2,4]. Konačno, sadržaj Poglavlja 5 odgovara člancima [1,3].

Datum i ovjera (pečat i potpis odgovorne osobe)

U (navesti grad),
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1. Potvrdu o predaji doktorske disertacije organizacionoj jedinici
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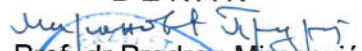
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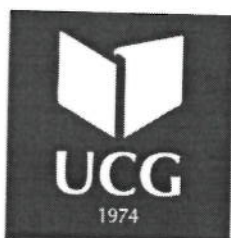
P O T V R D A

MSc Aladin Crnkić, student doktorskih studija na Prirodno-matematičkom fakultetu u Podgorici, dana **10.12.2018. godine** dostavio je ovom Fakultetu doktorsku disertaciju pod nazivom „**Kolektivno ponašanje i samoorganizacija u mrežama povezanih oscilatora**“, na dalji postupak.

DEKAN


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Na osnovu člana 69 statuta Univerziteta Crne Gore i člana 41 Pravila doktorskih studija, na XXVII sjednici Vijeća održanoj 21.12.2018. godine donijeta je

ODLUKA

I

Predlažemo Centru za doktorske studije i Senatu Univerziteta da imenuje komisiju za ocjenu doktorske disertacije "Kolektivno ponašanje i samoorganizacija u mrežama povezanih oscilatora", kandidata Aladina Crnkića, u sastavu:

1. Prof. dr Predrag Miranović, Prirodno-matematički fakultet Univerziteta Crne Gore (Naučna oblast: Fizika)
2. Prof. dr David Kaljaj, Prirodno-matematički fakultet Univerziteta Crne Gore (Naučna oblast: Matematika);
3. Prof. dr Vladimir Jaćimović, Prirodno-matematički fakultet Univerziteta Crne Gore (Naučna oblast: Matematika)
4. Prof. dr Igor Đurović, Elektrotehnički fakultet Univerziteta Crne Gore; (Naučna oblast: Računarske nauke);
5. Prof. dr Marija Mitrović Dankulov, Institut za Fiziku, Beograd, Srbija (Naučna oblast: Fizika);

Obrazloženje

Aladin Crnkić je predao doktorsku disertaciju pod nazivom "Kolektivno ponašanje i samoorganizacija u mrežama povezanih oscilatora". Vijeće Prirodno-matematičkog fakulteta je utvrdilo da su ispunjeni uslovi iz člana 38 Pravila doktorskih studija, da

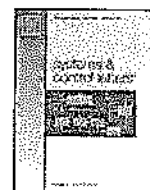
kandidat Aladin Crnkić ima, kao prvi autor, rad sa rezultatima iz teze objavljen u časopisu sa SCI/SCIE liste. Samim tim su se stekli uslovi da se imenuje komisija za ocjenu pomenute doktorske disertacije.

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Swarms on the 3-sphere with adaptive synapses: Hebbian and anti-Hebbian learning rule

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ABSTRACT

We introduce and analyze several models of swarm dynamics on the sphere S^3 with adaptive (state-dependent) interactions between agents. The equations describing the interaction dynamics are variations of the classical Hebbian principle from Neuroscience. We study asymptotic behavior in models with various realizations of Hebbian and anti-Hebbian learning rules. The swarm with the Hebbian rule and strictly nonnegative (attractive) interactions evolves towards consensus; if the Hebbian rule allows both attractive and repulsive interactions the swarm converges to bipolar configuration. The most interesting is the model with anti-Hebbian learning rule with both attractive and repulsive interactions. This model displays a rich variety of dynamical regimes and stationary formations, depending on the number of agents and system parameters. We prove that the model with such anti-Hebbian rule evolves towards a stable stationary configuration if the system parameter is above a certain bifurcation threshold. Finally, some simulation results are presented demonstrating how these theoretical results can be applied to coordination of rotating bodies in 3D space; this is done by mapping the trajectories from S^3 to special orthogonal group $SO(3)$.

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1. Introduction

The broad area of cooperative and distributed control comprises problems of consensus, balancing, formation keeping and coordinated motion in multi-agent systems. Special subdiscipline, named *Geometric consensus theory* (or, more generally, *Geometric coordination theory*) deals with these problems on non-Euclidean manifolds [1,2]. Coordination algorithms are naturally formulated on homogeneous spaces [2,3]. However, coordination of agents depends greatly on geometric and algebraic properties of the underlying manifold, and algorithms on different homogeneous spaces perform differently. Hence, although the recent theoretical advances developed a fairly universal approach to coordination problems on arbitrary homogeneous spaces, there are few universal results regarding convergence to desired equilibrium configurations.

Some problems of Geometric coordination theory are formulated as minimization of appropriate objective functions on specific homogeneous spaces. Gradient descent methods for these minimization problems provide distributed coordination protocols. One example of this kind is consensus problem on the circle S^1

over the weighted graph G . The gradient flow for the disagreement cost function yields the system of ODE's on S^1 . This system of ODE's is essentially the famous Kuramoto model of identical oscillators that are coupled through the graph G . This observation indicates an intriguing relation between consensus on homogeneous spaces and universal natural phenomenon of synchronization of coupled oscillators; [4]. Coordination algorithms on S^1 have straightforward applications in cooperative control of collective planar motions [5].

In the present paper we consider the coordination of the swarm on the 3-sphere S^3 . There are two good reasons to focus on this specific manifold. First, geometry of spheres S^n (with the exception of the circle S^1) favors consensus. Indeed, consensus algorithms on S^n perform much better than their analogues on special orthogonal groups $SO(n)$ [6]. Second, S^3 is, along with S^1 , the only sphere with the group property. This makes S^3 a very convenient setting for various coordination algorithms [6,7].

The inspiration for this paper to some extent stems from the fact that gradient flows for consensus problems on higher-dimensional homogeneous spaces are known in Physics as *non-Abelian Kuramoto models*, see [8]. This observation extends the relation between consensus and synchronization: while the consensus on S^1 corresponds to synchronization in the classical (Abelian) Kuramoto model, the consensus on S^3 corresponds to *quantum synchronization* in non-Abelian Kuramoto model [8,9].

Most papers on Geometric coordination theory deal with the swarm of agents that communicate through the constant graph

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of interactions. In the present paper we study the swarm with adaptive (state-dependent) interactions. One natural principle for introducing state-dependent synapses is inspired by the classical Hebbian rule from Neuroscience. This rule states that the pairs of agents whose states are close to each other strengthen their synapses, or: *the cells that fire together, wire together*. Kuramoto model with synapses that obey certain variations of Hebbian (or anti-Hebbian) learning rule has been studied in several papers; see [10–12]. To our best knowledge, non-Abelian Kuramoto models with Hebbian learning rule have not been considered so far. Notice, however, that swarms on S^n with state-dependent synapses have been studied in [13,14]. Also, the paper [15] deals with the swarming on sphere with various cooperative rules in discrete-time models. In this paper we demonstrate that swarms on S^3 with some natural and simple learning rules for synapses exhibit various behavioral patterns and stable configurations.

In the next Section we describe the model of swarm dynamics on S^3 and recall some existing results for the case of time-constant interactions. In Section 3 we introduce the specific Hebbian learning rule that allows both attractive and repulsive interactions. As one might expect, such swarm tends to a bipolar configuration. In Section 4 the model with anti-Hebbian learning rule is studied; this model allows for a variety of nontrivial dynamical regimes and equilibria depending on number of agents and the system parameters. Coordination algorithms on S^3 can be mapped to algorithms on the group $SO(3)$ of 3D rotations using the double cover map $S^3 \rightarrow SO(3)$. This is equivalent to representation of 3D rotations by unit quaternions. It is well-known that such representation may produce some peculiar effects, due to the fact that two antipodal quaternions correspond to the same rotation. However, in many cases such approach has significant advantages over designing methods directly on $SO(3)$. In Section 5 we provide several short videos in order to visualize different dynamical regimes on the group $SO(3)$. Finally, Section 6 contains a brief conclusion.

2. Swarms on the 3-sphere with constant interactions

Consider the swarm of N agents evolving on the 3-sphere S^3 . In order to introduce the coordinates on S^3 we will use the algebra of unit quaternions. The states of agents at the moment t will be described by unit quaternions $q_1(t), \dots, q_N(t)$ or by corresponding unit vectors in \mathbb{R}^4 : $x_1(t), \dots, x_N(t)$.

We suppose that the evolution of each agent is described by the following quaternion-valued ODE on S^3 :

$$\dot{q}_j = q_j f_j q_j - \bar{f}_j, \quad j = 1, \dots, N. \quad (1)$$

Here, $q_j(t)$ is a unit quaternion, describing the position of the j th agent on S^3 and $f_j = f_j(q_1, \dots, q_N)$ are quaternionic functions called *coupling functions* or *communication protocols* depending on the context. The notion \bar{a} stands for quaternionic conjugation of quaternion a (the conjugate of a unit quaternion coincides with the inverse one: $\bar{a} = a^{-1}$).

Notice that Eqs. (1) preserve S^3 ; this is easily verified by checking that $q_j(0) \in S^3$ for all $j = 1, \dots, N$ implies $q_j(t) \in S^3$ for all $t > 0$.

One can introduce different forms of coupling functions f_j , depending on specific goals. In this paper we consider the coupling functions of the following form:

$$f_j = -\frac{\alpha}{2N} \sum_{k=1}^N w_{jk} \bar{q}_k, \quad (2)$$

where the bar denotes quaternionic conjugation as before.

Plugging (2) into (1) yields:

$$\dot{q}_j = \alpha \sum_{k=1}^N w_{jk} (q_j q_k q_j - \bar{q}_k), \quad j = 1, \dots, N. \quad (3)$$

The coefficients w_{jk} in (2) are interpreted as strengths (weights) of interactions between agents. In this paper we will deal with the case when interactions can be of any sign (attractive, repulsive or time-dependent). If w_{jk} are strictly nonnegative, the system (3) provides distributed consensus algorithm.

The system (3) can be equivalently written in real coordinates with unit vectors $x_1(t), \dots, x_N(t) \in \mathbb{R}^4$:

$$\dot{x}_j = \alpha \sum_{k=1}^N w_{jk} (x_k - \langle x_j, x_k \rangle x_j), \quad j = 1, \dots, N. \quad (4)$$

The system in the form (4) has already been studied in various contexts: it has been interpreted as the swarm on sphere in [13], opinion dynamics in [16] and gradient descent system for consensus problem on sphere in [6].

The system of matrix ODE's on special orthogonal groups $SO(n)$ that are analogous to (3) appear in [2,3] as gradient flows for consensus problems on $SO(n)$.

Definition 2.1. The configuration of agents with

$$q_1 = \dots = q_N \quad (5)$$

is called *consensus*.

Remark 2.1. There is a certain ambiguity in the literature regarding the terminology. For instance, in [3] the configurations that satisfy (5) are called *synchronization*, while the term *consensus* has broader meaning. In such terminology, synchronization is a partial case of consensus. On the other hand, most authors do not distinguish between the two terms and use them as synonyms for configurations (5). In the present paper we will also stick to this convention. In addition, in some papers the terms *alignment* and *rendezvous* are also used with the same meaning.

Definition 2.2. The point

$$m(q) = \frac{1}{N} \sum_{j=1}^N q_j$$

is called the *center of mass* (or *centroid*) of agents' states q_1, \dots, q_N .

Obviously, $m(q)$ is a point in 4-dimensional unit ball.

The central notions in the theory of coupled oscillators are the state coherence and the order parameter.

Definition 2.3. The number $r \equiv r_{glob} = |m(q)|$, $0 \leq r \leq 1$ is called the *global order parameter* of the configuration.

Along with the global order parameter, we also introduce the notion of angular order parameters. For that purpose we use the Hopf coordinates on S^3 . To this aim, we represent quaternions q_k in the Cayley–Dickson form: $q_k = z_k + v_k j$, where z_k and v_k are complex numbers. Furthermore, as q_k are unit quaternions, z_k and v_k can be written as follows: $z_k = e^{i\varphi_k} \sin \theta_k$, $v_k = e^{i\psi_k} \cos \theta_k$, with $\varphi, \psi \in (0, 2\pi)$, $\theta \in (0, \frac{\pi}{2})$.

Now, introduce the following complex numbers:

$$r_\varphi(t) e^{i\varphi(t)} = \frac{1}{N} \sum_{k=1}^N e^{i\varphi_k(t)} \quad \text{and} \quad r_\psi(t) e^{i\psi(t)} = \frac{1}{N} \sum_{k=1}^N e^{i\psi_k(t)}. \quad (6)$$

Obviously, $r_\varphi(t)$ and $r_\psi(t)$ are real numbers in $[0, 1]$.

Definition 2.4. We refer to r_φ and r_ψ as *angular order parameters*.

Definition 2.5. The configuration q_1, \dots, q_N is called *balanced*, if $m(q) = 0$ (or, equivalently, $r = 0$).

We further outline several known facts regarding the consensus and balancing in the model with constant interactions.

Theorem 2.1. Suppose that interactions w_{jk} in (3) are constant and symmetric, i.e. $w_{jk} = w_{kj}$ for all $1 \leq j, k \leq N$. Then (3) is the gradient descent system for the following potential function:

$$V = \frac{1}{N^2} \sum_{j,k} w_{jk} \left(1 - \frac{1}{2}(\bar{q}_j q_k + \bar{q}_k q_j)\right).$$

Theorem 2.2 ([16]). Suppose that interactions w_{jk} in (3) are constant and symmetric. Then the system converges to a stationary equilibrium.

Theorem 2.3 ([6]). Consider the swarm on the sphere S^n for $n \geq 2$ with agents interacting through connected and undirected communication graph. Then the set of consensus configurations is almost globally stable for the system. In other words, the set of all initial data $q_1(0), \dots, q_N(0)$ for which the swarm does not converge towards consensus has zero Lebesgue measure on $S^n \times \dots \times S^n$.

Theorem 2.4 ([7]). Suppose that each pair of agents is connected with equal negative synaptic strength $w_{jk} = w < 0$. In addition, suppose that the initial points $q_1(0), \dots, q_N(0)$ are all distinct. Then (3) converges to a balanced configuration.

Remark 2.2. Theorems 2.1 and 2.2 hold for any sphere S^n , while Theorem 2.3 holds for any sphere S^n with $n \geq 2$.

Remark 2.3. The gradient flow system on $SO(n)$ analogous to (3) is a distributed protocol for consensus problem on $SO(n)$, see [2,3]. However, Theorems 2.2–2.4 do not hold for the system on $SO(n)$ when $n \geq 2$. This remark emphasizes the essential difference between consensus and balancing algorithms on spheres and those on special orthogonal groups. This difference can be roughly summarized by stating that geometry of spheres (with the exception of the circle) is much more favorable for consensus than geometry of rotation groups.

3. Swarm on S^3 with Hebbian learning rule

There are several ways to design the state-dependent interactions in accordance with the Hebbian learning principle. In order to obtain some equilibria different from consensus, we introduce the learning rule that results in both attractive and repulsive interactions w_{jk} . Consider the swarm governed by the system (3) with coefficients w_{jk} obeying the following system of ODE's:

$$\dot{w}_{jk} = \varepsilon(\bar{q}_j q_k + \bar{q}_k q_j - \mu w_{jk}), \quad \mu > 0, \quad j = \overline{1, N}, \quad k = \overline{1, N} \quad (7)$$

with the initial conditions satisfying $w_{jk}(0) = w_{kj}(0)$ for all $1 \leq j, k \leq N$. This ensures that interactions $w_{jk}(t)$ are symmetric for any further moment $t > 0$.

The parameter ε in (7) is interpreted as learning rate.

The system (7) defines a kind of Hebbian rule: in order to understand this, notice that the quaternionic expression $\frac{1}{2}(\bar{q}_j q_k + \bar{q}_k q_j)$ is real-valued and equals $\cos \rho_{jk}(t) = \langle x_j, x_k \rangle(t)$, where $\rho_{jk}(t)$ is an angle between the unit vectors $x_j(t)$ and $x_k(t)$. Hence, the equilibrium interaction w_{jk} between agents j and k is maximally attractive when agents j and k are aligned ($q_j = q_k$). On the contrary, the interaction is maximally repulsive if the agents are positioned in antipodal points, $q_j = -q_k$.

It is not surprising that the system (3), (7) ends up in bipolar (antipodal) configuration on S^3 . In Fig. 1 we depict the evolution of pairwise cosines $\cos \rho_{jk}(t) = \langle x_j, x_k \rangle(t)$ for some pairs x_j, x_k .

Remark 3.1. In all simulations the initial conditions for agents' states $q_1(0), \dots, q_N(0)$ are chosen randomly from the uniform distribution on S^3 . The interactions are initially set to zero, $w_{jk}(0) = 0$, for all j, k , meaning that network does not exist at $t = 0$.

Definition 3.1. The configuration of agents, such that

$$q_1 = \dots = q_m = -q_{m+1} = \dots = -q_N, \quad \text{where } 0 \leq m \leq N$$

is called bipolar (or antipodal).

Notice that the set of bipolar configurations include all consensus configurations. In other words, consensus is bipolar configuration with $m = N$.

Proposition 3.1. The set of bipolar configurations is asymptotically stable for the model (3), (7).

Proof. Denote by $\hat{X} = (\pm \hat{x}_1, \dots, \pm \hat{x}_N)$ bipolar equilibrium configuration with the corresponding interactions $\hat{W} = (\pm 1, \dots, \pm 1)$. The dimensions of vectors \hat{X} and \hat{W} are equal to $4N$ and $N^2 - N$ respectively.

Write the linearization of (4), (7) around the point (\hat{X}, \hat{W}) :

$$\dot{\hat{X}} = L\hat{X} + T\hat{W} + D(\hat{X})\hat{X}, \quad \dot{\hat{W}} = P\hat{X} + M\hat{W} + G(\hat{X})\hat{X}.$$

Then the matrix of linear approximation of (4), (7) in the vicinity of (\hat{X}, \hat{W}) is:

$$J = \begin{pmatrix} L & T \\ P & M \end{pmatrix}.$$

Here, $T = \frac{\partial f}{\partial w_{jk}}(\hat{X}, \hat{W})$, $M = \frac{\partial g}{\partial w_{jk}}(\hat{X}, \hat{W})$, where f and g are right hand sides of (3) and (7):

$$f_j(\hat{X}, \hat{W}) = \alpha \sum_{k=1}^N w_{jk}(\bar{q}_j q_k q_j - \bar{q}_k) = \alpha \sum_{k=1}^N w_{jk}(x_k - \langle x_j, x_k \rangle x_j)$$

and

$$g_{jk}(\hat{X}, \hat{W}) = \varepsilon(\bar{q}_j q_k + \bar{q}_k q_j - \mu w_{jk}) = \varepsilon(2\langle x_j, x_k \rangle - \mu w_{jk}).$$

Simple differentiation gives:

$$\frac{\partial f_j}{\partial w_{jk}} = \alpha(x_j - \langle x_k, x_j \rangle x_k),$$

This expression equals zero at bipolar configuration, when $x_j = \pm x_k$, that is:

$$\frac{\partial f}{\partial w_{jk}}(\hat{X}, \hat{W}) = 0, \quad \text{for all } j \text{ and } k.$$

Also, by differentiating g :

$$\frac{\partial g_{jk}}{\partial w_{st}} = \begin{cases} -\varepsilon\mu, & \text{if } j = s, k = t; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the matrix T consists of zeros and M is $(N^2 - N) \times (N^2 - N)$ diagonal matrix with diagonal elements $-\mu$.

The characteristic polynomial of J reads

$$\det(L - tI) \det(M - tI) = \det(L - tI)(-\mu - t)^{N^2 - N}.$$

It follows that antipodal equilibrium (\hat{X}, \hat{W}) of (4), (7) is stable whenever \hat{X} is a stable equilibrium of (4) with corresponding interactions $w_{jk} = \pm 1$ for all j and k .

To conclude the proof we refer to results of [13,16] stating that antipodal configuration is asymptotically stable equilibrium of (4) if interactions w_{jk} are symmetric. \square

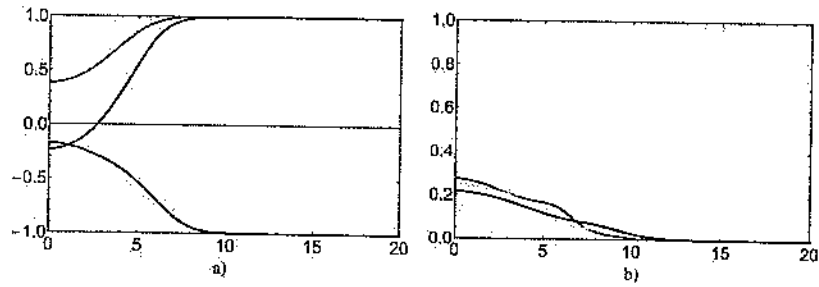


Fig. 1. Model with Hebbian learning (3), (7) with $N = 10$ agents. Parameter value: $\varepsilon = 0.2$. Zero initial values for synapses $w_{jk}(0) = 0$ and random initial values for $q_j(0)$ (see Remark 3.1). (a) inner products $\langle x_j, x_k \rangle$ for some j, k and (b) order parameters: global order parameter (black), φ -angle order parameter (red) and ψ -angle order parameter (yellow). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

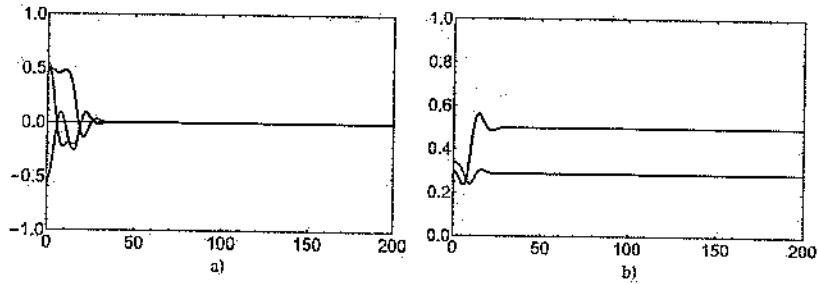


Fig. 2. Model with anti-Hebbian learning (3), (8) with $N = 4$ agents. Parameter values: $\varepsilon = 0.2$, $\mu = 2$. Initial values are explained in Remark 3.1. (a) inner products $\langle x_j, x_k \rangle$ for some j, k and (b) order parameters: global order parameter (black), φ -angle order parameter (red) and ψ -angle order parameter (yellow). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4. Swarm on S^3 with anti-Hebbian learning rule

In this Section we consider the model (3) with slightly modified rule for the dynamics of synapses:

$$\dot{w}_{jk} = \varepsilon(-(\bar{q}_j q_k + \bar{q}_k q_j) - \mu w_{jk}), \quad \mu > 0, \quad j = \overline{1, N}, \quad k = \overline{1, N}. \quad (8)$$

As in Section 3, we again suppose that initial conditions $w_{jk}(0)$ are symmetric, ensuring that $w_{jk}(t) = w_{kj}(t)$ for any $t > 0$.

The system (8) defines the rule that is in certain sense opposite to the Hebbian: the interaction between two agents becomes repulsive when their states are sufficiently close.

The anti-Hebbian model (3), (8) exhibits a rich variety of equilibrium configurations and dynamical regimes depending on the parameter μ and the number of agents N . For this reason, it is more difficult to obtain some theoretical results. We start with some numerical simulations for different number of agents N .

4.1. Case A: $N \leq 4$ agents

This case is very simple: the swarm evolves towards the configuration where all interactions w_{jk} vanish. Such disposition is achieved when the system of vectors x_1, \dots, x_N is orthonormal. Fig. 2 illustrates the evolution of 4 agents towards this stable configuration.

4.2. Case B: $N = 5$ agents

For $N = 5$ the situation is more involved. The agents tend to occupy orthonormal system of vectors on S^3 , so that all interactions w_{jk} vanish. However, such configuration is not possible, since the dimension of the ambient space is insufficient and there is one redundant agent. Hence, they seek for a compromise, and if the interactions can be made sufficiently weak, this compromise turns out to be stable.

In Fig. 3 we depict the evolution towards the stable configuration when the parameter μ is sufficiently large.

In fact, for any $\mu > \mu_0 \approx 1.39$ the same stable equilibrium configuration (with cosines of all angles between vectors equal to ± 0.25) is achieved.

On the whole, simulations suggest that for any $\mu > 1.39$ there exists essentially unique (up to rotation on S^3 and remuneration of agents) stable equilibrium.

For $\mu < 1.39$ this equilibrium is unstable and the whole system oscillates; this is illustrated in Fig. 4. Hence, as μ decreases, the system undergoes an oscillatory bifurcation at $\mu \approx 1.39$.

4.3. Case C: $N = 6$ agents

The model with 6 agents has bifurcation value μ_1 approximately at 1.59. For $\mu > \mu_1$ the system tends to a stable equilibrium. There is an essential difference with the previous case: this equilibrium is not unique and the final configuration depends on initial conditions. In other words, the system with 6 agents and $\mu > \mu_1 \approx 1.59$ is *multistable*, with several (essentially different) stable equilibria. The convergence towards one stable configuration is shown in Fig. 5.

For $\mu < \mu_1$ the system exhibits oscillatory behavior, see Fig. 6.

Summarizing the above discussion, the simulation results indicate that the agents will achieve a compromise if the parameter μ is sufficiently large. In other words, for sufficiently large μ there exists either a unique (modulo 4D rotations and remuneration of agents) stable equilibrium (when $N = 5$), or several essentially different stable equilibria (as with $N = 6$ agents). The following theorem claims that this is valid for arbitrary number of agents.

Theorem 4.1. *For μ sufficiently large the system (3) with anti-Hebbian learning rule (8) tends to a stable stationary equilibrium.*

Proof. For simplicity, we will work with real vectors x_1, \dots, x_N and the system in the form (4).

First, introduce the vectors:

$$\alpha_j = \sum_{k \neq j} w_{jk} x_k, \quad j = 1, \dots, N.$$

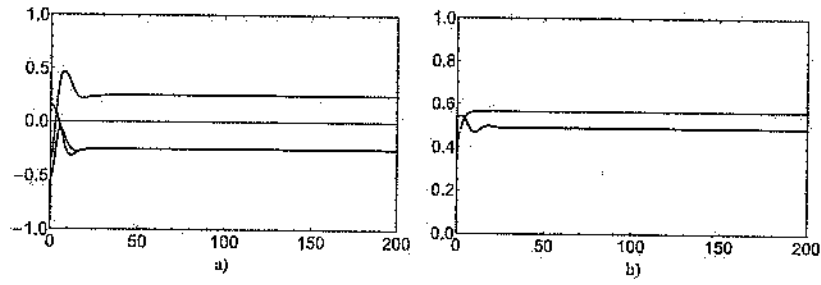


Fig. 3. Model with anti-Hebbian learning (3), (8) with $N = 5$ agents. Parameter values: $\varepsilon = 0.2, \mu = 2$. Initial values are explained in Remark 3.1. (a) inner products $\langle x_j, x_k \rangle$ for some j, k and (b) order parameters: global order parameter (black), φ —angle order parameter (red) and ψ —angle order parameter (yellow). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

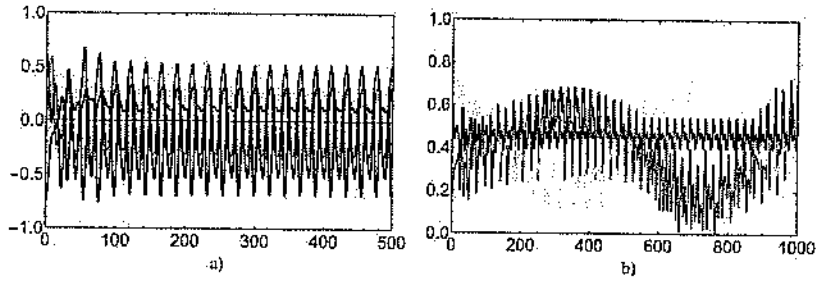


Fig. 4. Model with anti-Hebbian learning (3), (8) with $N = 5$ agents. Parameter values: $\varepsilon = 0.2, \mu = 1$. Initial values are explained in Remark 3.1. (a) inner products $\langle x_j, x_k \rangle$ for some j, k and (b) order parameters: global order parameter (black), φ —angle order parameter (red) and ψ —angle order parameter (yellow). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

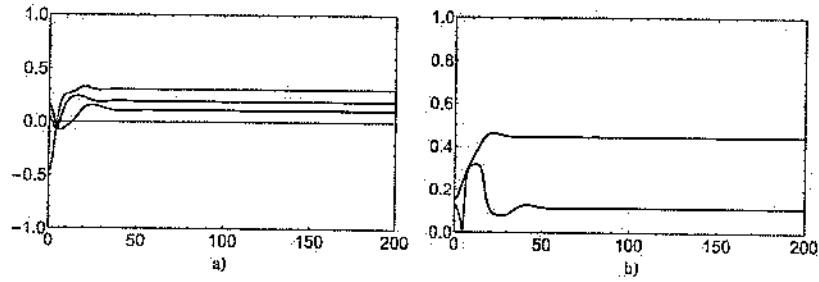


Fig. 5. Model with anti-Hebbian learning (3), (8) with $N = 6$ agents. Parameter values: $\varepsilon = 0.2, \mu = 2$. Initial values are explained in Remark 3.1. (a) inner products $\langle x_j, x_k \rangle$ for some j, k and (b) order parameters: global order parameter (black), φ —angle order parameter (red) and ψ —angle order parameter (yellow). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

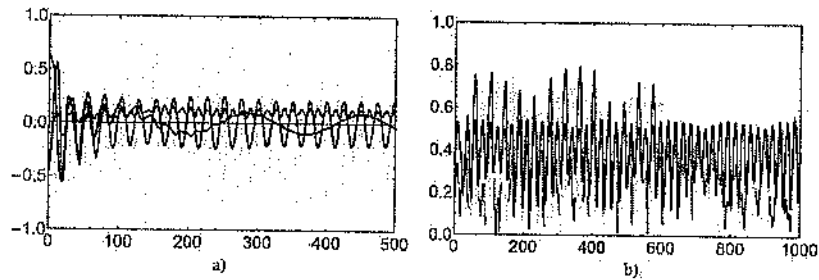


Fig. 6. Model with anti-Hebbian learning (3), (8) with $N = 6$ agents. Parameter values: $\varepsilon = 0.2, \mu = 1$. Initial values are explained in Remark 3.1. (a) inner products $\langle x_j, x_k \rangle$ for some j, k and (b) order parameters: global order parameter (black), φ —angle order parameter (red) and ψ —angle order parameter (yellow). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Vector α_j can be interpreted as the total influence on agent j from all other agents.

Substituting α_j into (4) we obtain:

$$\dot{x}_j = \alpha_j - \langle \alpha_j, x_j \rangle x_j.$$

Define the kinetic energy of the system: $E(t) = \frac{1}{2} \sum_{j=1}^N |\dot{x}_j(t)|^2$.

Taking into account that x_j are unit vectors it is easy to check that $E(t) = \frac{1}{2} \sum_j \langle \dot{x}_j, \alpha_j \rangle$.

Rewrite the system (8) in real vectors:

$$\dot{w}_{jk} = -\langle x_j, x_k \rangle - \mu w_{jk}, \quad \mu > 0. \quad (9)$$

For the proof we need the following auxiliary

Lemma 4.1. For μ sufficiently large it holds

$$\sum_{j=1}^N \sum_{k=1}^N \dot{w}_{jk}^2(t) \leq E(t) + \epsilon(t),$$

where $\epsilon(t)$ is a differentiable function that converges to zero when $t \rightarrow \infty$.

The proof of this Lemma is provided in Appendix A.

In order to prove the Theorem consider the following function:

$$F(t) = \sum_{j=1}^N \langle x_j, \alpha_j \rangle + \frac{1}{2} \mu \sum_{j=1}^N \sum_{k=1}^N w_{jk}^2.$$

Differentiation of $F(t)$ along the trajectories of (4), (9) yields:

$$\begin{aligned} \frac{d}{dt} F(t) &= \sum_j \langle \dot{x}_j, \alpha_j \rangle + \sum_j \langle x_j, \dot{\alpha}_j \rangle + \mu \sum_j \sum_k \dot{w}_{jk} \dot{w}_{jk} = \\ &= \sum_j \langle \dot{x}_j, \alpha_j \rangle + \sum_j \sum_k \langle x_j, w_{jk} \dot{x}_k \rangle + \sum_j \sum_k \langle x_j, \dot{w}_{jk} x_k \rangle - \\ &= \mu \sum_j \sum_k w_{jk} \langle x_j, x_k \rangle - \mu^2 \sum_j \sum_k w_{jk}^2 = \\ &= \sum_j \langle \dot{x}_j, \alpha_j \rangle + \sum_k \left(\sum_j w_{kj} x_j, \dot{x}_k \right) + \sum_k \left(\sum_j \dot{w}_{kj} x_j, x_k \right) - \\ &= \mu \sum_k \left(\sum_j w_{kj} x_j, x_k \right) - \mu^2 \sum_j \sum_k w_{jk}^2 = \end{aligned}$$

(using (9) and the symmetricity of interactions $w_{jk}(t)$ for all t)

$$\begin{aligned} &= \sum_j \langle \dot{x}_j, \alpha_j \rangle + \sum_k \langle \alpha_k, \dot{x}_k \rangle + \sum_j \sum_k (-\langle x_j, x_k \rangle - \mu w_{jk}) \langle x_j, x_k \rangle - \\ &= \mu \sum_k \left(\sum_j w_{kj} x_j, x_k \right) - \mu^2 \sum_j \sum_k w_{jk}^2 = 2 \sum_j \langle \dot{x}_j, \alpha_j \rangle - \\ &= \sum_j \sum_k \langle x_j, x_k \rangle^2 - \mu \sum_k \langle \alpha_k, x_k \rangle - \mu \sum_j \langle x_j, \alpha_j \rangle - \\ &= \mu^2 \sum_j \sum_k w_{jk}^2 = 4E(t) - \sum_j \sum_k \langle x_j, x_k \rangle^2 - \\ &= 2\mu \sum_j \sum_k w_{jk} \langle x_j, x_k \rangle - \mu^2 \sum_j \sum_k w_{jk}^2 = \\ &= 4E(t) - \sum_j \sum_k (-\langle x_j, x_k \rangle - \mu w_{jk})^2 = 4E(t) - \sum_j \sum_k \dot{w}_{jk}^2. \end{aligned}$$

The kinetic energy $E(t)$ is a nonnegative function. Applying Lemma 4.1 we now obtain that $\frac{d}{dt} F(t) \geq 3E(t) + \epsilon(t) \geq \epsilon(t)$ where $\epsilon(t) \rightarrow 0$ when $t \rightarrow \infty$.

Further, the function $F(t)$ is bounded, since

$$\begin{aligned} |F(t)| &\leq \left| \sum_j \langle x_j, \alpha_j \rangle \right| + \mu \left| \sum_j \sum_k w_{jk}^2 \right| = \\ &= \left| \sum_j \sum_k w_{jk} \langle x_j, x_k \rangle \right| + \mu \left| \sum_j \sum_k w_{jk}^2 \right| \leq \sum_j \sum_k (|w_{jk}| + \mu w_{jk}^2) \end{aligned}$$

and the functions $w_{jk}(t)$ are bounded as solution of (9) with $\mu > 0$.

Hence, $F'(t) \geq \epsilon(t)$ and $|F(t)| < c$ and we conclude that $F'(t) \rightarrow 0$ when $t \rightarrow \infty$.

One now has that $3E(t) + \epsilon(t) \leq F'(t) \rightarrow 0$ as $t \rightarrow \infty$ and $E(t) \geq 0$. Hence, $E(t) \rightarrow 0$ and the system tends to a stationary equilibrium when $t \rightarrow \infty$. \square

5. Applications and visualization

As emphasized in Introduction, the geometry of S^3 makes it very convenient underlying space for the design of consensus and

coordination algorithms. This suggests that in many cases, it is advantageous to design algorithms on rotation groups $SO(3)$ and $SO(4)$ by using group homomorphisms of S^3 . This is achieved by running simulations on S^3 and mapping the trajectories of all agents from S^3 onto $SO(3)$ (and from $S^3 \times S^3$ to $SO(4)$). It is well known that in certain situations this approach may cause some undesired effects. For example, bipolar configuration on S^3 corresponds to consensus on $SO(3)$.

In Appendix B, we provide several short videos illustrating different dynamical regimes in swarms with 5 and 6 agents with anti-Hebbian learning rule (8). These videos are obtained by mapping trajectories from S^3 to $SO(3)$.

The first two videos illustrate the convergence towards equilibrium configurations with 5 and 6 agents respectively. The parameter value is $\mu = 2$ in both simulations.

Further, we illustrate oscillatory regimes that occur when μ is below bifurcation threshold. The third and fourth videos demonstrate an oscillatory behavior in swarms with 5 and 6 agents respectively. The parameter value is set at $\mu = 1$.

6. Conclusion

Recent papers [6, 15] brought a significant progress in understanding consensus algorithms on spheres over arbitrary communication graphs. In particular, it has been shown in [6] that consensus set is almost globally stable for swarms on spheres consisting of agents communicating through connected undirected graph.

In the present paper we have analyzed swarm dynamics on sphere with state-dependent interactions that satisfy certain realizations of Hebbian and anti-Hebbian learning rules. The model with anti-Hebbian rule can exhibit various stable formations or oscillatory behavior depending on system parameters. We have proven the theorem stating that the swarm tends towards a stable equilibrium provided that the system parameter is sufficiently large.

Underline that all theoretical results of this paper hold for any sphere S^n (and not for S^3 exclusively). We have focused on S^3 since this particular case is of special importance in applications. In addition, group property of S^3 makes it convenient to introduce different communication protocols and algorithms and to present simulation results more transparently. For instance, the notion of angular order parameters introduced in this paper makes sense for S^3 only.

Acknowledgments

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Appendix A

This Appendix contains the proof of Lemma 4.1. In order to simplify notations, set: $b(t) = -\langle x_j, x_k \rangle(t)$, $x(t) = w_{jk}(t)$. Then Eq. (9) is written as

$$\dot{x} = b(t) - \mu x.$$

The solution of this ODE is written as follows

$$x(t) = x(0)e^{-\mu t} + \int_0^t b(s)e^{-\mu(t-s)} ds =$$

$$x(0)e^{-\mu t} + \int_0^t b(s)e^{-\mu(t-s)} ds = x(0)e^{-\mu t} + e^{-\mu t} \int_0^t b(s)e^{\mu s} ds =$$

(using integration by parts) =

$$x(0)e^{-\mu t} + \frac{1}{\mu}e^{-\mu t} \left(b(t)e^{\mu t} - b(0) - \int_0^t b'(s)e^{\mu s} ds \right) = \frac{1}{\mu}b(t) - \frac{1}{\mu}e^{-\mu t} \int_0^t b'(s)e^{\mu s} ds + e^{-\mu t} \left(x(0) - \frac{1}{\mu}b(0) \right).$$

Multiplying this equality by μ we obtain

$$\mu x(t) - b(t) = -e^{-\mu t} \int_0^t b'(s)e^{\mu s} ds + \epsilon(t),$$

where $\epsilon(t)$ denotes a function that converges to zero when $t \rightarrow \infty$.

By taking limit when $t \rightarrow \infty$ and using L'Hospital's rule we get

$$\mu x(t) - b(t) \rightarrow -\frac{b'(t)e^{\mu t}}{\mu e^{\mu t}} = -\frac{b'(t)}{\mu}, \text{ when } t \rightarrow \infty.$$

Evaluating the squares of the above equality

$$(\mu x(t) - b(t))^2 = \frac{b'(t)^2}{\mu^2} + \epsilon(t).$$

Turning back to the original notations yields

$$(\mu x(t) - b(t))^2 = (\mu w_{jk}(t) + \langle x_j, x_k \rangle(t))^2 =$$

$$\dot{w}_{jk}(t)^2 = \frac{\left(\frac{d}{dt} \langle x_j, x_k \rangle(t) \right)^2}{\mu^2} + \epsilon(t).$$

Now, it is easy to check that $\left(\frac{d}{dt} \langle x_j, x_k \rangle(t) \right)^2 \leq cE(t)$ for some positive constant c . Indeed:

$$\begin{aligned} \left(\frac{d}{dt} \langle x_j, x_k \rangle(t) \right)^2 &= \langle \dot{x}_j, x_k \rangle^2 + \langle x_j, \dot{x}_k \rangle^2 + 2\langle \dot{x}_j, x_k \rangle \langle x_j, \dot{x}_k \rangle \leq \\ &|\dot{x}_j|^2 \cdot |x_k|^2 + |x_j|^2 \cdot |\dot{x}_k|^2 + 2|\dot{x}_j| \cdot |x_k| \cdot |x_j| \cdot |\dot{x}_k| = \\ &|\dot{x}_j|^2 + |\dot{x}_k|^2 + 2|\dot{x}_j| \cdot |\dot{x}_k| \leq c \sum_i |\dot{x}_i|^2 = cE(t). \end{aligned}$$

Hence,

$$\sum_j \sum_k \dot{w}_{jk}^2 = \frac{1}{\mu^2} \sum_j \sum_k \left(\frac{d}{dt} \langle x_j, x_k \rangle(t) \right)^2 + \epsilon(t) \leq \frac{cN^2}{\mu^2} E(t) + \epsilon(t)$$

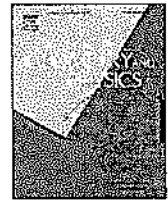
and for μ sufficiently large: $\sum_j \sum_k \dot{w}_{jk}^2 \leq E(t) + \epsilon(t)$.

Appendix B. Supplementary material

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.sysconle.2018.10.004>. This material contains short videos illustrating the swarm dynamics with 5 and 6 agents under the anti-Hebbian learning rule. See Section 5 for details.

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Modelling mean fields in networks of coupled oscillators

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ABSTRACT

We present the mathematical model of mean fields in complex networks of identical Kuramoto oscillators. In this framework, mean fields in the network are represented by the set of points in the unit disc with hyperbolic metric (Poincaré disc model). This set of points characterizes the network topology. The simulations for some random and regular graphs are presented.

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1. Introduction

The classical Kuramoto model, studied in his seminal paper [1], describes globally (all-to-all) coupled heterogeneous population of phase oscillators:

$$\dot{\varphi}_j = \omega_j + \frac{K}{N} \sum_{i=1}^N \sin(\varphi_i - \varphi_j), \quad j = 1, \dots, N. \quad (1)$$

Here, $\varphi_j(t)$ and ω_j denote the phase and the frequency of oscillator j , and K is a global coupling strength, the same for each pair of oscillators.

Probably the most important feature of this model is its *mean-field character*. In fact, Kuramoto started the analysis of (1) by introducing order parameter $r(t)$ and mean phase $\mu(t)$, defined by:

$$r(t)e^{i\mu(t)} = \frac{1}{N} \sum_{j=1}^N e^{i\varphi_j(t)}. \quad (2)$$

In new variables, (1) can be rewritten in its *mean-field form*:

$$\dot{\varphi}_j = \omega_j + Kr \sin(\mu - \varphi_j), \quad j = 1, \dots, N. \quad (3)$$

This form unveils the mean-field effect of the global coupling: it acts like all oscillators would be coupled to the same external field $\mu(t)$ with the common coupling strength $Kr(t)$ (see also [2]). This property makes the classical Kuramoto model so mathematically tractable. On the other hand, it is not trivial; its intrigue stems from the fact that both mean phase $\mu(t)$ and the strength of mean field $Kr(t)$ vary in time and depend on states of all oscillators at each instant of time.

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The same mean-field argument is valid for more general types of coupling, such as time-delayed coupling with global delay τ , phase-shifted coupling (Kuramoto–Sakaguchi model), time-dependent coupling $K(t)$, or noisy coupling with the common noise. It is essential that the coupling is global — that is, the same between each pair of oscillators.

It is natural to try the similar reasoning when dealing with the population of oscillators coupled through the complex network of interactions. Given the sufficiently large network, it can be conceived that contributions of all (or the greater part of) oscillators to collective dynamics are infinitesimally small (negligible) and interactions between oscillators can be approximated by introducing one or more mean fields. However, the situation is obviously far more complicated in this case, as the mean field does not have the same effect on all oscillators. More precisely, dynamics of different oscillators is governed by different mean fields. The challenge is to choose appropriate mathematical objects that describe mean fields in complex network.

In this paper, we introduce the mathematical model of mean fields in networks of coupled oscillators. Our approach, inspired by the paper [3], is a continuation and extension of [4] and can be used to characterize the network topology, detect communities, compare different networks, etc.

The concepts introduced in this paper are particularly transparent when applied to globally coupled population. In this case, there exists a global mean field that can be represented by a unique Möbius transformation at any moment. This is explained in Section 2. In Section 3 we discuss how this idea can be adapted when dealing with complex networks. In Section 4, the mathematical framework for description of mean fields in complex networks of oscillators is introduced. Based on it, we develop the statistical method of computation of mean fields and apply it to characterize some regular and random networks. Finally, in Section 5, we draw some conclusions and briefly discuss potential applications of our approach.

2. MMS principle and mean fields in globally coupled population

We start from the particularly simple case of the homogeneous population with global coupling, i.e. model (1) with all oscillators having the same intrinsic frequency $\omega_j \equiv \omega$. Following [3], rewrite (1) in more general form:

$$\dot{\phi}_j = f e^{i\phi_j} + \omega + \bar{f} e^{-i\phi_j}, \quad j = 1, \dots, N, \quad (4)$$

where f is a global complex-valued coupling function. Introducing the new variable $z_j(t) = e^{i\phi_j(t)}$, we represent the state of oscillator j as a point on the unit circle S^1 in the complex plane.

Denote by G the (sub)group of Möbius transformations that preserve the unit disc \mathbb{D} in the complex plane. The general transformation belonging to G can be written as:

$$\mathcal{M}(z) = \frac{e^{i\psi} z + \alpha}{1 + \bar{\alpha} e^{i\psi} z}, \quad (5)$$

for some $\psi \in [0, 2\pi]$ and $\alpha \in \mathbb{C}$, $|\alpha| < 1$.

In [3], it is shown that the states $z_j(t)$ of globally coupled population of identical oscillators (4) evolve by the action of one-parametric family of transformations (5) with the parameters ψ and α satisfying the following system of ODEs [5]:

$$\begin{aligned} \dot{\alpha} &= i(f(t)\alpha^2 + \omega\alpha + \bar{f}(t)); \\ \dot{\psi} &= f(t)\alpha + \omega + \bar{f}(t)\bar{\alpha}. \end{aligned} \quad (6)$$

We will refer to this result as *MMS principle*.

Remark 1. MMS principle states that (4) admits many constants of motion and can be reduced to three-dimensional dynamics (6) of global variables $\alpha(t)$ and $\psi(t)$. It is important for our further considerations to emphasize one consequence that follows from the Lie group theory: given the initial states of oscillators $z_1(0), \dots, z_N(0)$, their states $z_1(t), \dots, z_N(t)$ at each moment t are obtained by the action of certain disc-preserving Möbius transformation. In other words, for each $t > 0$, one has $z_j(t) = \mathcal{M}_t(z_j(0))$, $j = 1, \dots, N$ for some $\mathcal{M}_t \in G$. Notice however, that it is impossible to specify this transformation \mathcal{M}_t a priori, as it depends on coupling function f and states of all oscillators at each instant of time.

There are several possible ways to derive MMS principle. In [3] two methods are exposed, one based on analytic and another on algebraic and geometric arguments. Different method with more algebraic details is exposed in [6]. It is based on observation that the formal substitution $z_j(t) = e^{i\phi_j(t)}$ in (4) yields complex Riccati ODEs:

$$\dot{z}_j = i(fz_j^2 + \omega z_j + \bar{f}). \quad (7)$$

First, notice that (7) with $\omega \in \mathbb{R}$ defines the flow on the unit circle — that is, given the initial condition $z_j(0)$ on S^1 , one has that $z_j(t) \in S^1$ for any $t > 0$. Furthermore, it can be shown that Eq. (7) defines one-parametric family of disc-preserving Möbius transformations. Indeed, Poincaré maps of Riccati equations are Möbius transformations, see for instance [7,8].

However, there is an important nuance in the above reasoning. By referring to (7) as Riccati ODE, we implicitly assume that the coupling function f depends on t only, and not on z_1, \dots, z_N . This is essentially mean-field approximation; by adopting it we conceive that equations for the states of oscillators are coupled only through some common complex-valued function $f(t)$.

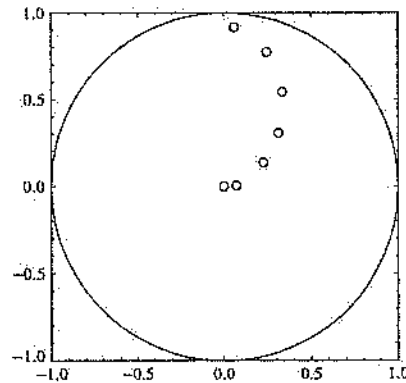


Fig. 1. Evolution of the mean field on time interval $t \in [0, 4]$ in population of $N = 500$ oscillators with global coupling strength $K = 2$.

For sufficiently large globally coupled population this approximation is valid and we can treat (7) as Riccati flow on S^1 with time-dependent coefficients $f(t) : \mathbb{R} \rightarrow \mathbb{C}$ and $\omega \in \mathbb{R}$. The situation is more involved if coupling is not global, i.e. if oscillators are coupled through the complex network. This will be discussed in the next section. We start by introducing the mathematical model of a (unique) mean field in globally coupled population.

Definition 1. Mean field in globally coupled population (4) at the moment $T \geq 0$ is a Möbius transformation $\mathcal{M}_T \in G$, such that states of all oscillators at T are given by $z_j(T) = \mathcal{M}_T(z_j(0))$, $j = 1, \dots, N$.

It follows from MMS principle (see Remark 1) that such Möbius transformation exists for any $T \geq 0$.

With each disc-preserving Möbius transformation (5) we associate the corresponding parameter $\alpha \in \mathbb{C}$, $|\alpha| < 1$. In this way, mean field in globally coupled population (4) at $t > 0$ is represented by the point in the unit disc $\alpha(t)$, satisfying the first ODE in (6), see Fig. 1.

Remark 2. Geometric meaning of parameter $\alpha(t)$ is also clearly exposed in [3]. The evolution of states of oscillators is governed by a certain one-parametric family of Möbius transformations $\mathcal{M}_t \in G$. Then, $\alpha(t)$ turns out to be the image of the centre under the action of the family \mathcal{M}_t : $\alpha(t) = \mathcal{M}_t(0)$. In an exceptional case, when the initial distribution of phases is uniform on $[0, 2\pi]$, $\alpha(t)$ is simply the centroid of oscillators' states at the moment t . Another parameter of \mathcal{M}_t , the angle $\psi(t) \in [0, 2\pi]$, is the overall rotation of oscillators on time interval $[0, t]$ and is essentially irrelevant for characterization of mean field.

Clearly, mean field evolves with the time and its evolution depends both on coupling strength and initial states of oscillators.

3. Möbius transformations in complex network of oscillators

In Section 2 we have introduced mathematical definition (in fact, the model) of a mean field in the population of globally coupled oscillators. Our next goal is to extend this model in order to study the complex networks of oscillators. Consider the (sufficiently large) network \mathcal{N} of identical Kuramoto oscillators:

$$\dot{\varphi}_j = \omega + \frac{1}{N} \sum_{i=1}^N K_{ij} \sin(\varphi_i - \varphi_j), \quad j = 1, \dots, N. \quad (8)$$

The network of interactions \mathcal{N} is given by the matrix K_{ij} . Matrix K_{ij} can be asymmetric, moreover, the whole approach is valid also for the networks with phase-shifted, time-delayed or noisy coupling.

For the sake of consistency of exposition, suppose that initial phases of oscillators are chosen from the uniform distribution on $[0, 2\pi]$:

$$\varphi_j(0) \in U[0, 2\pi], \quad j = 1, \dots, N.$$

Rewrite (8) in more general form:

$$\dot{\varphi}_j = f_j e^{i\varphi_j} + \omega + \bar{f}_j e^{-i\varphi_j}, \quad j = 1, \dots, N. \quad (9)$$

As above, the state of oscillator j is represented by the unit complex number (point on S^1) $z_j(t) = e^{i\varphi_j(t)}$. Substitution in (9) yields the system of complex ODEs:

$$\dot{z}_j = i(f_j z_j^2 + \omega z_j + \bar{f}_j), \quad j = 1, \dots, N \quad (10)$$

for some coupling functions $f_j(t, z_1, \dots, z_N)$ and $\omega \in \mathbb{R}$.

Fix oscillator j and consider his equation (10). Strictly speaking, it is not Riccati equation, since the function f_j depends on z_1, \dots, z_N . However, by conceiving that the contribution of j to network dynamics is infinitesimally small, we can neglect the dependence of functions f_1, \dots, f_N on z_j and regard (10) as the Riccati flow equation on S^1 :

$$\dot{z}_j = i(f_j(t)z_j^2 + \omega z_j + \bar{f}_j(t)). \quad (11)$$

Obviously, in globally coupled (and large) population, all oscillators satisfy the same Riccati equation (7). In complex networks different oscillators in general satisfy different equations (11). Our approach is based on detection of the groups of oscillators that (approximately) satisfy the same Riccati equation.

In whole, if oscillator j satisfies the Riccati flow equation of the form (11), his state evolves by the action of one-parametric families of Möbius transformations with parameters $\alpha(t)$ and $\psi(t)$ satisfying (6). However, knowing the dynamics of one oscillator at time interval $t \in [0, T]$ is not enough to determine if it evolves by the action of Möbius group. The action of Möbius transformation can be detected only by observing dynamics of groups consisting of $p \geq 4$ oscillators and computing cross ratios (see [4]). We will use this method in simulations in the next section.

4. Mean fields in complex networks of Kuramoto oscillators

Consider the model (8) of oscillators coupled through the complex network \mathcal{N} with initial phases $\varphi_j(0)$ sampled from the uniform distribution on $[0, 2\pi]$.

Definition 2 ([4]).

1. We say that four oscillators i, j, k, l agree, if at any time $t \geq 0$, there exists a Möbius transformation \mathcal{M}_t that maps points $z_i(0), z_j(0), z_k(0), z_l(0)$ to points $z_i(t), z_j(t), z_k(t), z_l(t)$, respectively.
2. We say that the group of $p \geq 4$ oscillators i_1, \dots, i_p live in the common field, if at any time $t \geq 0$, there is a Möbius transformation \mathcal{M}_t that maps points $z_{i_1}(0), \dots, z_{i_p}(0)$ to points $z_{i_1}(t), \dots, z_{i_p}(t)$, respectively.

Notice that the above concepts depend both on network topology and initial phases of oscillators. In other words, four oscillators can agree in the network \mathcal{N} for certain initial conditions and disagree in the same network for some other initial conditions. Underline also that two sets of oscillators living in two common fields can intersect, since one oscillator can live in more than one common field.

Remark 3. Given four oscillators in the network \mathcal{N} , an efficient way to check if they agree is provided by the concept of cross ratio of four points on S^1 , see [4].

Definition 3.

1. We say that Möbius transformation \mathcal{M} is present in the network \mathcal{N} at the moment $T \geq 0$, if there exist four oscillators i, j, k, l , such that $z_i(T) = \mathcal{M}(z_i(0))$, $z_j(T) = \mathcal{M}(z_j(0))$, $z_k(T) = \mathcal{M}(z_k(0))$, $z_l(T) = \mathcal{M}(z_l(0))$. In other words, \mathcal{M} is present in the network \mathcal{N} at the moment T , if one can find four oscillators in \mathcal{N} that agree on \mathcal{M} at T .
2. Assume that \mathcal{M} is present in \mathcal{N} at $T \geq 0$. Then, the weight of \mathcal{M} is $\frac{m}{4N}$, where $m \leq N$ is the total number of oscillators that live in the common field \mathcal{M} at T .

Definition 4. Mean fields in the network \mathcal{N} at the moment $T \geq 0$ are all Möbius transformations that are present in \mathcal{N} at T . Weights of mean fields in \mathcal{N} are weights of corresponding Möbius transformations.

Fix the set of initial states $z_1(0), \dots, z_N(0)$ and the moment $T > 0$. Denote by Y_4 the set of all quadruples $\{i, j, k, l\}$ that agree. Obviously, Y_4 is finite, as it has at most $\binom{N}{4}$ elements. Each element from Y_4 defines a unique Möbius transformation at T . Consider the function $g_{\mathcal{N}} : Y_4 \rightarrow G$, mapping quadruples to Möbius transformations. Domain of $g_{\mathcal{N}}$ is the finite set $P \subseteq G$ consisting of all Möbius transformations that are present in \mathcal{N} at T .

Each Möbius transformation from P is represented by the corresponding parameter α . In this way, mean fields in \mathcal{N} at the moment $T \geq 0$ are represented by the finite set ("cloud") of (weighted) points in \mathbb{D} .

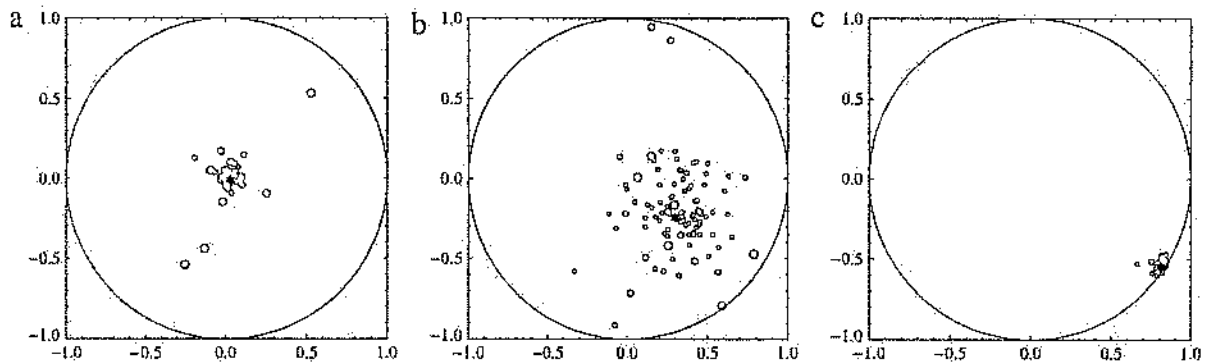
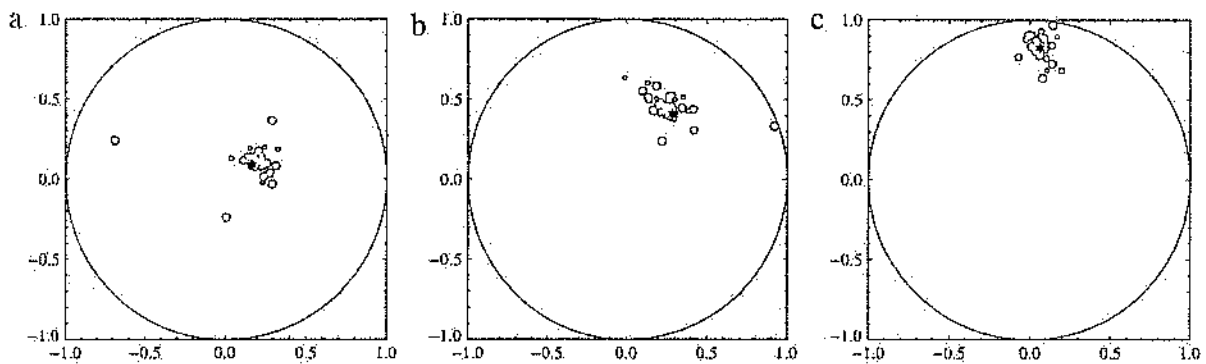
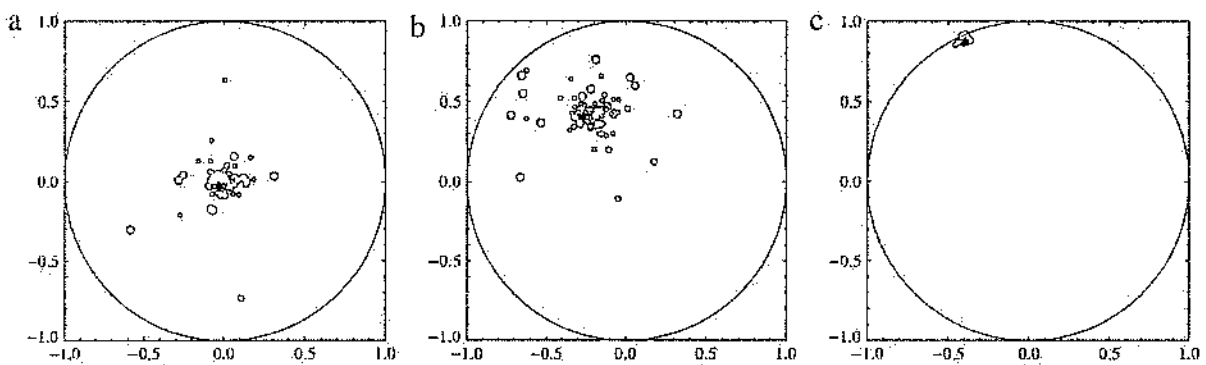
Suppose that mean fields in the network \mathcal{N} at the moment $T \geq 0$ are represented by the "cloud" of points $\alpha_1, \dots, \alpha_p$ in \mathbb{D} with corresponding weights w_1, \dots, w_p . Our next steps require notion of distance between points $\alpha_1, \dots, \alpha_p$. Recalling that these points are images of the zero point under some Möbius transformations $\mathcal{M}_1, \dots, \mathcal{M}_p$, it is natural to use hyperbolic distance (Poincaré disc model, see [9]).

Denote $w = w_1 + \dots + w_p$. Define the function:

$$f(\xi) = \sum_{k=1}^p \frac{w_k}{w} d^2(\xi, \alpha_k),$$

where

$$d(\xi, \alpha) = \ln \left(\frac{|\bar{\alpha}\xi - 1| + |\xi - \alpha|}{|\bar{\alpha}\xi - 1| - |\xi - \alpha|} \right)$$

Fig. 2. Mean fields in graph ER 0.1 at moments $T = 2, 25, 50$.Fig. 3. Mean fields in graph ER 0.9 at $T = 2, 3, 4$.Fig. 4. Mean fields in ER graphs with (a) $d = 0.1$, (b) $d = 0.5$ and (c) $d = 0.9$ at $T = 5$.

is a hyperbolic distance in \mathbb{D} between points ξ and α . Function f has the unique minimum B in \mathbb{D} , called the *Poincaré barycentre* (or, in more general context, *Karcher mean*, see [10,11]) of the set $\{\alpha_1, \dots, \alpha_p\}$.

Definition 5. The average mean field in the network \mathcal{N} at the moment $T \geq 0$ is the Poincaré barycentre $B(T)$ of the set $\{\alpha_1, \dots, \alpha_p\} \in \mathbb{D}$.

Denote by $\kappa(T)$ the value of function f at the minimum point $B(T)$.

Definition 6. The h -coherence of the network \mathcal{N} at the moment $T \geq 0$ is $r(T) = \frac{1}{1+\kappa(T)}$.

Remark 4. Notice that h -coherence (hyperbolic coherence) of the network differs from the notion of coherence, introduced in [4].

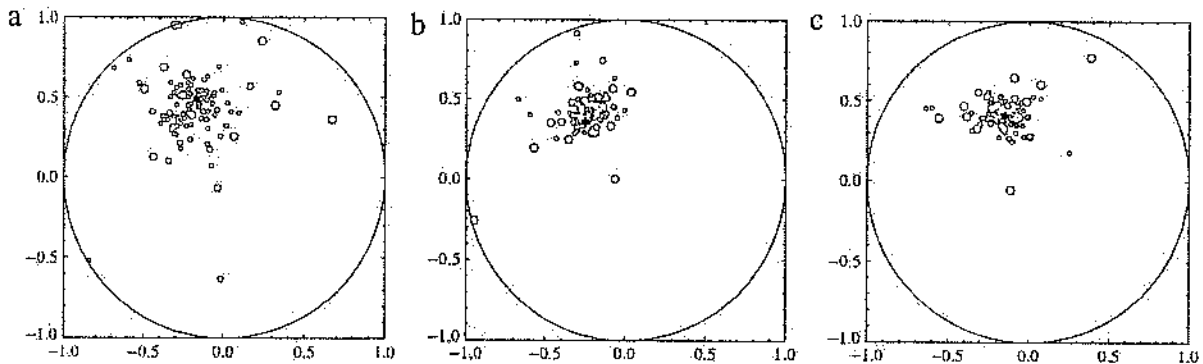


Fig. 5. Mean fields in WS graphs with $k = 250$ and (a) $\beta = 0.1$, (b) $\beta = 0.5$ and (c) $\beta = 0.9$ at $T = 5$.

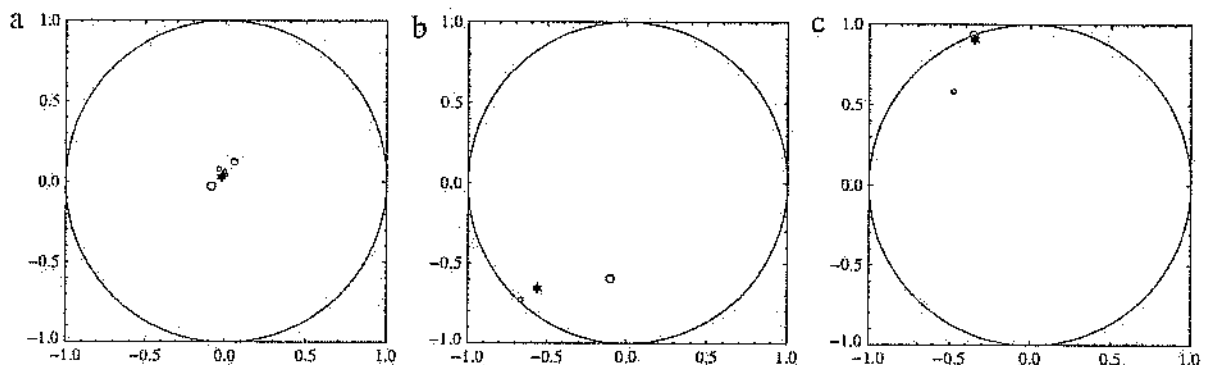


Fig. 6. Mean fields in the network consisting of two perfectly coherent communities with intra-community coupling $K = 0.675$ and inter-community coupling $\nu = 0.325$ at $T = 50, 200, 250$.

Remark 5. For a given network \mathcal{N} , average mean field $B(T)$ and h-coherence $r(T)$ are functions of time.

Remark 6. It is obvious that for the globally coupled population, studied in Section 2, $\kappa(T) = 0$ for any T . Therefore, its h-coherence equals one at any T . We say this is *perfectly h-coherent network*. For any other network, one has $0 < r(T) < 1$. Typically, h-coherence is a decreasing function of time.

In Figs. 2–6 mean fields in different networks are depicted. The method of simulations is based on the above definitions and nice properties of mathematical objects we use. However, in order to study complex networks, some adaptations and relaxations are necessary and we briefly explain the method (some of the following steps are explained more thoroughly in [4]).

1. Pick the random initial phases $\varphi_1(0), \dots, \varphi_N(0)$ from the uniform distribution on $[0, 2\pi]$.
2. Solve (8) on $t \in [0, T]$ with these initial phases to obtain phases of oscillators at the moment T : $\varphi_1(T), \dots, \varphi_N(T)$.
3. Substitute $z_j(0) = e^{i\varphi_j(0)}$ and $z_j(T) = e^{i\varphi_j(T)}$ for $j = 1, \dots, N$ to obtain states of oscillators at moments $t = 0$ and $t = T$ as points on S^1 .
4. Pick randomly 1000 quadruples of oscillators. For each quadruple i, j, k, l check if they agree. This is verified easily by comparing cross ratios of points z_i, z_j, z_k, z_l at moments $t = 0$ and $t = T$. If the cross ratio is preserved, then i, j, k, l agree. However, we need some flexibility at this point as the cross ratio is almost never perfectly preserved in complex network. Therefore, we allow some small error by supposing that oscillators agree if cross ratio is approximately preserved up to $\varepsilon = 10^{-2}$.
5. Suppose that oscillators i, j, k, l (approximately) agree. Then, using cross ratio, find all the oscillators that (approximately) agree with the triple i, j, k . Denote by $m \leq N$ the total number of oscillators that agree with i, j, k (counting them as well).
6. For the quadruple i, j, k, l , find their Möbius transformation \mathcal{M} (with some small error). The weight of \mathcal{M} is $\frac{m}{4N}$.
7. Write \mathcal{M} in the form (5) in order to find the corresponding α .
8. Depict α in \mathbb{D} with the corresponding weight.

Small error is tolerated at the step 4, and consequently in all subsequent steps. However, since our method is essentially statistical, this approximation does not affect simulation results significantly.

Table 1.
h-coherence of Watts–Strogatz graphs (coupling strength is set to $K = 7$ and $T = 3$).

WS graphs with $k = 50$	$\beta = 0$	$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.9$	$\beta = 1$
h-coherence	0.943087	0.959091	0.967391	0.974499	0.974103
WS graphs with $k = 250$	$\beta = 0$	$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.9$	$\beta = 1$
h-coherence	0.253563	0.27055	0.723713	0.798361	0.801395
WS graphs with $k = 450$	$\beta = 0$	$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.9$	$\beta = 1$
h-coherence	0.473923	0.480417	0.60308	0.638407	0.685943

Mean fields and h-coherence depend both on network topology and initial distribution of phases of oscillators. In this paper we explicitly assume uniform distribution of initial phases in order to focus on topology only. However, there are other possibilities. One might compare mean fields in one network for different distributions of initial phases (for instance, by taking uniform and von Mises initial distributions in two simulations).

In all simulations the same set of initial phases is used; this is convenient when comparing between two networks in order to avoid the ambiguity with random mean phase $\mu(t)$. Otherwise, when considering two networks one should compare two sets of points modulo group $SO(2)$ of planar rotations.

We present the results of simulations for some Erdős–Renyi (ER) and Watts–Strogatz (WS) graphs (see [12,13]). In Figs. 2 and 3, the evolution of mean fields in ER graphs with linking probabilities $d = 0.1$ and $d = 0.9$ are illustrated by depicting snapshots at different moments. Mean fields in different ER and WS graphs at $T = 5$ are shown in Figs. 4 and 5. The coupling strength on the link is set to be $K = 2$ in all simulations.

The average mean field (barycentre $B(t)$) is represented by the black star in all figures.

In general, random networks are more h-coherent than the regular ones. WS networks get more coherent as the probability of rewiring β increases, from minimally coherent with $\beta = 0$ (perfectly regular network), to maximally coherent when $\beta \rightarrow 1$ (yielding perfectly random network, i.e. ER graph); see Table 1.

In Fig. 6, the evolution of mean fields in the network with two perfect communities (all-to-all coupling, with intra-communities coupling $K = 0.675$, stronger than inter-community coupling $\nu = 0.325$) are depicted. This model with phase-shift $\gamma = \frac{\pi}{2} - 0.1$ in the coupling is the simplest network exhibiting chimera state, see [14]. As expected, Fig. 6 shows presence of two dominant mean fields, created inside two communities. When chimera state occurs, these two fields diverge, one evolving towards the circle S^1 and another performing (quasi)-periodic evolution in the interior of the disc \mathbb{D} . This example shows that mean fields can be very sensitive to small fluctuations in initial conditions.

5. Conclusion

In this paper we suggest that the group of Möbius transformations can serve as an adequate mathematical object to describe mean fields in networks of coupled oscillators. This approach enables us to take advantage of rich mathematical theory related to Möbius transformations, including classical concepts of Complex Analysis, Projective Geometry and Algebra. For instance, cross ratio is used to detect Möbius transformations based on dynamics of four oscillators. Also, the Lie group properties of the set of Möbius transformations are essentially used to visualize mean fields and measure the distance between them. Our method is statistical and applicable for the networks that contain hundreds of oscillators, or more.

The idea is particularly transparent for globally coupled population (complete graph). In this network, at each moment there is a global mean field, represented by one Möbius transformation and further visualized by a single point in \mathbb{D} . We say that this network is perfectly coherent.

Mean fields in the network are represented by the set of points in disc \mathbb{D} with hyperbolic metric. This set of points provides characterization of the network topology. Based on this set, we have introduced notions of average mean field and h-coherence of the network. The h-coherence decreases if the total coupling strength over the network grows. On the other hand, if the total coupling strength is fixed, the h-coherence increases with number of links. Therefore, it is natural to compare the h-coherence in networks with equal (expected) number of links and equal total coupling strength. In this setting, the h-coherence can be seen as a measure of randomness of the network. This is confirmed by the results presented in Table 1. For weighted networks, the h-coherence is maximal if the weights are uniformly distributed (equal) over all links.

This model can be applied to detect communities in networks. Kuramoto model has been used to study the network topology in number of papers (see [15,16] for more details); these methods are typically based on observing the process of synchronization in the network. In contrast, our idea is based on detecting mean fields. We say that the group of oscillators $\{i_1, \dots, i_p\}$ is coherent, if the set of mean fields $\{\alpha_1, \dots, \alpha_p\}$ where $\{i_1, \dots, i_p\}$ live, is contained in the small disc inside \mathbb{D} . (In particular, the group is perfectly coherent, if the set of mean fields is a single point). Coherent groups can be sparsely or densely interconnected (the latter are regarded as communities). However, in our approach they are clearly distinguished: densely interconnected groups correspond to the points that are distant from the centre of \mathbb{D} , while sparsely interconnected ones correspond to the points that are close to the centre.

As another potential application, we mention the interesting possibility of measuring similarity between networks by comparing the corresponding sets of points in Poincaré disc model.

Finally, underline that the whole approach (including all definitions of this paper) is valid for networks with various types of interactions, including noisy interactions, time-delayed interactions, etc.

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Low-dimensional dynamics in non-Abelian Kuramoto model on the 3-sphere

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Low-dimensional dynamics in non-Abelian Kuramoto model on the 3-sphere

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This paper deals with the low-dimensional dynamics in the general non-Abelian Kuramoto model of mutually interacting generalized oscillators on the 3-sphere. If all oscillators have identical intrinsic generalized frequencies and the coupling is global, the dynamics is fully determined by several global variables. We state that these generalized oscillators evolve by the action of the group $G_{\mathbb{H}}$ of (quaternionic) Möbius transformations that preserve S^3 . The global variables satisfy a certain system of quaternion-valued ordinary differential equations, that is an extension of the Watanabe-Strogatz system. If the initial distribution of oscillators is uniform on S^3 , additional symmetries arise and the dynamics can be restricted further to invariant submanifolds of (real) dimension four. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5029485>

Low-dimensional dynamics in the Kuramoto model with global coupling is well known and extensively studied. The first theoretical result of this kind has been obtained by Watanabe and Strogatz in 1994. In 2009, Marvel, Mirollo, and Strogatz provided a group theoretical explanation of this low-dimensional dynamics. It has been shown that Kuramoto oscillators evolve by the action of the group $G_{\mathbb{C}}$ of Möbius transformations that preserve S^1 . Moreover, parameters of these Möbius transformations are the global variables; they satisfy the system of ordinary differential equations (ODE's) known as the Watanabe-Strogatz system. Consequently, the dynamics takes place on the 3-dimensional invariant submanifolds that lie in the orbit of $G_{\mathbb{C}}$. In this paper, we report an analogous result for the non-Abelian Kuramoto model on S^3 . Non-Abelian Kuramoto models have been recently introduced as extensions of the Kuramoto model on S^1 . It is shown that oscillators in S^3 -model evolve by the action of the group $G_{\mathbb{H}}$ of Möbius transformations that preserve S^3 . The evolution of global variables is described by a certain system of quaternion-valued ODE's; this system is an extension of the Watanabe-Strogatz system. Finally, the evolution of the distribution of oscillators is restricted on invariant submanifolds of a real dimension 10. Along with theoretical interest, this study can have various applications in Science and Engineering; some of them are briefly pointed out in the paper.

I. INTRODUCTION

Synchronization in large ensembles of coupled oscillators is a universal phenomenon with great variety of manifestations in Physics and Life Sciences and applications in Engineering.¹ A paradigmatic model of this kind was introduced by Kuramoto² in 1975:

$$\dot{\varphi}_j = \omega_j + \frac{K}{N} \sum_{i=1}^N \sin(\varphi_j - \varphi_i), \quad j = 1, \dots, N. \quad (1)$$

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Here, N is the total number of oscillators, $\varphi_j(t)$ and $\omega_j \in \mathbb{R}$ are, respectively, the phase and the intrinsic frequency of the j -th oscillator, and K is the global coupling strength.

One can also consider a more general form of the Kuramoto model:

$$\dot{\varphi}_j = f e^{i\varphi_j} + \omega_j + \bar{f} e^{-i\varphi_j}, \quad j = 1, \dots, N, \quad (2)$$

with a global complex-valued coupling function $f = f(t, \varphi_1, \dots, \varphi_N)$.

The system (1) is obtained from (2) for the specific choice of coupling function:

$$f = -i \frac{K}{2N} \sum_{i=1}^N e^{-i\varphi_i}.$$

It is important for our further exposition to make a distinction, and we will refer to (1) as the *basic Kuramoto model* and to (2) simply as the *Kuramoto model*.

Notice that the form (2) includes various generalizations, such as phase-shifted coupling (Kuramoto-Sakaguchi model), delayed or noisy coupling, etc.

In this paper, we assume a simple setup with an ensemble consisting of identical oscillators with a global (all-to-all) coupling. Then, (2) is rewritten as follows:

$$\dot{\varphi}_j = f e^{i\varphi_j} + \omega + \bar{f} e^{-i\varphi_j}, \quad j = 1, \dots, N. \quad (3)$$

Individual oscillators in models (1)–(3) are called *Kuramoto oscillators* (or *sinusoidally coupled oscillators*). This term is used to stress that (a) amplitudes of oscillators are neglected and (b) the coupling depends only on the first harmonics $e^{i\varphi_j}$ and $e^{-i\varphi_j}$ (and not on higher harmonics).

At the beginning of the 1990s, it has been numerically observed^{3–6} that (3) exhibits seemingly low-dimensional dynamics, for instance, neutrally stable solutions such as so-called *splay states*.

These experimental findings have been followed by the theoretical result: Watanabe and Strogatz⁷ reported a special transformation of variables reducing (3) to the three-dimensional dynamical system and found $N - 3$ constants

of motion. In the end of the 2000s, new insight into low-dimensional behavior of (2) has been achieved^{8–10} in several papers.¹¹

The paper of Marvel *et al.*¹⁰ is particularly enlightening since it offers a group-theoretic explanation of this low-dimensional dynamics: the evolution of oscillators is governed by the action of Möbius transformations. In order to explain this, we introduce new complex variables $z_j(t) = e^{i\varphi_j(t)}$ representing the states of oscillators. In the absence of coupling, each oscillator performs rotations on the unit circle S^1 in the complex plane:

$$\dot{z}_j = i\omega z_j, \quad j = 1, \dots, N, \quad \omega \in \mathbb{R}. \quad (4)$$

Substituting $z_j(t) = e^{i\varphi_j(t)}$ into (3) yields complex-valued Riccati differential equations:

$$\dot{z}_j = i(fz_j^2 + \omega z_j + \bar{f}), \quad j = 1, \dots, N, \quad (5)$$

with a global coupling $f = f(t, z_1, \dots, z_N)$ and a common intrinsic frequency $\omega \in \mathbb{R}$.

Complex-valued Riccati equations are closely related to Möbius transformations in the extended complex plane. In addition, the equation of the specific form (5) preserves the unit circle, i.e., if $z_j(0) \in S^1$, then $z_j(t) \in S^1$ for all $t > 0$. (This is easily verified by showing that $z(0)\bar{z}(0) = 1$ implies $\frac{d}{dt}[z(t)\bar{z}(t)] = 0$.) Henceforth, we deal with the subgroup $G_{\mathbb{C}}$ of Möbius transformations that preserve the unit disc in the complex plane.

The general S^1 -preserving Möbius transformation can be written in the following parametrization:

$$\mathcal{M}(z) = \frac{e^{i\psi}z + \alpha}{1 + \bar{\alpha}e^{i\psi}z}, \quad (6)$$

with parameters $\psi \in [0, 2\pi]$ and $\alpha \in \mathbb{C}$, $|\alpha| < 1$.

The set of all S^1 -preserving Möbius transformations is a three-parametric subgroup $G_{\mathbb{C}}$ of the group of all Möbius transformations in the complex plane. Marvel *et al.* have shown that the states z_j of oscillators evolve by the action of $G_{\mathbb{C}}$. More precisely, $z_j(t) = \mathcal{M}_t[z_j(0)]$ for any $t > 0$ and $j = 1, \dots, N$, with \mathcal{M}_t being a one-parametric family of Möbius transformations in $G_{\mathbb{C}}$. Thus, the evolution of the distribution of oscillators is restricted to a three-dimensional submanifold lying in the space of distributions on S^1 . Moreover, parameters ψ and α in (6) satisfy the system that was obtained earlier by Watanabe and Strogatz:⁷

$$\begin{cases} \dot{\alpha} = i(f\alpha^2 + \omega\alpha + \bar{f}), \\ \dot{\psi} = f\alpha + \omega + \bar{f}\bar{\alpha}. \end{cases} \quad (7)$$

This line of research brings a new intrigue into the study of Kuramoto oscillators (along with traditional focus on frequency synchronization and phase transitions from/to coherent states): relations with hyperbolic geometry, complex analysis, and implications. A major step in this direction was made in the recent paper¹² by Chien *et al.* Among other things,

they have demonstrated that a wide class of systems of the form (3) exhibits the gradient flow dynamics in the Poincaré disc. As one consequence, conformal barycenters of probability measures on S^1 can be computed by solving (1) with the negative coupling strength ($K < 0$).¹³

The result of Marvel *et al.* relies to a great extent on geometric reasoning, especially on Lie group properties of both the unit circle S^1 and the group $G_{\mathbb{C}}$ of Möbius transformations acting on it.¹⁰ This observation is the main motivation for the present study: the analogous analysis can be conducted on some higher dimensional Lie groups (at least on the sphere S^3) with intriguing interpretations.

It turns out that the corresponding models have already been introduced in 2009 by Lohe.¹⁴ Lohe considered the extension of the classical Kuramoto model (1) to the group $U(n)$ of unitary matrices:

$$i\dot{U}_j U_j^* = H_j - \frac{iK}{2N} \sum_{l=1}^N (U_l U_l^* - U_j U_j^*), \quad j = 1, \dots, N. \quad (8)$$

This is the system of matrix ODE's for complex $n \times n$ matrices U_j , and H_j are given Hermitian matrices. The notion U_j^* stands for the conjugate matrix of U_j . Notice that (8) preserves $U(n)$, i.e., if the initial conditions satisfy $U_j(0) \in U(n)$, one has that $U_j(t) \in U(n)$ for all t .

Following the terminology of Lohe, we will refer to (8) for $n > 1$ as the *basic non-Abelian Kuramoto model*. Alternatively, researchers also call it the *Lohe model*. The model describes the collective motion of mutually interacting generalized oscillators, whose states are given by unitary matrices $U_j(t)$. These generalized oscillators are referred to as *Kuramoto-Lohe oscillators*. In the absence of coupling ($K = 0$), these oscillators satisfy simple equations:

$$\dot{U}_j = -iH_j U_j, \quad j = 1, \dots, N. \quad (9)$$

The matrices iH_j belong to the Lie algebra $\mathfrak{su}(n)$, we will consider them as *intrinsic generalized frequencies* of the Kuramoto-Lohe oscillators. For $n = 1$, (8) defines the motion on the group $U(1)$, which is precisely the basic Abelian Kuramoto model (1).

In this paper, we focus on the particular case $n = 2$ corresponding to the motions on $SU(2)$ with the group manifold S^3 . Intrinsic generalized frequencies $iH_j \in \mathfrak{su}(2)$ are skew-Hermitian zero-trace matrices.

In whole, there are two exciting ongoing research directions. The study of low-dimensional dynamics in the Abelian Kuramoto model started in 1994, and in 2009, relations with hyperbolic geometry and complex analysis are exposed. The second line of research, initiated by Lohe in 2009, works with non-Abelian Kuramoto models on unitary groups $U(n)$ and elaborates new paradigms of quantum synchronization.^{15,16} The present paper is intended as a continuation and combination of these two research directions.

In Sec. II, we introduce the non-Abelian Kuramoto model, which is a general version of the Lohe model on S^3 . One technical nuance is that the governing equations are written down in quaternion-valued variables.

In Sec. III, we provide a group-theoretic description of low-dimensional dynamics in the model as the motion on orbits of the group of (quaternionic) Möbius transformations. Parameters of Möbius transformations are global variables that determine the evolution of the whole system. The dynamics of these global variables is given by the generalized Watanabe-Strogatz (WS) system. Furthermore, this reasoning allows us to identify symmetries and constants of motion. In Sec. IV, we pass to Hopf (angular) coordinates on S^3 and identify a special form of the coupling function which turns the system into the basic non-Abelian Kuramoto model. In Sec. V, we introduce an additional assumption that the initial distribution of oscillators is uniform on S^3 . Similar as with the classical Kuramoto model, this assumption leads to a further reduction of the dimension and a simple ODE describing the evolution of the order parameter. In Sec. VI, we point out some applications of the model in Mathematics and Engineering. Finally, we conclude the paper with a brief discussion on some theoretical issues and interconnections that are still to be addressed.

II. THE MODEL OF COUPLED GENERALIZED OSCILLATORS ON S^3

There are different ways to introduce coordinates on S^3 . For our purposes, the algebra of quaternions is probably the most convenient one. Following an analogy with (4), consider a single generalized oscillator whose motion is described by the quaternion-valued ordinary differential equation (QODE):

$$\dot{q} = wq + qu, \quad (10)$$

where $q(t)$ is a unit quaternion and w and u are “pure” quaternions [meaning that $\text{Re}(w) = \text{Re}(u) = 0$]. The set of unit quaternions is identified with the Lie group S^3 with the corresponding Lie algebra consisting of pure quaternions. Hence, if initial point $q(0)$ for (10) is a unit quaternion, then the solution $q(t)$ will be a unit quaternion for all t . In other words, the motion (10) is restricted on S^3 . We say that the particle described by (10) is a Kuramoto-Lohe (KL) oscillator with intrinsic frequencies w and u .

Furthermore, conceive an ensemble consisting of N mutually interacting identical KL oscillators that satisfy quaternion-valued Riccati differential equation:

$$\dot{q}_j = q_j f q_j + w q_j + q_j u - \bar{f}, \quad j = 1, \dots, N. \quad (11)$$

Here, $f = f(t, q_1, \dots, q_N)$ is a quaternionic function, representing the coupling between KL oscillators. The notion \bar{f} in (11) stands for the conjugate quaternion, i.e., if $f = f_1 + if_2 + jf_3 + kf_4$, then $\bar{f} = f_1 - if_2 - jf_3 - kf_4$.

The system (11) is non-Abelian Kuramoto model on S^3 (or Lohe model for $n=2$). As we will see in Sec. IV, it is the generalization of (8) in the same way as (2) is the generalization of (1). In this paper, the main focus is on low-dimensional dynamics and group-theoretic properties of (11). Underline that we assume highly an idealized setup, with an ensemble consisting of identical (w and u do not depend on j) and globally coupled (f does not depend on j) KL oscillators.

One can verify that $q_j(0)\bar{q}_j(0) = 1$ implies $\frac{d}{dt}[q_j(t)\bar{q}_j(t)] = 0$, meaning that (11) preserves S^3 .

In conclusion, we mention that quaternion-valued Riccati differential equations of the form (11) have been studied in several papers.^{17–19}

III. LOW-DIMENSIONAL DYNAMICS OF GLOBALLY COUPLED GENERALIZED OSCILLATORS ON S^3

Denote by \mathbb{H} the algebra of quaternions. The set of Möbius (fractional linear) transformations acting on 4-dimensional sphere $\mathbb{H} \cup \{\infty\}$ is the group $GL(2, \mathbb{H})$. Then Lie algebra of $GL(2, \mathbb{H})$ consists of matrices that generate vector fields of the form $qfq + bq + qc + g$, where f, b, c , and $g \in \mathbb{H}$ are arbitrary quaternions.

Consider the subgroup $G_{\mathbb{H}} \subset GL(2, \mathbb{H})$ of all transformations that preserve the unit sphere S^3 . This subgroup has the Lie algebra consisting of matrices that generate vector fields with $g = -f$ and $\text{Re}(b) = \text{Re}(c) = 0$.²⁰ Therefore, the dynamics (11) can be described in terms of the actions of Möbius transformations that preserve S^3 .

In order to explore this in more detail, consider a generic Möbius transformation that preserves S^3 :

$$\mathcal{M}(q) = p(1 - q\bar{a})^{-1}(q - a)r,$$

for some unit quaternions p and r and a quaternion $a \in \mathbb{H}$, $|a| < 1$. The inverse transformation yields a slightly different parametrization:

$$\mathcal{G}(q) = (\bar{p}q\bar{r} + a)(1 + \bar{a}\bar{p}q\bar{r})^{-1}. \quad (12)$$

Theorem 1. Consider an ensemble of N coupled particles whose dynamics is governed by (11) with the initial conditions $q_1(0), \dots, q_N(0)$. Then, $q_j(t) = \mathcal{G}_t[q_j(0)]$ for some one-parametric family \mathcal{G}_t of transformations belonging to $G_{\mathbb{H}}$.

Moreover, parameters $a(t)$, $p(t)$, and $r(t)$ of \mathcal{G}_t satisfy the following set of QODE's:

$$\begin{cases} \dot{a} = afa + wa + au - \bar{f}; \\ \dot{p} = -p(w + af - \bar{a}\bar{f}); \\ \dot{r} = -r(u - af + \bar{a}\bar{f}). \end{cases} \quad (13)$$

The system (13) is an analogue (or extension) of the Watanabe-Strogatz system (7).

The above theorem establishes low-dimensional behavior of (11): the dynamics is restricted to $B^4 \times S^3 \times S^3$, which is a submanifold of a real dimension 10. (Here, B^4 denotes the unit ball in the 4-dimensional space.) The system (13) is obtained by the direct derivation of \mathcal{G}_t , corresponding calculations are shown in the Appendix.

Remark 1. The Möbius transformation \mathcal{G} is parametrized in such a way that $\mathcal{G}(0) = a$ [see formula (12)]. Furthermore, notice that the QODE for $a(t)$ in (13) is the same as Eq. (11) for q_j . By considering the extension of \mathcal{G}_t from S^3 to the unit ball B^4 , we readily see that $a(t)$ is an image of the origin under the action of \mathcal{G}_t , i.e., $a(t) = \mathcal{G}_t(0)$.

Now, it is not difficult to guess what might be the constants of motion of (11). Indeed, the quantity that is preserved under Möbius transformations is cross-ratio. In our case, the

quaternionic cross-ratios

$$Q(q_i, q_j, q_k, q_l) = (q_j - q_i)^{-1}(q_l - q_i)(q_l - q_k)^{-1}(q_j - q_k), \quad 1 \leq i, j, k, l \leq N, \quad (14)$$

are constants of motion. However, the question of how many functionally independent cross-ratios one has here is quite subtle and we cautiously go around it. Quaternionic Möbius transformation is not uniquely determined by its action on four points, and some additional invariants (along with cross-ratios) exist.²¹

IV. DYNAMICS IN HOPF COORDINATES AND SOME SPECIAL COUPLING FUNCTIONS

In this section, we will use the Cayley-Dickson form that allows us to represent quaternions by pairs of complex numbers, that is, $q = q_1 + q_2j$, $f = f_1 + f_2j$, where q_1, q_2, f_1 , and f_2 are complex numbers. As for pure quaternions, we denote $w = iw_a + w_2j$, $u = iu_a + u_2j$, where $w_a, u_a \in \mathbb{R}$, and w_2, u_2 are complex numbers. Unit quaternion q can also be written in angle variables:

$$q = q_1 + q_2j \text{ where } q_1 = e^{i\varphi} \cos \theta, \quad q_2 = e^{i\psi} \sin \theta,$$

with $\varphi, \psi \in (0, 2\pi)$, $\theta \in (0, \frac{\pi}{2})$. Substitution into (11) yields the following system of ODE's for $j = 1, \dots, N$:

$$\begin{cases} \dot{\varphi}_j = w_a + u_a + \frac{1}{2}[\bar{u}_2 \tan(\theta_j) e^{-i(\varphi_j - \psi_j)} - \bar{w}_2 \tan(\theta_j) e^{i(\varphi_j + \psi_j)} - u_2 \tan(\theta_j) e^{i(\varphi_j - \psi_j)} + w_2 \tan(\theta_j) e^{-i(\varphi_j + \psi_j)}] + \frac{1}{\cos(\theta_j)}(\bar{f}_1 e^{-i\varphi_j} - f_1 e^{i\varphi_j}); \\ \dot{\psi}_j = w_a - u_a + \frac{1}{2}[\bar{u}_2 \cot(\theta_j) e^{-i(\varphi_j - \psi_j)} + \bar{w}_2 \cot(\theta_j) e^{i(\varphi_j + \psi_j)} - u_2 \cot(\theta_j) e^{i(\varphi_j - \psi_j)} - w_2 \cot(\theta_j) e^{-i(\varphi_j + \psi_j)}] + \frac{1}{\sin(\theta_j)}(\bar{f}_2 e^{i\psi_j} - f_2 e^{-i\psi_j}); \\ \dot{\theta}_j = \frac{1}{2}[\bar{u}_2 e^{-i(\varphi_j - \psi_j)} + \bar{w}_2 e^{i(\varphi_j + \psi_j)} + u_2 e^{i(\varphi_j - \psi_j)} + w_2 e^{-i(\varphi_j + \psi_j)}] + \sin(\theta_j)(\bar{f}_1 e^{i\varphi_j} + \bar{f}_2 e^{-i\psi_j}) + \cos(\theta_j)(f_2 e^{-i\psi_j} + f_1 e^{i\varphi_j}). \end{cases} \quad (15)$$

This substitution is valid for $\theta \neq 0$ and $\theta \neq \frac{\pi}{2}$. Despite seemingly complicated expressions, the ODE's for angles provide some feeling about dynamics. First, notice that the coupling (terms containing f_1 or f_2) enters the equations in a pretty simple way, resembling the classical Kuramoto model (3). Second, observe that the angle φ_j is coupled to angles φ_i for $i \neq j$ only indirectly, through θ_j . Remarkably, the coupling quaternion $f = f_1 + f_2j$ splits: the complex number f_1 is responsible for the coupling of angles φ_j , while f_2 stands for the coupling of ψ_j .

These observations provide a hint on how to choose the particular form of the quaternion-valued function f depending on specific goals. For instance, set:

$$f = -\frac{K}{2N} \sum_{i=1}^N \bar{q}_i = -\frac{K}{2N} \langle \bar{q} \rangle, \quad (16)$$

where $\langle q \rangle = \sum q_i \in B^4$ denotes the centroid (center of mass) of the points $q_1, \dots, q_N \in S^3$. With this specific choice of the

coupling function f , the system (15) is rewritten as follows:

$$\begin{cases} \dot{\varphi}_j = w_a + u_a + \frac{1}{2}[\bar{u}_2 \tan(\theta_j) e^{-i(\varphi_j - \psi_j)} - \bar{w}_2 \tan(\theta_j) e^{i(\varphi_j + \psi_j)} - u_2 \tan(\theta_j) e^{i(\varphi_j - \psi_j)} + w_2 \tan(\theta_j) e^{-i(\varphi_j + \psi_j)}] + \frac{K}{N \cos(\theta_j)} \sum_{i=1}^N [\sin(\varphi_i - \varphi_j + \theta_i) + \sin(\varphi_i - \varphi_j - \theta_i)]; \\ \dot{\psi}_j = w_a - u_a + \frac{1}{2}[\bar{u}_2 \cot(\theta_j) e^{-i(\varphi_j - \psi_j)} + \bar{w}_2 \cot(\theta_j) e^{i(\varphi_j + \psi_j)} - u_2 \cot(\theta_j) e^{i(\varphi_j - \psi_j)} - w_2 \cot(\theta_j) e^{-i(\varphi_j + \psi_j)}] + \frac{K}{N \sin(\theta_j)} \sum_{i=1}^N [\cos(\psi_i - \psi_j - \theta_i) - \cos(\psi_i - \psi_j + \theta_i)]; \\ \dot{\theta}_j = \frac{1}{2}(\bar{u}_2 e^{-i(\varphi_j - \psi_j)} + \bar{w}_2 e^{i(\varphi_j + \psi_j)} + u_2 e^{i(\varphi_j - \psi_j)} + w_2 e^{-i(\varphi_j + \psi_j)}) - \frac{K \sin(\theta_j)}{N} \sum_{i=1}^N [\cos(\varphi_i - \varphi_j - \theta_i) + \cos(\varphi_i - \varphi_j + \theta_i)] + \frac{K \cos(\theta_j)}{N} \sum_{i=1}^N [\sin(\psi_i - \psi_j + \theta_i) - \sin(\psi_i - \psi_j - \theta_i)]. \end{cases} \quad (17)$$

The coupling is attractive if $K > 0$, in this case the system evolves towards synchronization for any initial conditions (underline that we assume identical generalized frequencies and all-to-all coupling). It can be shown²² that synchronization (fully coherent state) is the only asymptotically stable configuration for $K > 0$.

The situation is more subtle for the case of repulsive coupling, $K < 0$. For nearly all initial conditions, the system evolves towards a fully incoherent state. In fact, the system converges towards a fully incoherent state whenever the initial distribution does not contain a *majority cluster*, meaning that there are no $N/2$ oscillators located at the same point. Given the initial distribution of KL oscillators, this fully incoherent state is unique (up to rotation), it is an attracting fixed point on the orbit of $G_{\mathbb{H}}$. Chen *et al.*¹² provided a detailed and rigorous mathematical description of fixed points on orbits of $G_{\mathbb{C}}$ for the basic Kuramoto model on S^1 .

In order to illustrate the above discussion, introduce the global real order parameter of the system in the standard way:

$$\rho = |\langle q \rangle| \quad \text{where} \quad \langle q \rangle = \frac{1}{N} \sum_{i=1}^N q_i.$$

Then, the values $\rho = 1$ and $\rho = 0$ of the global order parameter correspond to coherent state (synchronization) and fully incoherent state, respectively.

Along with the global order parameter in this model, it also makes sense to define the *angular order parameters* ρ_φ and ρ_ψ in the following way:

$$\rho_\varphi(t) e^{i\langle \varphi \rangle(t)} = \frac{1}{N} \sum_{i=1}^N e^{i\varphi_i(t)}, \quad \rho_\psi(t) e^{i\langle \psi \rangle(t)} = \frac{1}{N} \sum_{i=1}^N e^{i\psi_i(t)}.$$

Figure 1 illustrates the evolution towards coherent state for $K = 0.5$: both global order parameter (a) and angular order parameters (b) gradually increase towards 1. Initial conditions are sampled from the uniform distribution on S^3 .

The evolution towards a fully incoherent state for the repulsive coupling $K = -0.5$ is shown in Fig. 2: global order parameter (a) and angular order parameters (b) tend to

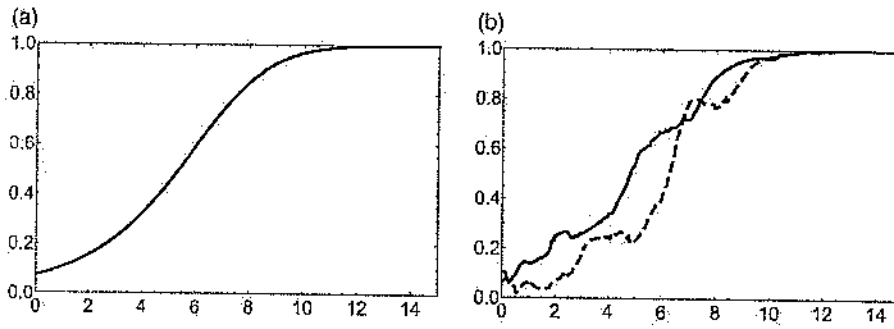


FIG. 1. The evolution of global order parameter ρ (a) and angular order parameters ρ_ψ (solid line) and ρ_ϕ (dashed line) (b) for the system (11) with the coupling function (16). The simulation is performed for $N = 100$ KL oscillators, with coupling strength $K = 0.5$. Initial conditions are sampled from the uniform distribution on S^3 .

zero. Initial conditions in this simulation are sampled from von Mises-Fisher distribution on S^3 with the mean direction $\mu = (1/2, 1/2, 1/2, 1/2)$ and the concentration parameter $\kappa = 2$.

Remark 2. Underline that coupling function in the form (16) is just one possibility for the model (11). In many situations, it might be interesting to consider some other coupling functions. For instance, one can introduce phase shifts along the angles φ and ψ in the following way:

$$f(q_1, \dots, q_N) = f_1(q_1, \dots, q_N) + f_2(q_1, \dots, q_N)j, \quad (18)$$

where complex-valued functions f_1 and f_2 are defined as follows:

$$f_1 = -\frac{K}{2N} \sum_{i=1}^N e^{-i(\varphi_i - \alpha)} \cos \theta_i,$$

$$f_2 = \frac{K}{2N} \sum_{i=1}^N e^{i(\psi_i - \beta)} \sin \theta_i.$$

Figure 3 illustrates the evolution of the global (a) and angular (b) order parameters for phase shifts $\alpha = \beta = \pi/2$. It can be seen that the global order parameter remains nearly constant around 0.2, while angular order parameters perform irregular oscillations approximately in anti-phase one with respect to another. The initial conditions for Fig. 3 are sampled from the uniform distribution.

Remark 3. By substituting (16) and setting $u = 0$ in (11), one obtains

$$\dot{q}_j = -\frac{K}{2N} \sum_{i=1}^N q_j \bar{q}_i q_i + w q_j + \frac{K}{2N} \sum_{i=1}^N q_i, \quad (19)$$

which is precisely the basic non-Abelian Kuramoto model (8) on S^3 .

In order to see this, multiply (8) by $-i$ and by matrices U_j from the right. Notice that $-iH_j$ are trace-zero skew-Hermitian 2×2 matrices comprising the Lie algebra $\mathfrak{su}(2)$. Using isomorphism between Lie groups, represent matrices $U_j \in SU(2)$ as unit quaternions and $-iH_j \in \mathfrak{su}(2)$ as pure quaternions. Hence, the model (8) is rewritten as (19).

V. UNIFORM INITIAL DISTRIBUTION YIELDS 4D DYNAMICS

Like with the Abelian Kuramoto model (3), there exists a special case when the dynamics (11) is completely determined by the single quaternionic parameter a in (12). In this case, equations for p and r in (13) decouple from the equation for a and the dynamics is restricted on the 4-dimensional invariant submanifold.

Consider (11) in the thermodynamic limit, $N \rightarrow \infty$, and suppose that the initial distribution of KL oscillators is uniform on S^3 . Then, the distribution evolves on the submanifold consisting of measures that are obtained as Möbius transformations of the uniform Lebesgue measure on S^3 . It is known that Möbius transformations of the uniform measure on S^3 yield harmonic measures (or, in probability terms, wrapped Cauchy distributions²³) with the density function:

$$\rho[y; a(t)] = \frac{1}{2\pi^2} \left[\frac{1 - |a(t)|^2}{|y - a(t)|^2} \right]^3, \quad y \in S^3,$$

where $a(\cdot) \in B^4$ is a parameter of the distribution.

Remark 4. Given the probability distribution on S^3 , one can focus on two important points in B^4 : centroid (center of mass) $\langle q \rangle(t)$ and the conformal barycenter²⁴ $a(t)$. There is a special feature of the case when the initial distribution is uniform: only in this case, these two points coincide. In other words, only in this case centroid is an image of zero under the

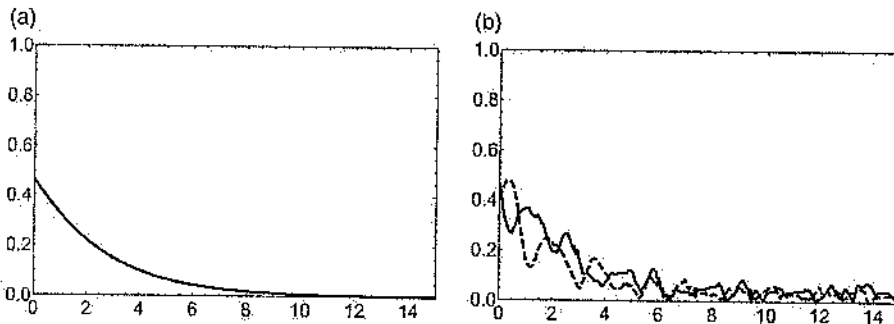


FIG. 2. The evolution of global order parameter ρ (a) and angular order parameters ρ_ψ (solid line) and ρ_ϕ (dashed line) (b) for (11) with the coupling function (16). The simulation is performed for $N = 100$ KL oscillators, with coupling strength $K = -0.5$. Initial conditions are sampled from the von Mises-Fisher on S^3 with mean direction $\mu = (0.5, 0.5, 0.5, 0.5)$ and the concentration $\kappa = 2$.

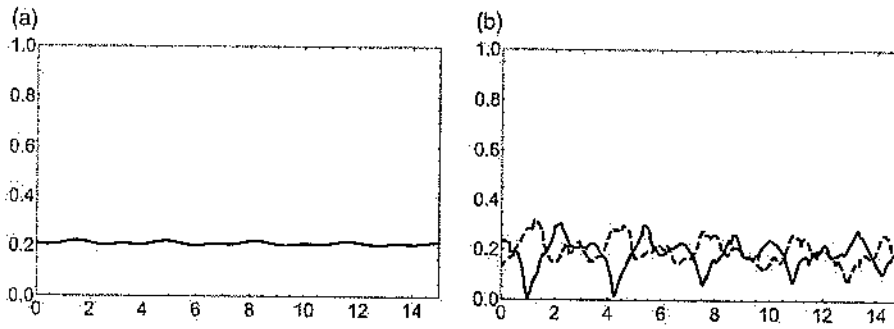


FIG. 3. The evolution of global order parameter ρ (a) and angular order parameters ρ_ψ (solid line) and ρ_ϕ (dashed line) (b) for (11) with the coupling function (18). The simulation is performed for $N = 50$ KL oscillators, with coupling strength $K = 2$ and phase shifts $\alpha = \beta = \pi/2$. Initial conditions are sampled from the uniform distribution on S^3 .

action of one-parametric family \mathcal{G}_t of Möbius transformations (cf. Remark 1).

Consider the basic non-Abelian Kuramoto model and assume the uniform initial distribution on S^3 . Only in this particular case, the evolution of the real order parameter $\rho(t) = |\langle q \rangle|$ is given by the simple (real-valued) ODE:

$$\frac{d}{dt}\rho = \frac{K}{2}(\rho - \rho^3).$$

VI. POTENTIAL APPLICATIONS

Variations of the classical Kuramoto model found interesting (and sometimes unexpected) applications in different fields.^{25–27} In this section, we point out some potential applications of the model (11).

We start by one purely mathematical application. Suppose that probability measure μ on S^3 is given. Assume in addition that μ does not contain atoms with the weight $\geq 1/2$. Then, the conformal barycenter of μ can be found by solving the basic non-Abelian Kuramoto system (17) with $K < 0$ and with the initial conditions sampled from the measure μ .²⁸ This further makes it possible to compute the Douady-Earle extension²⁴ of the homeomorphism $f : S^3 \rightarrow S^3$ to the self-map of the ball B^4 .

In order to proceed with some Engineering applications, we point out some relations between non-Abelian Kuramoto models and algorithms from *Geometric consensus theory*. Geometric consensus theory is a recently developed subdiscipline in the broad field of *Distributed and cooperative control*, which deals with consensus and coordination problems on certain non-Euclidean (notably homogeneous) spaces.^{22,29}

The setup for consensus problems is the swarm of identical individuals (agents) that communicate through the given undirected connected graph P . In such a setup, the consensus problem can be stated as minimization of a certain potential cost function.

The first example is consensus on the unit circle S^1 . Then Kuramoto model with zero frequencies appears as the gradient descent system for minimization of a certain potential function that is defined on the torus $T^N = S^1 \times \dots \times S^1$. In particular, the system (1) with $w_j \equiv 0$ appears in the consensus problem with the complete communication graph P . For the case of general communication graph, one obtains the model of Kuramoto oscillators that are coupled through the graph P .

More generally, there is an important class of consensus problems on matrix Lie groups $SO(n)$ and $SU(n)$ with a number of applications in robotics, space navigation, and swarm control.^{30,31} The consensus algorithm is obtained by applying the gradient descent method to the problem of minimization of the potential cost function. This gradient descent system turns out to be precisely the non-Abelian Kuramoto model (8) with zero generalized frequencies $H_j \equiv 0$.

One particular case of special importance is consensus on Lie groups S^3 and $SO(3)$. This is due to the fact that quaternions provide a convenient way to work with rotations in the 3-space; in fact, this was the original motivation for Hamilton to introduce this algebra in 1843. The representation of 3D rotations by unit quaternions is based on the double cover map from S^3 to $SO(3)$. The consensus and coordination algorithms on S^3 and $SO(3)$ solve some important problems in space navigation such as attitude synchronization and formation flying. The system (11) with the coupling function (16) provides an algorithm for consensus (for $K > 0$) and balancing (for $K < 0$) problems. (Fully incoherent states are typically referred to as *balanced states* in Distributed control theory.) For some other coordination problems, one can consider different coupling functions f , one example being explained in Remark 2. Underline, however, that (11) with general coupling function f different from (16) does not exhibit potential dynamics, i.e., it is not a gradient descent system for any cost function.

Finally, we mention that quaternionic Möbius transformations are used in designing conformal deformations of meshes in the 3-space with applications in animation and graphics.³² This means that (11) provides a way of computation and dynamical generation of continuous conformal deformations of 3D meshes.

VII. CONCLUSION AND OUTLOOK

In this paper, we study the collective dynamics in an ensemble of identical generalized oscillators on S^3 with the global coupling. The governing equations are quaternion-valued Riccati differential equations [or, equivalently, matrix Riccati differential equations on the group $SU(2)$]. Then, an ensemble evolves by the action of the group of Möbius transformations in $\mathbb{H} \cup \{\infty\}$ that preserve S^3 . This is a group-theoretic explanation for the low-dimensional behavior in the model. This study is to a great extent inspired by the results on low-dimensional behavior in analogous classical Kuramoto model.^{8–10} In our model, this low-dimensional dynamics takes

place on invariant submanifolds of a real dimension 10. The dynamics of global variables is given by the system (13), the analogue of the Watanabe-Strogatz system for the model (3).

We believe that a significant value of the present study lies in potential applications in Mathematics, Machine learning, Data mining, and Robotics; some of them are mentioned in Sec. VI. For different applications, one needs to consider variations of the model (11) without the assumption that the coupling is global and/or that generalized oscillators are identical.

The system (11) has one possible physical interpretation as the system of coupled quantum oscillators^{15,16} or coupled time-evolution operators on $SU(2)$. It is also appealing to work with various non-Abelian Kuramoto models on compact Lie groups in order to elaborate a unified geometric point of view on self-organization mechanisms, such as synchronization and swarming. In our point of view, the model (11) on the particular manifold S^3 might have a central role in such interpretations.

ACKNOWLEDGMENTS

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APPENDIX: PROOF OF THEOREM 1

Here, we shortly provide the calculations used in the derivation of (13). Calculations are quite cumbersome due to the lack of commutativity in the algebra of quaternions.

Fix a base point $k \in S^3$, and let $q = \mathcal{G}(k)$.

From Eq. (12), we get

$$q(1 + \bar{a}pk\bar{r}) = \bar{p}k\bar{r} + a,$$

and then we differentiate the above equation directly, keeping k constant:

$$\begin{aligned} \dot{q} &= [\dot{\bar{p}}k\bar{r} + \bar{p}k\dot{\bar{r}} + \dot{a} - q(\dot{\bar{a}}pk\bar{r} + \bar{a}\dot{p}k\bar{r} + \bar{a}p\dot{k}\bar{r})] \\ &\quad \times (1 + \bar{a}pk\bar{r})^{-1} \\ &= (\dot{\bar{p}}k\bar{r} + \bar{p}k\dot{\bar{r}} + \dot{a} - q\dot{\bar{a}}pk\bar{r} - q\bar{a}\dot{p}k\bar{r} - q\bar{a}p\dot{k}\bar{r}) \\ &\quad \times (\bar{p}k\bar{r} + a)^{-1}q. \end{aligned} \quad (A1)$$

Inverting the equation for $q = \mathcal{G}(k)$ gives

$$(\bar{p}k\bar{r} + a)^{-1} = (1 - \|a\|^2)^{-1}(\bar{q} - \bar{a}).$$

Substituting this in (A1) yields

$$\begin{aligned} \dot{q} &= (1 - \|a\|^2)^{-1}[(1 - q\bar{a})\dot{\bar{p}}p(1 - q\bar{a})^{-1}(q - a) \\ &\quad + (q - a)\dot{r}\bar{r} + \dot{a} - q\dot{\bar{a}}(1 - q\bar{a})^{-1}(q - a)](\bar{q} - \bar{a})q \\ &= (1 - \|a\|^2)^{-1}\left[(1 - q\bar{a})\dot{\bar{p}}p\frac{1 - a\bar{q}}{\|1 - q\bar{a}\|^2}\|q - a\|^2q \right. \\ &\quad \left. + (q - a)\dot{r}\bar{r}(1 - \bar{a}q) + \dot{a}(1 - \bar{a}q) \right. \\ &\quad \left. - q\dot{\bar{a}}\frac{1 - \bar{a}q}{\|1 - q\bar{a}\|^2}\|q - a\|^2q\right]. \end{aligned}$$

Since $\|1 - q\bar{a}\|^2 = \|q\|^2 \cdot \|\bar{q} - \bar{a}\|^2 = \|q - a\|^2$, we have

$$\begin{aligned} \dot{q} &= (1 - \|a\|^2)^{-1}[\dot{\bar{p}}p\bar{q} - q\dot{\bar{a}}\dot{\bar{p}}p\bar{q} - \dot{\bar{p}}pa + q\dot{\bar{a}}\dot{\bar{p}}pa + q\dot{r}\bar{r} \\ &\quad - a\dot{r}\bar{r} - q\dot{r}\bar{r}\bar{a}q + a\dot{r}\bar{r}\bar{a}q + \dot{a} - \dot{\bar{a}}\bar{a}q - q\dot{\bar{a}}q + q\dot{\bar{a}}a]. \end{aligned}$$

Comparing this to (11), we get the following system:

$$\begin{cases} (1 - \|a\|^2)^{-1}(-\dot{\bar{a}}\bar{p}p - \dot{r}\bar{r}\bar{a} - \dot{a}) = \dot{f}; \\ (1 - \|a\|^2)^{-1}(\dot{\bar{p}}p + \dot{a}\bar{r}\bar{r}\bar{a} - \dot{\bar{a}}\bar{a}) = \omega; \\ (1 - \|a\|^2)^{-1}(\dot{\bar{a}}\bar{p}pa + \dot{r}\bar{r} + \dot{\bar{a}}\bar{a}) = u; \\ (1 - \|a\|^2)^{-1}(-\dot{\bar{p}}pa - a\dot{r}\bar{r} + \dot{a}) = -\dot{f}. \end{cases} \quad (A2)$$

The system (A2) can be algebraically rearranged to give the set of QODE's (13).

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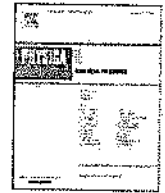
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Characterizing complex networks through statistics of Möbius transformations



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HIGHLIGHTS

- We study collective behavior of oscillators coupled through complex network.
- We introduce concepts of coherence of the network and correspondence (of one node).
- We characterize complex network by statistics of Möbius transformations.
- With each complex network, we associate two random variables.

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ABSTRACT

It is well-known now that dynamics of large populations of globally (all-to-all) coupled oscillators can be reduced to low-dimensional submanifolds (WS transformation and OA ansatz). Marvel et al. (2009) described an intriguing algebraic structure standing behind this reduction: oscillators evolve by the action of the group of Möbius transformations.

Of course, dynamics in complex networks of coupled oscillators is highly complex and not reducible. Still, closer look unveils that even in complex networks some (possibly overlapping) groups of oscillators evolve by Möbius transformations. In this paper, we study properties of the network by identifying Möbius transformations in the dynamics of oscillators. This enables us to introduce some new (statistical) concepts that characterize the network. In particular, the notion of coherence of the network (or subnetwork) is proposed.

This conceptual approach is meaningful for the broad class of networks, including those with time-delayed, noisy or mixed interactions.

In this paper, several simple (random) graphs are studied illustrating the meaning of the concepts introduced in the paper.

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1. Introduction

Extensive research of large populations of coupled oscillators led to remarkable progress since 1975 and seminal paper of Kuramoto [1]. In some cases, better understanding of dynamics and collective behavior of coupled oscillators unveiled unexpected relations to different mathematical theories. One idealistic model, for which the dynamics is particularly well understood, is a population of identical, globally (all-to-all) coupled phase oscillators with sinusoidal type of coupling function. In 1993, Watanabe and Strogatz [2,3] reported transformation of variables

that reduced dynamics of such population to low-dimensional submanifold. This indicates that such systems contain some hidden symmetries and admit many constants of motion. Furthermore, in 2008, Ott and Antonsen [4,5] reported new intriguing results for the case when initial phases of oscillators are uniformly distributed on $[0, 2\pi]$. For such initial data, the evolution equation describing distribution of phases admits solutions belonging to particularly fine class (that are precisely Poisson kernels, see [6]) with reduction of dynamics to the submanifold of dimension two.¹ These reductions to low-dimensional dynamics are now well known as WS and OA ansatz, respectively. In 2009, Marvel et al. [7]

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¹ OA ansatz applies also for nonidentical populations, for instance, if intrinsic frequencies of oscillators are chosen from Lorentzian distribution.

summarized these results and placed it in a broader mathematical context by explaining algebraic structure that stands behind them. They showed that such populations of oscillators dynamically generate automorphisms of unit disc as they are governed by the action of one-parametric family of Möbius transformations preserving the unit disc. There are many applications and consequences of this theoretical result (MMS principle²) that are still to be explored. In our point of view, the most important is in relating collective behavior of coupled oscillators to some fascinating mathematical objects that are extensively studied since XIX century up to today.

For this paper, it is of key importance to clearly emphasize under what conditions a population of coupled oscillators obeys MMS principle. First, oscillators are supposed to be identical, meaning that their intrinsic frequencies $\omega_i(t)$ are all equal. Second, each pair of oscillators is coupled with the same coupling strength $K(t)$. Third, the coupling function must be of sinusoidal type (sometimes also referred to as Kuramoto-type coupling), meaning that oscillators are coupled through the first harmonics only, and not through higher harmonics. Of course, these are very restrictive assumptions, especially the second one. One cannot expect that such a fine algebraic structure (and low-dimensional behavior) will be found in any complex network of oscillators.³ Nevertheless, in some papers, OA ansatz is applied successfully to study complex networks of coupled oscillators [8,9], heterogeneous [10], or hierarchically organized [11] populations of oscillators.

In this paper, we characterize complex networks by detecting collective behavior of oscillators in it. However, collective behavior is not observed through synchronization phenomena, but by presence and diversity of Möbius transformations in dynamics of coupled oscillators on given time intervals. We investigate some topological and structural properties of complex networks by measuring to what extent algebraic structure described in [7] persists in the network. Roughly, the more complex the network is, the less of this structure will be found in it.

The idea of applying coupled oscillators to study properties of complex networks is first examined in 2005–2007 in [12–14]. It is based on observation that gradual synchronization process in the complex network of Kuramoto oscillators (see [15]) reveals essential information about network topology and, in particular, helps to detect interconnected clusters. This idea has been pursued further in several papers that study network properties by observing synchronization process in it [16–18]. Methods of investigation of complex networks by studying appropriate model of coupled oscillators had certain impact and applications in the last decade, along with more traditional methods that are often based on classical results from Graph theory or random walks on networks [19,20].

In the next section, we recall several basic facts from Complex Analysis, introduce some new concepts and explain the main idea by studying idealistic models. In Section 3, we reconsider and upgrade these concepts to make them meaningful for random and realistic complex network. Finally, in Section 4, we draw some conclusions and raise some open questions for further study.

2. Concepts: coherence of the network, correspondence of the node

We consider Kuramoto model of identical phase oscillators coupled through the network of interactions:

$$\dot{\phi}_i = \omega + \frac{1}{N} \sum_{j=1}^N K_{ij}(t) \sin(\phi_j - \phi_i), \quad i = \overline{1, N}, \quad t \in [0, T]. \quad (1)$$

Here, $\phi_i \in [0, 2\pi)$ is a phase (state) of oscillator i , ω is an intrinsic frequency of all oscillators and K_{ij} is a matrix of pairwise interactions (coupling strengths) between oscillators i and j in the network \mathcal{N} .

We will assume that number of oscillators N is sufficiently large (say, $N > 100$). Another assumption is that initial phases of oscillators are random, chosen from some specific probability distribution on $[0, 2\pi]$:

$$\phi_i(0) = \phi_i^0, \quad i = \overline{1, N}. \quad (2)$$

For instance, in all examples throughout this paper, we will assume that initial phases of oscillators have uniform distribution on $[0, 2\pi]$. Hence, one might assume $\phi_i^0 \sim U[0, 2\pi]$.

In the sequel, it will be convenient to represent the phase of each oscillator by the point on unit circle S^1 in the complex plane: $z_i(t) = e^{i\phi_i(t)}$. Accordingly, the initial distribution of points $z_i(0)$, $i = \overline{1, N}$ is uniform on S^1 .

In the case of global coupling ($K_{ij}(t) = K(t)$, $\forall i, j = \overline{1, N}$), the points z_i evolve by the action of the group of Möbius transformations that preserve the unit disc [7]. At each given moment t , points are governed by the action of certain disc-preserving Möbius transformation. Then, from the theory of Lie groups, it follows that points $z_1(T), \dots, z_N(T)$ are obtained from $z_1(0), \dots, z_N(0)$ by applying some Möbius transformation (that is very difficult to specify a priori).

It is very well known that for two points z_i, w_i , there exists infinitely many Möbius transformations that map z_i into w_i . The same holds if one has two distinct points z_i, z_j and two images w_i, w_j . However, Möbius transformation is uniquely determined by its action on three distinct points:

Proposition 1. Let z_i, z_j, z_k be three distinct points on S^1 and w_i, w_j, w_k another three distinct points on S^1 . Then, there exists unique Möbius transformation that maps z_i to w_i , z_j to w_j and z_k to w_k .

The above proposition is a basic fact from Complex Analysis [6]. Based on it, we introduce certain terminology that will be used in the sequel. Let $z_1(0), \dots, z_N(0)$ be initial states of oscillators and $z_1(T), \dots, z_N(T)$ their states at the moment $t = T$ (governed by the system (1)).

- Definition 1.**
1. We say that four oscillators i, j, k, l agree, if at any time $T > 0$, there exists a Möbius transformation \mathcal{M}_T that maps points $z_i(0), z_j(0), z_k(0), z_l(0)$ to points $z_i(T), z_j(T), z_k(T), z_l(T)$, respectively.
 2. We say that $p > 4$ oscillators i_1, \dots, i_p live in the same field, if at any time $T > 0$, there exists a Möbius transformation \mathcal{M}_T that maps points $z_{i_1}(0), \dots, z_{i_p}(0)$ to points $z_{i_1}(T), \dots, z_{i_p}(T)$.

Remark 1. Notice that the concepts in the above definition depend both on network topology and initial states of all oscillators. In particular, four oscillators in the network can agree for certain initial conditions (2) and disagree in the same network for some other initial conditions. However, as simulations confirm, if four oscillators agree (disagree) at the certain time interval $(0, t_1)$, then they agree (disagree) at all time intervals (t_1, t_2) .

Remark 2. The common field can be created both by external force (influencing frequencies ω) and oscillators (network) themselves (through their mutual interactions K_{ij}).

Definition 2. Coherence of the network \mathcal{N} of coupled oscillators is a probability that randomly chosen four oscillators from \mathcal{N} agree.

We denote coherence of the network \mathcal{N} by $r_{\mathcal{N}}$, $0 \leq r_{\mathcal{N}} \leq 1$.

For the group of oscillators $\mathcal{H} \subseteq \{1, 2, \dots, N\}$, we define the coherence of the group \mathcal{H} in the network \mathcal{N} in the same way.

² We tend to refer to this result as WS ansatz (after Watanabe and Strogatz) or MMS principle (after Marvel, Mirollo and Strogatz).

³ In fact, MMS principle can be slightly extended to some other idealistic models (say, network with two ideal clusters), as we will mention below.

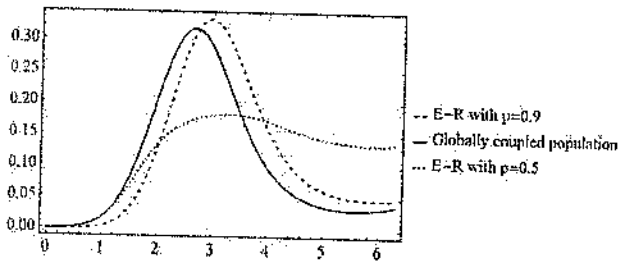


Fig. 1. Distribution of phases at moment $T = 1$ governed by the system (1) for uniformly distributed initial phases for three graphs. From OA ansatz and MMS principle it follows that the distribution for globally coupled population is a Poisson kernel with certain parameters. As expected, distribution of phases for Erdős–Rényi graph with $p = 0.9$ is closer to Poisson kernel than one with $p = 0.5$.

Remark 3. Being a statistical measure, coherence does not depend on initial states of individual oscillators (given the network is sufficiently large). In other words, one specific quadruple of oscillators might agree for some initial conditions and disagree for the another, but the probability that four randomly chosen oscillators agree will be approximately the same for fixed network topology and specific distribution of initial phases.

Now, fix an oscillator i and choose randomly three oscillators j_1, j_2, j_3 distinct from i .

Definition 3. The correspondence level of oscillator i in the network \mathcal{N} is a probability that quadruple i, j_1, j_2, j_3 agree, divided by the network coherence r .

We denote correspondence level of oscillator i by m_i .

Obviously, we can also consider correspondence level of an oscillator in any group \mathcal{H} of oscillators in the network \mathcal{N} .

It is obvious that in each network average correspondence level equals one.

The notion of correspondence can be used to identify important (influential) oscillators in the network. (By influential oscillators, we (roughly) mean those that influence existence of common fields significantly.) Indeed, oscillators that have extremely low or high correspondence levels are suspicious to be important (or marginal) ones.

As we have seen, it is important for our considerations to check if four oscillators agree. As one can guess, the most efficient way to check this is given by another basic geometric concept: cross ratio. Recall [21] that cross ratio of four points z_1, z_2, z_3, z_4 in the complex plane is given by

$$CR(z_1, z_2, z_3, z_4) = \frac{z_4 - z_1}{z_3 - z_1} \cdot \frac{z_3 - z_2}{z_4 - z_2}.$$

Proposition 2. Let z_1, z_2, z_3, z_4 be four distinct points in complex plane and w_1, w_2, w_3, w_4 another four distinct points. Then, Möbius transformation that maps z_i into w_i , $i = 1, 4$ exists if and only if $CR(z_1, z_2, z_3, z_4) = CR(w_1, w_2, w_3, w_4)$.

Remark 4. Cross ratio of four points z_1, z_2, z_3, z_4 is a real number if and only if z_1, z_2, z_3, z_4 are concyclic (or collinear).

In our case, all points lay on the unit cycle; therefore, all cross ratios will be real numbers.

Now, consider several simple models.

Model 1. Consider the model of globally coupled population from [7]. Each pair is connected by the link with the constant coupling strength $K_{ij} = K$. In [7], it is shown theoretically that at each time interval (t_1, t_2) all oscillators are governed by the same Möbius transformation. (Using above terminology, we say that they all live in the same field.) Therefore, coherence of this network is maximal, $r = 1$ and the correspondence level of each oscillator equals one. Of course, simulations support this. We say that this is a perfectly coherent network.

Remark 5. As a partial case of the above model, empty network of identical oscillators (i.e. no interactions/coupling, $K_{ij} \equiv K = 0$) is also perfectly coherent. Indeed, in this model, oscillators perform simple rotations on S^1 with equal frequencies ω . Such rotation is a trivial case of Möbius transformations.

Model 2. Consider the network consisting of two communities, where each pair of oscillators is coupled, if z_i and z_j belong to the same community, then coupling strength $K_{ij} = K$, whereas for two oscillators that belong to different communities $K_{ij} = v < K$. In other words, each pair of oscillators is coupled, but inter-community coupling K is stronger than intra-community coupling v . This model is studied in [22,23] as a simple example of population of coupled oscillators that can exhibit puzzling phenomena called *chimera state*. In [7], this model is not mentioned, but it is easy to show theoretically that the result therein can also be applied to this model [10]. Therefore, both communities evolve by actions of the group of Möbius transformations (at each moment two different Möbius transformations for two communities). In other words, there exist two fields here, the first is common for one community, the second for another.

Coherence of the network depends on the ratio of sizes of the two communities. Coherence is minimal if the communities are of the same size, then it equals $r = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, and the correspondence level of each oscillator is $m_i = 1$.

If one community is twice larger than another, then the coherence will increase: $r = \left(\frac{1}{3}\right)^4 + \left(\frac{2}{3}\right)^4 = \frac{17}{81}$. Let i be an oscillator belonging to the smaller and j to the bigger community. Then, for correspondence levels of i and j , one has, respectively,

$$m_i = \frac{\left(\frac{1}{3}\right)^3}{r} = \frac{\frac{1}{27}}{\frac{17}{81}} = \frac{3}{17}, \quad m_j = \frac{\left(\frac{2}{3}\right)^3}{r} = \frac{\frac{8}{27}}{\frac{17}{81}} = \frac{24}{17}.$$

In whole, this is not a perfectly coherent network ($r < 1$), but it contains two perfectly coherent groups.

Model 3. Now, we consider Erdős–Rényi (ER) random graph [19] where each pair of oscillators i, j is coupled with the probability $p = 0.9$. One might expect that dynamics (distribution of phases) will evolve close to the ideal network (Model 1), as only 10% of interactions (edges in the graph) are now missing (Fig. 1). However, we cannot hope for any theoretical result regarding existence of Möbius transformations in it and we fully rely on simulations. Simulations show that a random quadruple almost never perfectly agree, i.e. the cross ratio is almost never perfectly preserved. Therefore, we examine how often the cross ratio is almost preserved. We consider system (1) on $t \in [0, 1]$ and study relative difference of cross ratios given by

$$X = \frac{CR(1) - CR(0)}{CR(0)},$$

where $CR(0)$ and $CR(1)$ stand for cross ratios of four points taken at moments $t = 0$ and $t = 1$, respectively. We take approximation $\varepsilon = 0.01$, i.e. say that four oscillators (almost) agree, if $|X| < 0.01$. In this case, simulation shows that coherence of the network is approximately 0.2 (for the coupling strength $K = 4$).

3. Random networks

The last example demonstrates that we need to slightly modify and relax all definitions in order to obtain meaningful concepts for random (and realistic) networks.

Consider a network of coupled oscillators \mathcal{N} . Choose four random oscillators i, j, k, l from \mathcal{N} and consider their initial states (chosen from $U[0, 2\pi]$) $z_i(0), z_j(0), z_k(0), z_l(0)$ and their states at the moment $t = T$ governed by the system (1): $z_i(T), z_j(T), z_k(T), z_l(T)$. Denote the corresponding cross ratios at moments $t = 0$

and $t = T$ by $CR(0)$ and $CR(T)$, respectively. Consider relative difference of cross ratios:

$$X_N = \frac{CR(T) - CR(0)}{CR(0)}.$$

As oscillators i, j, k, l are chosen randomly, one can treat X_N as a random variable. In this way, with each network N we associate random variable X_N .

1. For Model 1, X_N is completely concentrated at zero, its density function is delta function $\delta_0(x)$.
2. For Model 2, X_N has a nonzero probability of zero point (for the case of equally large communities the probability of zero equals $\frac{1}{8}$). Hence, in this case, distribution of X_N is also singular (i.e. not absolutely continuous).

Conclusion. If random variable X_N has positive probability of point zero, then the network N contains a perfectly coherent group. On the other side, random variables associated with random networks turn to be absolutely continuous.

We now introduce modified definitions of coherence and correspondence level that will be meaningful for random networks as well. Fix small $\varepsilon > 0$ and $T > 0$.

Definition 4. For a given $\varepsilon > 0$, coherence of the network N is $r_N = P\{-\varepsilon < X_N < \varepsilon\}$.

As before, we introduce notion of coherence of the group of oscillators in the same way.

Remark 6. Notice that in the above definition, coherence depends on both ε and time T . In simulations, coherence decreases with T , since the deviation of cross ratios (or computation error) accumulates on longer intervals. Therefore, when comparing two networks it is important to fix the same ε and T for both. Moreover, coherence also depends on initial distribution of phases. This is the reason that in Section 2 we specified initial distribution of phases to be uniform on $[0, 2\pi]$.

One can also introduce another possible definition of coherence that is independent on ε :

Definition 5. S-coherence of the network N is an entropy of the corresponding random variable X_N .

Now, consider one oscillator i . Pick randomly three oscillators distinct from i , denote them by j, k, l . Denote their states at each moment t by the points $w_i(t), w_j(t), w_k(t), w_l(t)$ at the circle S^1 . Cross ratio of these four points at each moment t is a real number, denote it by $CR_w(t)$. Set $p_i^* = P\{-\varepsilon < \frac{CR_w(t) - CR_w(0)}{CR_w(0)} < \varepsilon\}$. (In other words, p_i^* is probability that w_i and three random oscillators agree up to small $\varepsilon > 0$.)

Definition 6. Correspondence level of an oscillator i in the network N is $m_i = \frac{p_i^*}{r_N}$.

Here r_N stands for coherence of the network N . Notice that the correspondence level of an oscillator depends both on his interactions (i.e. network topology) and initial states (phases) of the entire system.

Choosing random oscillator i from the network, we consider his correspondence level m_i as a random variable. In this way, we introduce one more random variable that is associated with the network N . Denote it by Y_N .

Remark 7. We briefly explain two random variables X_N and Y_N that characterize the network N . Random variable X_N provides information about coherence of the network. Concentration of

distribution of X_N around zero indicates presence of coherent groups in the network. On the other hand, Y_N characterizes distribution of roles (influence) of oscillators in the network. For specified network interactions K_{ij} and fixed initial conditions (2), these random variables can take finite set of possible values ($\binom{N}{4}$ and N values, respectively). However, as we suppose that initial phase of each oscillator is random (uniformly distributed on $[0, 2\pi]$), we obtain that both random variables take infinite set of possible values. Domain of X_N is a real line, while Y_N takes positive real numbers.

Samples from two random variables X_N and Y_N can be generated numerically using Monte Carlo method. In the examples below, we depict empirical probability density function for the two random variables and briefly comment the figures.

Remark 8. As X_N can be highly centered around zero for some networks, it is convenient to rescale it by considering random variable $-\log_{10}|X_N|$. We will plot density functions both for X_N and $-\log_{10}|X_N|$ for some examples studied below.

In examples that follow we set $\varepsilon = 0.01$ and $T = 1$ and choose random initial phases from the uniform distribution on $[0, 2\pi]$. Notice that it is necessary to choose T in such way that synchronization does not take place at interval $(0, T)$ and compare cross ratios before synchronization.

1. For Model 1 density functions are shown in Fig. 2(a)–(c). It can be seen that for this network p.d.f. for X_N is essentially delta function at zero, and p.d.f. for Y_N is delta function at 1.
2. For Model 2 two cases are shown. Fig. 2(d)–(f) shows density functions for the case when communities are of equal size. In that case p.d.f. for X_N has weight (measure) that is equal $\frac{1}{8}$ at zero. P.d.f. for Y_N is the same as in Model 1. Fig. 2(g)–(i) shows p.d.f. for the network in which one community is twice smaller than another. Notice that in that case distribution of Y_N is concentrated at two points: $x_0 = \frac{3}{17}$ has probability $\frac{1}{3}$ and $x_1 = \frac{24}{17}$ has probability $\frac{2}{3}$.
3. Density functions for Erdős–Rényi (ER) networks with $p = 0.9, 0.5$ and $p = 0.1$ are shown in Fig. 3. Coherence of these networks are approximately 0.2, 0.12 and 0.2. As expected, network with $p = 0.5$ has lower coherence than the other two. We also notice that random variables Y_N are absolutely continuous and concentrated on interval $(0, 2)$ with center at $x = 1$.
4. Consider famous Watts–Strogatz (WS) networks [24,19] with $N = 500, k = 50$ (initial number of connections) and different p (p is probability of rewiring). Empirical p.d.f. for parameters $k = 50$ and $p = 0.1$ are shown in Fig. 4.

Simulations show that the coherence of WS network does not depend on probability of rewiring p and is approximately the same as one in ER networks with similar number of edges. For instance, coherence of WS network with $N = 500, k = 50$ (and arbitrary p) is close to one for ER graph with $p = 0.1$.

4. Conclusion, some open questions and further research

In this paper, we propose conceptual framework for characterization of complex network by treating it as a network of coupled oscillators (where nodes are oscillators and interactions/edges are couplings). The idea can be explained in the following lines. The network topology is characterized by collective behavior of oscillators. One way to identify collective behavior is to observe synchronization process in the network (as discussed in Introduction, see references therein). However, synchronization is a consequence of collective behavior that sometimes occurs. Therefore, we study collective behavior of oscillators before synchronization eventually takes place, using statistics of Möbius transformations. Presence of

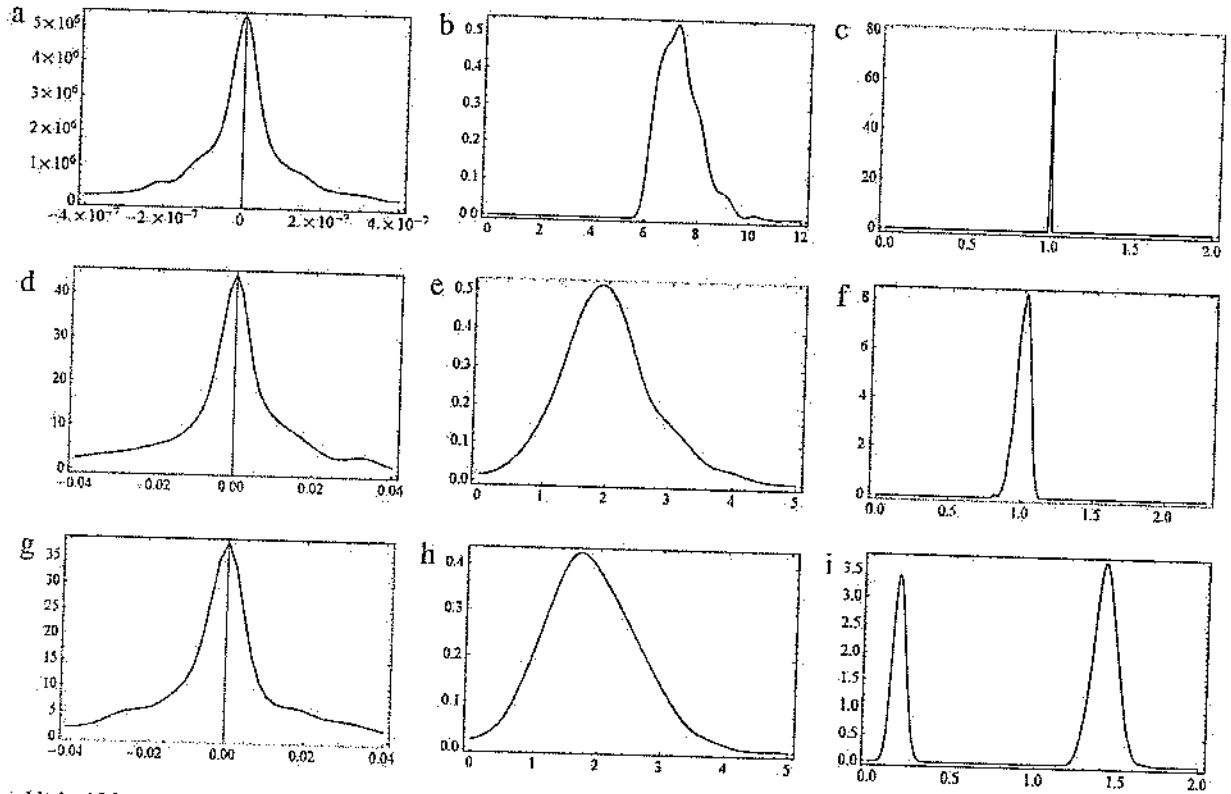


Fig. 2. Empirical p.d.f. for random variables X_N (first column), $-\log |X_N|$ (second column) and Y_N (third column) for three different models: (a), (b), (c) Globally coupled population (Model 1); (d), (e), (f) Two perfect communities of equal size (Model 2); (g), (h), (i) Two perfect communities of different (2 : 1) sizes.

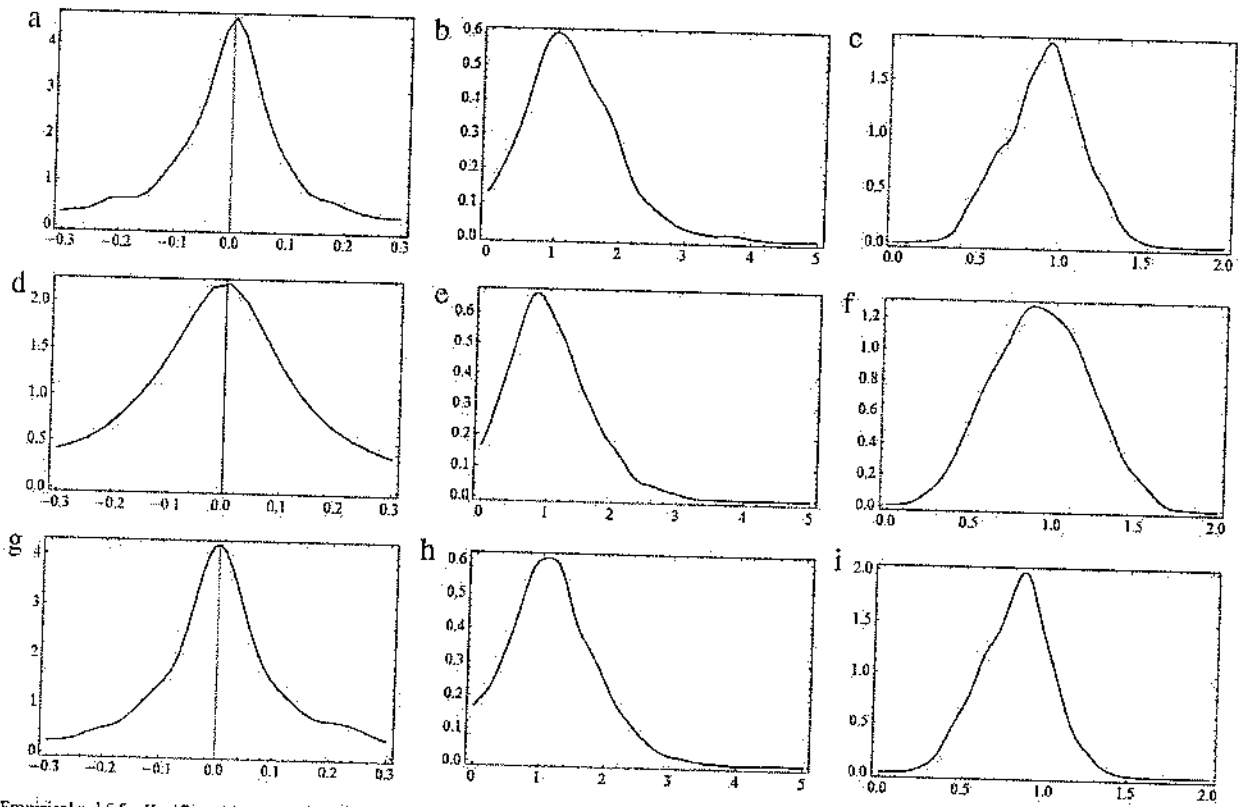


Fig. 3. Empirical p.d.f. for X_N (first column), $-\log |X_N|$ (second column) and Y_N (third column) for three different Erdős-Rényi graphs with $N = 500$ nodes: (a), (b), (c) graph with $p = 0.9$; (d), (e), (f) E-R graph with $p = 0.5$; (g), (h), (i) E-R graph with $p = 0.1$.

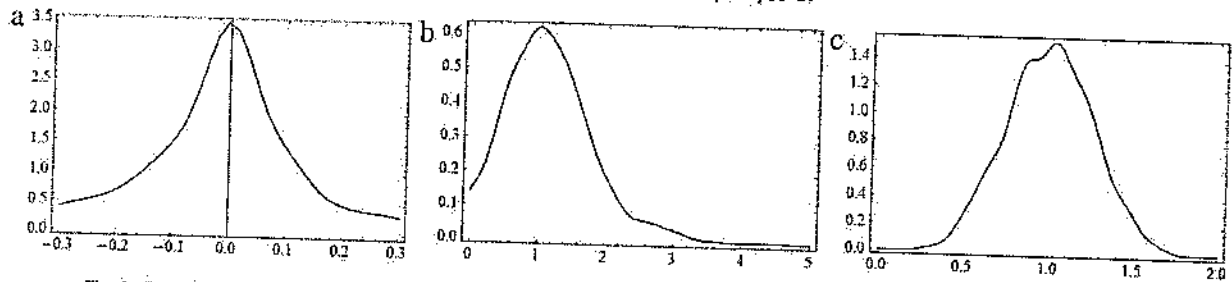


Fig. 4. Empirical p.d.f. for X_N , $-\log|X_N|$ and Y_N for Watts–Strogatz network with $N = 500$ nodes and parameters $k = 50$ and $p = 0.1$.

Möbius transformations in the network is detected using random cross ratios.

With each complex network \mathcal{N} we associate two random variables X_N and Y_N . Their empirical p.d.f. can be sampled by Monte Carlo method and contain important information about network \mathcal{N} and its subnetworks. Further, we introduce notions of coherence (of the group of nodes) and correspondence level (of one node).

The framework proposed here is suitable for broad class of networks (including those with repulsive, time-dependent, noisy, time-delayed and mixed interactions). Another potential advantage is that the method is based on measuring the state of nodes (phases of oscillators) only and does not rely on any information about interactions. This indicates possibility to characterize the network without information on interactions. However, this requires further study and new ideas.

We find it interesting (and probably difficult) to explore relations between notions of the network coherence and some other statistical measures on networks such as entropy and complexity of graphs, see [25–27].

Another interesting question is the following: which networks (or graphs) have the minimal coherence? In particular, how different types of interactions influence coherence? Does the presence of delayed interactions influence coherence? To what extent small noise in the interactions destroys coherence? In the further research, we will address these questions theoretically (following [7]) with subsequent simulations.

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Collective motions of globally coupled oscillators and some probability distributions on circle

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ABSTRACT

In 2010 Kato and Jones described a new family of probability distributions on circle, obtained as Möbius transformation of von Mises distribution. We present the model demonstrating that these distributions appear naturally in study of populations of coupled oscillators. We use this opportunity to point out certain relations between Directional Statistics and collective motion of coupled oscillators.

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1. Introduction

Collective dynamics of large populations of coupled oscillators is a subject of great theoretical interest and various applications. The most important model in this topic, that has served for decades as a paradigm of coupled oscillators systems in Physics and Biology, was introduced by Kuramoto [1]. This model describes large population consisting of N phase oscillators with global (all-to-all) coupling:

$$\dot{\varphi}_j = \omega_j + \frac{K}{N} \sum_{i=1}^N \sin(\varphi_i - \varphi_j), \quad j = 1, \dots, N, \quad t \geq 0. \quad (1)$$

Here, $\varphi_j(t)$ and ω_j are the phase and the frequency of oscillator j . It is assumed that each pair of oscillators is connected with the same (global) coupling strength K . Another assumption is that oscillators are coupled through the sine function only (and not through higher harmonics); this type of coupling is usually referred to as *sinusoidal coupling* or *Kuramoto coupling*.

Kuramoto started his analysis of (1) by introducing two global variables: order parameter $r(t)$ and average phase $\mu(t)$, defined by equation

$$r(t)e^{i\mu(t)} = \frac{1}{N} \sum_{j=1}^N e^{i\varphi_j(t)}.$$

In new variables system (1) can be rewritten in *mean-field* form (see [1,2]):

$$\dot{\varphi}_j = \omega_j + Kr(t) \sin(\mu - \varphi_j), \quad j = 1, \dots, N. \quad (2)$$

This form unveils mean-field character of Kuramoto model (1): mutual global coupling results in the same dynamics as all oscillators would be coupled to the mean phase μ with the coupling strength $Kr(t)$.

Kuramoto paper stimulated enduring interest of researchers in low-dimensional behavior of large populations of globally coupled oscillators. Remarkable progress in this topic has been achieved in papers of Ott and Antonsen [3], Marvel et al. [4] and Pikovsky and Rosenblum [5]. This progress brought new insight and paradigms into study of collective behavior and self-organization of coupled oscillators.

Ott and Antonsen in [3] have shown that in some cases (1) admits many constants of motion and performs low-dimensional dynamics for large N and some special initial conditions $\varphi_j(0)$, $j = 1, \dots, N$. Remarkably, explicit ODE describing the evolution of complex order parameter is derived not only for the case of identical oscillators ($\omega_j \equiv \omega$), but also if the intrinsic frequencies ω_j are chosen from Cauchy (Lorentzian) distribution (and some other distributions) on the real line. The density function of phase distribution $\rho(\varphi, t)$ evolves on invariant low-dimensional submanifolds

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in the space of all density functions. Ott–Antonsen (OA) ansatz is particularly transparent when all oscillators have equal intrinsic frequencies, in that case the density of oscillators' phases evolves on two-dimensional invariant submanifold consisting of Poisson kernels. Underline however that OA ansatz holds under restrictive assumptions on initial conditions, including the case when initial distribution of phases is uniform on $[0, 2\pi]$. In most papers on the topic this assumption is adopted, mainly because it yields particularly tractable complex ODE for order parameter. Besides, this assumption on initial phases seems quite natural in many models.

Marvel et al. [4] exposed an algebraic structure standing behind low-dimensional dynamics of globally coupled populations. Substituting $z_j(t) = e^{i\varphi_j(t)}$, the state of each oscillator is represented by the point $z_j(t)$ on the unit circle S^1 . If oscillators are identical ($\omega_j = \omega$), then their states evolve by the action of one-parametric family of Möbius transformations that preserve unit disc. Then, from the Lie group theory, it follows that the density function of oscillators' states $\rho(z, t)$ at each moment t is a certain Möbius transformation of the initial distribution $\rho(z, 0)$ on S^1 . In the sequel we will refer to this result as *MMS principle*.¹ Since the (sub)group of disc-preserving Möbius transformations can be parametrized by three (real) parameters, it follows that $\rho(z, t)$ evolves on invariant three-dimensional submanifold in the space of distributions. MMS principle shed additional light on Ott–Antonsen result: solutions in the form of Poisson kernels arise as Möbius transformations of circular uniform distribution. Therefore, for the case of globally coupled population, MMS principle is generalization of OA ansatz for arbitrary initial distribution of phases. However, unlike OA ansatz, MMS principle does not extend (at least, so far) to the case of nonidentical oscillators.

At this point we briefly emphasize some analogous concepts in Directional Statistics: distributions in the form of Poisson kernels have been studied for a long time as wrapped Cauchy distributions. McCullagh in [6] introduced reparametrization of wrapped Cauchy distributions using complex numbers thus representing them as Möbius transformations of circular uniform distribution. This analogy indicates that findings of Directional Statistics can be useful in understanding dynamics of large populations of coupled oscillators. In particular, one might ask what other circular distributions (apart from circular uniform and wrapped Cauchy) arise in dynamics of coupled oscillators.

Another family of distributions that is central in Directional Statistics consists of von Mises distributions $vM(\mu, \kappa)$. Their p.d.f. are given by (see [7,8]):

$$f_{vM}(\varphi) = \frac{1}{2\pi \mathcal{I}_0(\kappa)} \exp\{\kappa \cos(\varphi - \mu)\}, \quad 0 \leq \varphi \leq 2\pi, \quad (3)$$

where \mathcal{I}_0 is an order zero modified Bessel function of the first kind. In study of coupled oscillators von Mises distribution is mentioned mostly in papers on statistical detection and estimation of coupling between noisy oscillators, see [9–11]. In our context here it is more important that von Mises distributions arise as stationary distributions in the some simple models of coupled oscillators with noisy identically distributed frequencies, see Section 2.

In 2010 Kato and Jones described a new family of probability distributions on the circle obtained as Möbius transformations of von Mises distributions. For the fixed mean μ and concentration κ in von Mises distribution, this family constitutes a three-dimensional submanifold in the space of distributions. For brevity,

we will refer to distributions belonging to this family as *K–J distributions*.

In this paper we study the evolution of globally coupled population of oscillators in the case when initial distribution of their states is von Mises. This situation is not covered by OA ansatz, since it does not apply to this family of initial distributions. Instead, we rely on Watanabe–Strogatz ansatz (in the form of MMS principle) for this case. Referring to MMS principle, we conclude that the states of globally coupled identical oscillators will evolve on three-dimensional submanifold consisting of K–J distributions. This observation, along with some consequences and applications, makes the main motivation for this paper.

In Section 2, we argue that the assumption of von Mises distribution of initial states is quite natural in various models. In Section 3, we introduce the model of globally coupled population whose density function $\rho(z, t)$ evolves on the invariant submanifold consisting of K–J distributions. In Section 4, we discuss non-trivial relation between model parameters and parameters of resulting K–J distributions. In Section 5, the results of Kato and Jones [12] are used to make qualitative predictions of possible scenarios of collective dynamics in our model and relate qualitatively different scenarios to corresponding value sets of model parameters. This is followed by some simulations illustrating different scenarios and supporting our predictions. Questions about modality and skewness of resulting distributions are also tackled in Section 5. In Section 6, we provide explicit expressions for moments of K–J distributions using formulae from [4]. Finally, some conclusions and open questions are briefly emphasized in Section 7.

2. Coupled oscillators and von Mises distribution

Synchronization process is typically studied as a transition from fully incoherent state (with the initial value $r(0)$ of the order parameter set to zero), through partially coherent states ($0 < r(t) < 1$), towards full synchronization ($r(t) \rightarrow 1$). This setting implies that the initial distribution of oscillators' phases is uniform on $[0, 2\pi]$. For the globally coupled population (and some more general cases, as shown in [13,14]), OA ansatz provides a powerful method to treat such systems. Although this assumption sounds quite reasonable, in some processes certain degree of coherence in the population is present before the mean-field coupling is established. Therefore, in such cases some other distributions appear as initial for oscillators' phases. In this section we briefly explain how von Mises distributions emerge as the result of combined effects of coupling and noise.

One general observation is that von Mises distributions $vM(\mu, \kappa)$ are the maximum entropy distribution on S^1 for the fixed concentration parameter κ (the special case $\kappa = 0$ yields uniform distribution).

Furthermore, von Mises distributions arise as the stationary distributions of a drift and diffusion processes on the circle with a preferred orientation, see [15]. In particular, consider the population of noisy oscillators that is coupled to the external field μ :

$$\dot{\varphi}_j = R \sin(\mu - \varphi_j) + \xi_j, \quad j = 1, \dots, N, \quad t > 0, \quad (4)$$

where ξ_j are realizations of Gaussian white noise with the intensity $D > 0$:

$$\langle \xi_j(t) \rangle = 0, \quad \langle \xi_j(t) \xi_i(t') \rangle = 2D \delta_{ij} \delta(t - t').$$

The stationary distribution of this process is von Mises. This can be shown by solving Fokker–Planck equation, see [15,16].

In [16], the directional motions of swimming microorganisms with a preferred orientation μ is described by (4). Another example is the problem of estimation of phase coupling between

¹ MMS principle can be seen as reformulation of the older result, known as Watanabe–Strogatz ansatz. However, this reformulation revealed hidden symmetries and mathematical objects standing behind low-dimensional behavior of globally coupled populations of identical oscillators thus providing completely new insights into this phenomena.

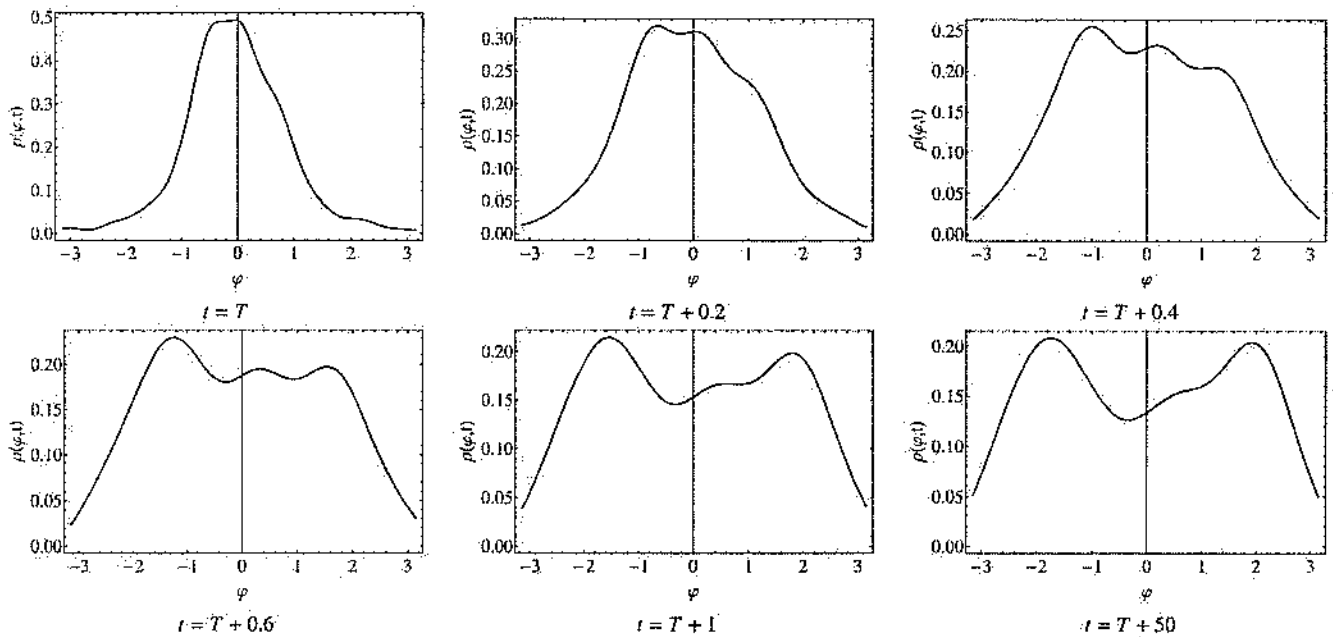


Fig. 1. Evolution of densities for parameter values $R = 0.135$, $D = 0.1$, $\mu = 0$ (yielding $\kappa \approx 2$) and $K = -4$, $\nu = 0$, $\tau = 0$. The distribution evolves towards stationary bimodal configuration.

noisy oscillators, the differences of oscillators' phases $\Delta_{jk}(t) = \varphi_j(t) - \varphi_k(t)$ are governed by (4) with $\mu = 0$, see [9].

Notice, that mean-field model (1) with identically distributed noisy frequencies $\omega_j = \omega + \xi_j$ also yields stationary von Mises distribution (with rotating mean angle $\mu(t)$), this is discussed in [11]. In [17] the circadian clocks in cyanobacteria population have been studied by using this model for intercellular communication.

3. Model

Consider the population of identical oscillators with global coupling:

$$\dot{\varphi}_j(t) = \omega + \frac{K}{N} \sum_{i=1}^N \sin(\varphi_i(t - \tau) - \varphi_j(t) - \nu), \quad (5)$$

$$j = 1, \dots, N, t \geq T.$$

Here ω is a noiseless frequency common for all oscillators and K is a global coupling strength. Notice that we do not require global coupling K to be positive thus allowing negative (repulsive) coupling as well.

We consider the process (5) on time interval $t \in [T, \infty)$ for some $T \geq 0$. Suppose that the initial states of oscillators $z_1(T) = e^{i\varphi_1(T)}, \dots, z_N(T) = e^{i\varphi_N(T)}$ are chosen from von Mises distribution $\nu M(\mu, \kappa)$.

Underline that OA ansatz is not applicable to this set of initial conditions (unless $\kappa = 0$, yielding uniform distribution). In order to apply MMS principle (or WS ansatz) we imposed an important constraint, that all oscillators have identical intrinsic frequencies ω . Another key assumption is that the coupling is global, meaning that the phase shift ν and the delay τ are the same in all pairwise interactions. The systems of the form (5) are referred to as Kuramoto-Sakaguchi model ([18]).

Putting all together, our model can be viewed as the process taking place in two stages, described by (4) up to the moment T and by (5) after T . It is assumed that T is sufficiently large, so that the process at the first stage reaches stationary distribution.

Another assumption is that the noise vanishes at the moment T , i.e. oscillators are noisy on $t \in [0, T]$ and noiseless after T .

Although this assumption may sound artificial at the first glance, we argue that this two-stage dynamics can serve as an adequate model in various situations. In particular, it can be conceived that the external field is deactivated at the moment T , thus realizing pairwise interactions and suppressing the noise ξ_j .

As one concrete and intriguing example, we refer to the experiment on progression of population of cells through the cell cycle. At the first stage population was treated two times with thymidine blocks to arrest all cells at (approximately) the same phase, that is the beginning of S-phase of the cell cycle. At the moment T the thymidine was washed out and the collective progression through the cell cycle was observed. Some details on this experiment and its results are exposed in the textbook of Morgan [19], page 25. In this particular situation, one might conceive that the dynamics of cells is described by (4) at the first stage (before thymidine was washed out), and by (5) afterwards.

By applying MMS principle to (5), we obtain that the distribution of phases at each moment $t > T$ belongs to K-J family. This is to be investigated further in the next section.

4. Dynamics of parameters of K-J distributions

The set of all Möbius transformations that preserve the unit disc D forms a subgroup. The general disc-preserving Möbius transformation can be written in the following form:

$$\mathcal{M}(z) = e^{i\mu} \frac{z + \alpha}{1 + \bar{\alpha}z}, \quad (6)$$

for some angle $0 < \mu < 2\pi$ and complex number $\alpha \in D$, $\alpha = re^{i\psi}$.

Let the random circular variable z has von Mises distribution $\nu M(0, \kappa)$. Then the image of z under Möbius transformation (6) is a circular random variable with the following p.d.f. (see [12]):

$$f_{KJ}(\varphi) = \frac{1 - r^2}{2\pi I_0(\kappa)} \frac{1}{1 + r^2 - 2r \cos(\varphi - \gamma)} \times \exp \left\{ \frac{\kappa (\xi \cos(\varphi - \eta) - 2r \cos \nu)}{1 + r^2 - 2r \cos(\varphi - \gamma)} \right\}, \quad (7)$$

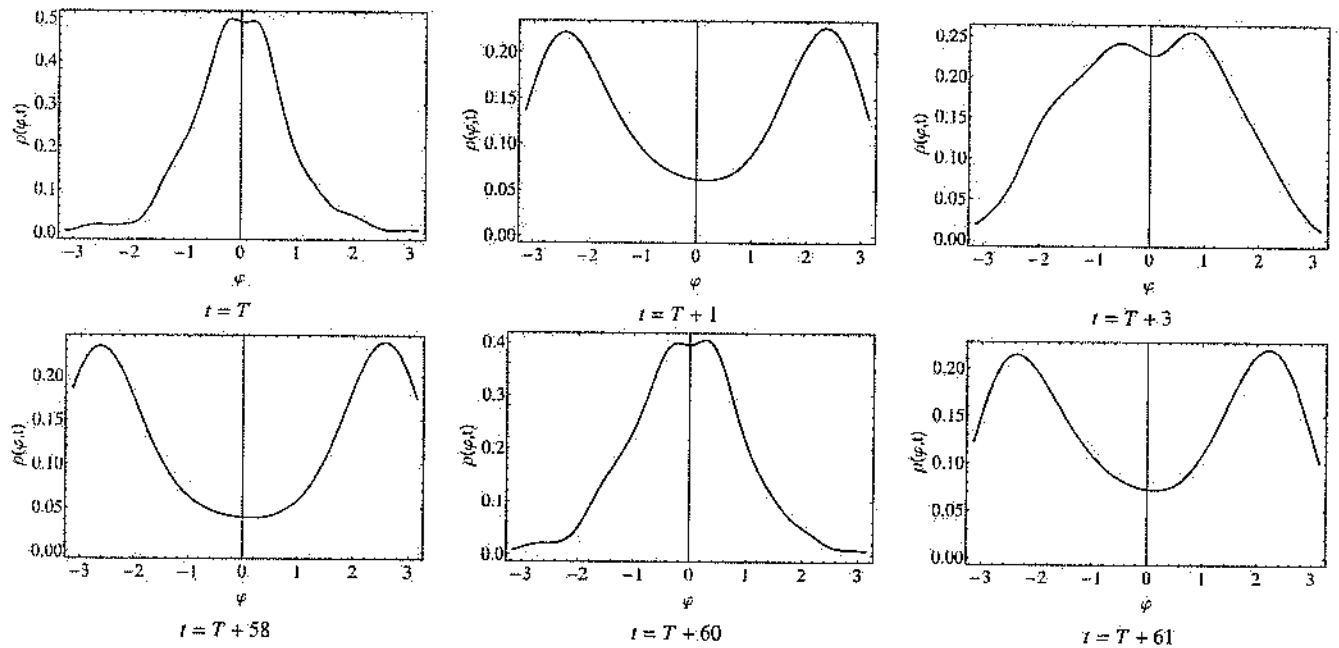


Fig. 2. Evolution of densities for parameter values $R=0.135$, $D=0.1$, $\mu=0$ (yielding $\kappa \approx 2$) and $K=-3$, $\nu=0$, $\tau=1$. In the case of time-delayed coupling, the modality alternates (unimodal to bimodal and vice-versa).

where $\gamma = \mu + \nu$, $\xi = \sqrt{r^4 + 2r^2 \cos(2\nu) + 1}$ and $\eta = \mu + \arg(r^2 \cos(2\nu) + 1 + ir^2 \sin(2\nu))$.

This distribution depends on four parameters: $\kappa \geq 0$, $0 \leq \mu$, $\nu < 2\pi$ and $0 \leq \tau < 1$. Parameter κ is inherited from von Mises distribution, while the remaining three come from Möbius transformation (6).

For the fixed value of parameter κ , functions of the form (7) constitute the three-dimensional submanifold in the space of probability distribution.

As explained in Section 2, the phases $\varphi_1(t), \dots, \varphi_N(t)$ at each moment $t > T$ evolve by the action of disc-preserving Möbius transformation (6) with the parameters $\mu(t) \in [0, 2\pi]$ and $\alpha(t) \in \mathbb{C}$, satisfying the following system of ODE's ([4,20]):

$$\begin{aligned} \dot{\alpha} &= i(f(t)\alpha^2 + g(t)\alpha + \bar{f}(t)); \\ \dot{\mu} &= f(t)\alpha + g(t) + \bar{f}(t)\bar{\alpha}. \end{aligned} \quad (8)$$

Here, f is a coupling function; $f(t) = -i \frac{K}{2N} \sum_{j=1}^N e^{-i(\varphi_j(t-\tau)-\nu)}$ and $g(t) = \omega$.

From the theory of Lie groups, one has that at each moment $t > T$ there exists a Möbius transformation \mathcal{M}_t mapping the phases $\varphi_1(T), \dots, \varphi_N(T)$ to $\varphi_1(t), \dots, \varphi_N(t)$.

The parameter κ is proportional to the ratio $\frac{R}{D}$. The remaining three parameters of K-J distribution are related to K , ν , ω and τ from (5). However, this relation is very subtle and is mediated through (8) and the action of corresponding flow of Möbius transformations.

In fact, this dependence is even more complicated, since the coupling function $f(t)$ in (8) depends also on states of all oscillators at the moment t . Therefore, parameters τ , ν and μ depend not only on K , ν , ω and τ , but also on R and D (through κ).

Therefore, it is very difficult to estimate the evolution of parameters of K-J distributions if the model parameters are given. That evolution can be partially understood only by physical intuition, simulations and expressions for moments and comparison with analysis of Kato and Jones [12]. This will be studied in the next sections.

5. Evolution of the shape on invariant submanifold of K-J distributions

In the paper of Kato and Jones it has been shown that distributions (7) have different shapes depending on parameters. In this section we study the evolution of the shape of phase density function in the model (5) for different parameter values. This study is supported by simulation results for some illustrative cases. In Figs. 1–3 the evolution of density function $\rho(\varphi, t)$ of oscillators' phases for three different cases is shown by depicting snapshots at different moments t . All densities have been obtained by solving the system (4)–(5) for the population consisting of $N=500$ oscillators. In all simulations, the moment of transition from the first stage to the second is $T=30$.

We will simulate the effects of attractive and repulsive coupling, time-delayed coupling with the global delay τ and phase-shifted coupling. Notice that MMS principle is valid for all types of the coupling mentioned above, hence all density functions depicted in figures belong to the family of K-J distributions.

The family (7) contains both symmetric and asymmetric, as well as unimodal and bimodal distributions. On the other hand, von Mises distributions are unimodal and symmetric. In our context this means that unimodal and symmetric initial phase distribution can evolve into bimodal and/or asymmetric one under the mean-field coupling. It is interesting to understand how different effects (such as negative coupling, time-delayed coupling, phase-shifted coupling) influence the shape of distributions. We distinguish several qualitatively different scenarios:

1. First consider the model without delay and phase shift: $\tau=0$, $\nu=0$.
 - a) If the coupling is absent at the first stage ($R=0$) and repulsive at the second ($K \leq 0$), then the density function $\rho(\varphi, T)$ is constant and remains constant for all $t > T$, meaning that states of oscillators have circular uniform distribution ($\kappa=0, r=0$).
 - b) If $R=0$ and $K > 0$, the density function $\rho(\varphi, T)$ is constant and evolves further on two-dimensional invariant subman-

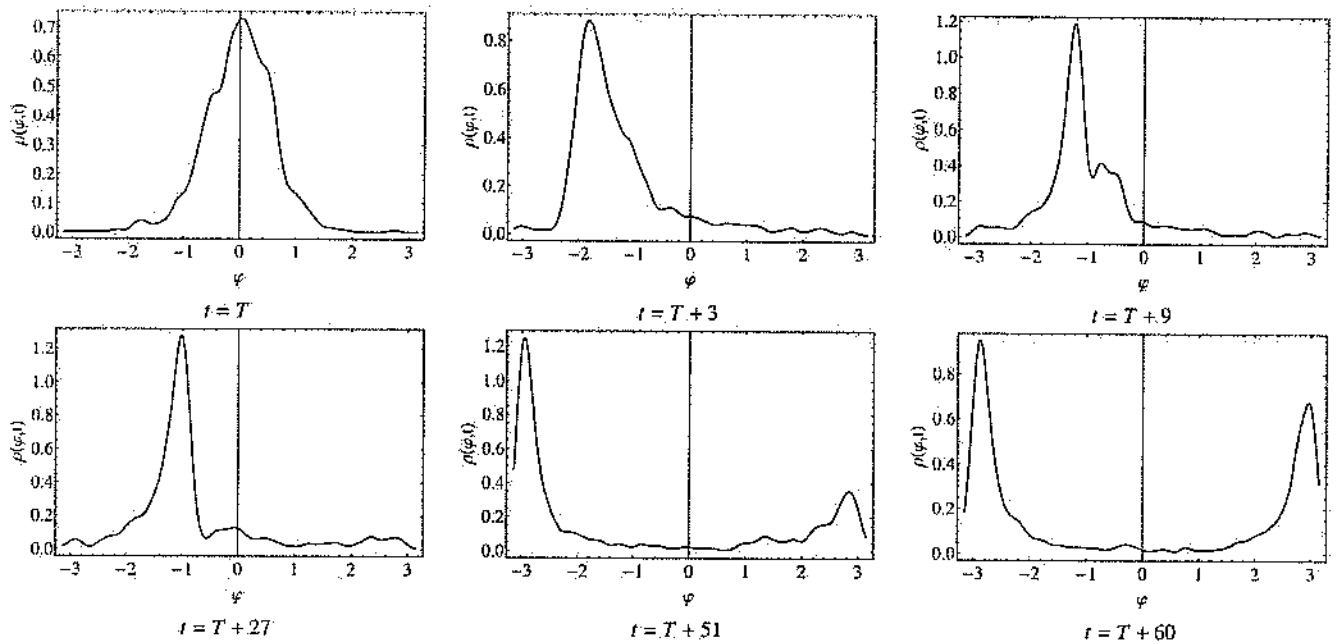


Fig. 3. Evolution of densities for parameter values $R=0.185$, $D=0.1$, $\mu=0$ (yielding $\kappa \approx 3$) and $K=2$, $\nu = \frac{\pi}{2}$, $\tau=0$. The phase-shift increases the skewness of distributions.

ifold consisting of wrapped Cauchy densities ($\kappa=0$) towards gradual synchronization ($r \rightarrow 1$).

- c) if the coupling is attractive at both stages, $K, R > 0$, no changes in modality occur. We obtain unimodal K-J densities for all $t > T$ and gradual synchronization ($\kappa > 0$, $r \rightarrow 1$).
 - d) If the attractive coupling at the first stage $R > 0$, is followed by the repulsive one $K < 0$, the system evolves towards stationary bimodal configuration with two concentration points separated by the angle π , this scenario is shown in Fig. 1. Here, $\kappa > 0$, $0 < r \rightarrow a < 1$.
2. Presence of the global time-delayed coupling ($\tau=1$) with $R > 0$ and $K < 0$, yields oscillatory dynamics; distribution changes modality (from unimodal to bimodal and vice versa) several (or infinitely many) times, see Fig. 2. Here, $\kappa > 0$, while r oscillates. The question of long-time behavior for this case is quite subtle (in fact, it depends on intensity of the repulsive coupling K).
 3. Presence of the phase shift $\nu \approx \frac{\pi}{2}$ yields strongly asymmetric distributions, Fig. 3. Thus, phase shift is the parameter that decidedly influences the skewness of distributions. Notice that all distributions depicted are slightly asymmetric. In fact, K-J distributions are symmetric only for several special cases: $\kappa=0$ (wrapped Cauchy), $r=0$ (von Mises) and $\nu=0$ or $\nu=\pi$.

We briefly address the question regarding bimodal stationary configurations mentioned in point 1d) above. Engelbrecht and Mirolo in [21] used MMS principle to prove that the only possible limit sets for (5) are single points. Therefore, despite appearance of bimodal distributions, the system can never synchronize at two distinct points. In fact, K-J family does not contain distributions with high concentrations at two points, densities are only moderately bimodal, such as those depicted in figures. This means that the stationary configuration of phase distribution depends only weakly on how peaked is initial von Mises distribution. In other words, stationary bimodal configurations obtained for the case 1d) differ only slightly for $\kappa=0.5$ and for κ very large.

6. Moments

Marvel et al. in [4] have also derived formulae for the moments of phase distribution of coupled oscillators. By simply plugging Fourier coefficients of von Mises distribution $vM(0, \kappa)$ into formulae (55)–(56) from there, we obtain an expression for the dynamical evolution of m -th moment:

$$\langle z^m \rangle(t) = \alpha^m(t) + \sum_{k=0}^{m-1} \frac{(1 - |\alpha(t)|^2)^{k+1}}{k!} \times \sum_{n=0}^{\infty} (-1)^n \frac{(n+k)!}{n!} \frac{\mathcal{I}_{n+k+1}(\kappa)}{\mathcal{I}_0(\kappa)} e^{i(m+n)\psi(t)} \bar{\alpha}^n(t), \quad (9)$$

where \mathcal{I}_l is a modified Bessel function of the first kind of integer order l . As before, $\alpha(t)$ and $\psi(t)$ are solutions to the system (8). Their geometric meaning is also unveiled in [4]: $\psi(t)$ is the overall counterclockwise rotation of the initial distribution up to the time t , and $\alpha(t)$ is the image of the center (zero) under the action of the flow of Möbius transformations.

In particular, the formula for centroid of the distribution yields:

$$\langle z \rangle(t) = \alpha(t) + (|\alpha(t)|^2 - 1) \sum_{n=1}^{\infty} (-1)^n \frac{\mathcal{I}_{n+1}(\kappa)}{\mathcal{I}_0(\kappa)} e^{in\psi(t)} \bar{\alpha}^{n-1}(t). \quad (10)$$

It is interesting to compare the above formulae with the expression for moments given in the paper of Kato and Jones [12].

7. Conclusion

In this paper we presented the model demonstrating how results of Directional Statistics can contribute to the qualitative understanding of collective dynamics of coupled oscillators. On the other hand, collective motions of coupled oscillators provide various interpretations and justifications for probability distributions on circles (and tori). Therefore, we hope that this contribution can be of interest for researchers from both fields in different ways. The most significant conclusions for statisticians might be the following:

- It is shown that K–J (along with wrapped Cauchy and von Mises) distributions are relevant in study of coupled oscillators.
- We suggested possibility that K–J distributions can model results of some real-life experiments. Back to the experiments on cell cycle mentioned in Section 2, figures in Morgan [19] demonstrate that density of cells progressing through the cell cycle transforms from unimodal and symmetric into asymmetric and bimodal with the time. In fact, the initial density depicted there looks pretty much like von Mises with κ relatively high, while the evolution of densities is intriguingly similar to what is shown in Fig. 3 here.

On the other hand, there are also several points that might be of interest for study of coupled oscillators:

- It is demonstrated how unimodal symmetric distribution can transform into bimodal and asymmetric one under the effect of mean-field coupling. We have specified what effects in the coupling might be responsible for changes in modality and/or skewness.
- Using the analysis of Kato and Jones ([12]), we explained some possible long-time evolution scenarios of phase distribution under the influence of mean field.

In conclusion, one might ask what other probability distributions on the circle (or torus) are expected to arise in dynamics of coupled oscillators. One notable example is presented in the paper [22], where it is shown that so-called *hyperbolic von Mises distribution* is a stationary distribution for Kuramoto–Sakaguchi model with multiplicative Gaussian white noise. Furthermore, bivariate (and multivariate) wrapped Cauchy distributions, studied in [23,24], are related with the model of two (or more) perfect communities with different inter and intra-community coupling strengths, introduced in [25] and frequently studied as paradigmatic for so-called *chimera states*.

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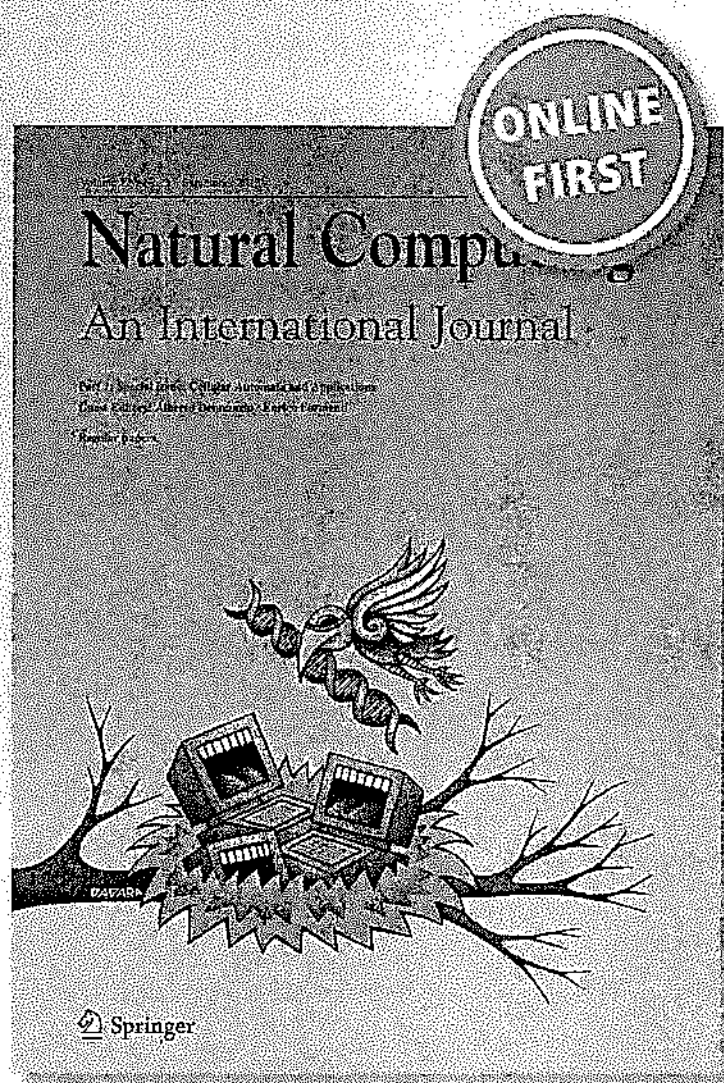
Data clustering based on quantum synchronization

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Data clustering based on quantum synchronization

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Abstract

There exists a specific class of methods for data clustering problem inspired by synchronization of coupled oscillators. This approach requires an extension of the classical Kuramoto model to higher dimensions. In this paper, we propose a novel method based on so-called non-Abelian Kuramoto models. These models provide a natural extension of the classical Kuramoto model to the case of abstract particles (called Kuramoto–Lohe oscillators) evolving on matrix Lie groups $U(n)$. We focus on the particular case $n = 2$, yielding the system of matrix ODE's on $SU(2)$ with the group manifold S^3 . This choice implies restriction on the dimension of multivariate data: in our simulations we investigate data sets where data are represented as vectors in \mathbb{R}^k , with $k \leq 6$. In our approach each object corresponds to one Kuramoto–Lohe oscillator on S^3 and the data are encoded into matrices of their intrinsic frequencies. We assume global (all-to-all) coupling, which allows to greatly reduce computational cost. One important advantage of this approach is that it can be naturally adapted to clustering of multivariate functional data. We present the simulation results for several illustrative data sets.

Keywords Data clustering · Quantum synchronization · Non-Abelian Kuramoto model · Kuramoto–Lohe oscillators

1 Introduction

Clustering is probably the most important *unsupervised learning* problem. It deals with detecting a structure in a collection of unlabeled data. There is no unique or universal definition of cluster. A loose definition could sound that cluster is a set of objects which are similar (in certain sense) to each other and less similar to other objects from the set. Therefore, clustering methods aim to determine the intrinsic grouping in a set of unlabeled data. Clustering does not use category labels that tag objects by prior identifiers, i.e. class labels. The absence of category information makes the difference between data clustering (unsupervised learning) and classification or discriminant analysis (supervised learning).

One of the most popular approaches to data clustering, called *K-means*, has been proposed around 1960. Since then, a great number of various methods and algorithms have been developed. The most popular approaches are based on distance connectivity between data, calculation of centroids, statistical distributions and/or density analysis. There is vast and constantly growing literature on clustering methods, techniques and applications, we recommend (Jain et al. 1999; Jain 2010) for concise overviews.

There also exists a less mainstream approach to clustering, inspired by self-organization in complex systems, notably by synchronization in Kuramoto model of coupled oscillators (Kuramoto 1975). The classical Kuramoto model over the complex network reads:

$$\dot{\phi}_j = \omega_j + \frac{1}{N} \sum_{i=1}^N K_{ij} \sin(\phi_i - \phi_j), \quad j = 1, \dots, N. \quad (1)$$

This model describes a population of N phase oscillators that are coupled through the complex networks of interactions. The state of each oscillator is given only by his phase $\phi_j(t)$. Intrinsic frequency of the j -th oscillator is a real number ω_j and the network of interactions is given by the matrix K_{ij} . Individual oscillators in this model are called *Kuramoto oscillators*.

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Variations of the classical Kuramoto model have been used for different computing applications, see surveys (Arenas et al. 2008; Rodrigues et al. 2016) and references therein. As one notable example, we mention the method of community detection in complex networks based on Kuramoto model (1) proposed by Arenas et al. (2006). Moreover, it has been shown that the synchronization process unveils essential information about the network topology. In the paper by Jaćimović and Crnkčić (2017) Kuramoto model has been used to characterize networks by associating certain random variables to them.

The idea of Arenas et al. (2006) could also be adapted for data clustering. However, efficient clustering of multivariate data requires extensions of the classical Kuramoto model to higher dimensions. One such method has been proposed in Miyano and Tsutsui (2007): multivariate data are encoded into vectors of natural “frequencies” in the specific dynamical system on \mathbb{R}^m that resembles the Kuramoto model. In the paper by Shao et al. (2013) (see also Novikov and Benderskaya 2014) “frequencies” have been removed from the model and multivariate data were encoded in initial conditions. However, both methods involve near-neighbors couplings and, in general, require the recalculation of pairwise distances between data vectors at each step. Thus, one might consider these methods as being based on distance connectivity of data.

In the present paper we propose the method that is based on so-called *non-Abelian Kuramoto models*. Such models has been first introduced by Lohe (2009):

$$\dot{U}_j = H_j U_j + U_j G_j + \frac{K}{2N} \sum_{i=1}^N (U_j U_i^* U_j - U_i), \quad j = 1, \dots, N. \quad (2)$$

Here $U_j(t) \in U(n)$ are unitary matrices, representing the states of particles. H_j and G_j belong to the Lie algebra $\mathfrak{u}(n)$ consisting of $n \times n$ skew-Hermitian matrices. These matrices are interpreted as intrinsic frequencies of particles. The notion U_j^* stands for the conjugate transpose matrix to U_j . Finally, K is a global coupling strength in the system.

It is important that (2) defines the dynamics on the group $U(n)$, meaning that $U_j(0) \in U(n)$ for all $j = 1, \dots, N$ implies that $U_j(t) \in U(n)$ for all $t > 0$. Furthermore, let \mathfrak{m} be subalgebra of $\mathfrak{u}(n)$ and M the corresponding subgroup of $U(n)$. Assume that $U_j(0) \in M$ for all $j = 1, \dots, N$, and that H_j and G_j belong to \mathfrak{m} . Then, dynamics of (2) takes place on the subgroup $M \subset U(n)$, that is $U_j(t) \in M$ for any $t > 0$.

Individual particles in (2) are called *Kuramoto–Lohe (K–L) oscillators*. Underline that we consider the population of non-identical K–L oscillators with the global coupling.

Notice that we have added an extra term into non-Abelian Kuramoto model: the second term on the right-hand side in (2) that is absent in the original model (cf. Lohe 2009, Eq. 2). In this way, each K–L oscillator has two matrices of intrinsic frequencies H_j and G_j . This slight generalization makes it possible to double the dimension of the data (as we encode the data into frequency matrices, see the next section).

Non-Abelian Kuramoto models of the form (2) are the most adequate extensions of Kuramoto model to higher dimensions. Here, we introduce a clustering method based on the model with the global coupling, leading to significant simplification in implementation and reduction of computational complexity.

From this point of view, the classical (Abelian) Kuramoto model (1) describes synchronization on the group $U(1)$ and (2) is a natural extension to $U(n)$. Transition to synchronization is pretty similar in both Abelian and non-Abelian models: there exists a critical coupling strength K_c (that depends on dispersion of intrinsic frequencies), such that synchronization effects take place for $K > K_c$, see Lohe (2009) and Lohe (2010).

To conclude, we argue that non-Abelian Kuramoto models provide a natural framework for synchronization-based clustering of multivariate data. In this setting we do not need to consider the near-neighbors and recalculate the pairwise distances between oscillators at each step. Instead, the coupling is global and constant, initial conditions are randomly chosen and the data are encoded in frequency matrices of K–L oscillators. The underlying idea is that K–L oscillators with close intrinsic frequencies will synchronize first thus unveiling hierarchical cluster structure in the data set.

In principle, depending on the dimension of data vectors, one can use (2) on $U(n)$ for arbitrary n . For simplicity, in the present paper we focus on the partial case $n = 2$, i.e. on the non-Abelian Kuramoto model on $SU(2)$. This implies a restriction on the dimension of multivariate data: we assume that the data can be represented as vectors in the space \mathbb{R}^k for $k \leq 6$. In other words, we identify clusters in data sets where each object is given by no more than 6 attributes.

2 Algorithm

In this section we will explain the method in more details.

Suppose that the data set contains N objects represented by vectors a_1, \dots, a_N in \mathbb{R}^k with $k \leq 6$. With each vector a_j we associate one K–L oscillator.

Step 1. Encode the vector a_j into frequency matrices H_j and G_j of the j -th oscillator. Matrices H_j and G_j

belong to the Lie algebra $\mathfrak{su}(2)$ of skew-Hermitian zero-trace matrices. Here, we use an isomorphism between the space $\mathfrak{su}(2)$ and \mathbb{R}^3 .

Step 2. Pick randomly the initial conditions $U_1(0), \dots, U_N(0)$ for the system (2) from the uniform distribution on S^3 . Using an isomorphism between $SU(2)$ and S^3 , transform these random points on S^3 into $SU(2)$ matrices.

Step 3. Solve (2) with global coupling strength K .

Step 4. Observe the synchronization process of K-L oscillators $U_1(t), \dots, U_N(t) \in SU(2)$ and detect (possibly hierarchical) structure of data.

However, there are two ambiguities remaining in the above algorithm:

Q1: How the coupling strength K is chosen?

Q2: What is the stopping criteria?

We do not have an exact answer to the first question. The appropriate coupling strength K should be determined based on several realizations of the algorithm. Obviously, if K is too low (below the critical strength K_c), then synchronization does not take place at all and no information can be gained. On the other hand, if K is too large, one might observe rapid transition towards complete synchronization, with similar result. Therefore, K should be chosen in such a way to ensure that gradual synchronization process takes place.

When it comes to the second question, we remark that the stopping criteria can be introduced in many ways. However, in most cases checking the stopping criteria at each step is computationally demanding. Therefore, we suggest to act in the following way: solve (2) and extract information about clusters (mutually synchronized groups of K-L oscillators) only at certain moments, say $t = 1, 5, 10, \dots$. In this way it is possible to identify hierarchical structure of the data without checking criteria at each step.

An alternative suggestion for the stopping criteria is based on the observation that synchronization process is hierarchical. Introduce an (Euclidean) average of matrices U_1, \dots, U_N :

$$\hat{U}(t) = \frac{1}{N} \sum_{j=1}^N U_j(t).$$

Notice that \hat{U} does not belong to $SU(2)$. However, determinant of matrix \hat{U} is real and $0 \leq \det \hat{U} \leq 1$, with the case $\det \hat{U} = 1$ corresponding to complete synchronization. Therefore, $\det \hat{U}$ is a global order parameter of the population. When the synchronization in some clusters occurs, the order parameter $\det \hat{U}$ grows rapidly and remains almost constant afterwards for a certain time until synchronization between neighboring clusters continues.

Therefore, simple and computationally inexpensive receipt might be: stop the algorithm (and extract information about clusters) when $|\det \hat{U}(t + \delta t) - \det \hat{U}(t)| < \varepsilon$ (for reasonably chosen δt and ε).

3 Case studies

In this section we present results of the method on two representative (and small) data sets.

As the first case study, we analyzed the famous Iris flower data set (Wikipedia contributors 2017), consisting of 50 samples from each of 3 species of Iris flower (*Iris setosa*, *Iris virginica*, *Iris versicolor*). Each flower is characterized by 4 attributes: the length and width of sepals and petals.

We solved the system of $N = 150$ matrix ODE's (2) with randomly chosen initial conditions from $SU(2)$. The four attributes of each flower have been encoded into frequency matrices H_j and G_j (the first 3 attributes into H_j , and the remaining one into G_j). The two redundant entries of G_j have been set to zero.

The method identifies two clusters: the first one corresponds precisely to the *Iris setosa* flowers, while the second cluster includes all *Iris virginica* and *Iris versicolor* flowers. This result is expected and natural from the point of view of unsupervised learning. In fact, it is known that without using category labels species *Iris virginica* and *Iris versicolor* are indistinguishable.

Repeated simulations on this data set always yield the same answer.

For the second study we collected the data characterizing society and economy in 28 different countries. With each country we associate the following 6 attributes: GDP per capita, birth rate, life expectancy at birth, infant mortality rate, Gini index (distribution of family income) and military expenditures (as percent of total GDP). The statistics has been extracted from the CIA world factbook (CIA 2017).

The data have been roughly scaled and encoded into the frequency matrices H_j and G_j (three attributes in each matrix). The system of 28 matrix ODE's (with randomly chosen initial conditions) has been solved. The algorithm found 9 clusters:

First cluster: Sweden, Germany, Canada, Denmark, Netherlands.

Second cluster: Serbia, Bosnia and Herzegovina, Montenegro, Albania, Bulgaria, Latvia, Indonesia, Malaysia, China, Thailand, Chile, Uruguay, Argentina.

Third cluster: Togo, Uganda, Senegal, Cameroon.
 Separate clusters: (countries that did not fall into any cluster): Saudi Arabia, Qatar, United Arab Emirates, Slovenia, Kyrgyzstan, South Africa.

For this data set results can slightly vary from one simulation to another. However, the fundamental clustering pattern remains the same regardless of the realization. For instance, the first cluster always consists of the same 5 countries. Also, the Balkans countries are always set into the same cluster.

Figures 1 and 2 illustrate the synchronization process for the two data sets respectively. Number of clusters decreases over time (Figs. 1a, 2a), while the global order parameter increases (Figs. 1b, 2b).

4 Clustering of functional data

The method we propose here allows for various extensions. In particular, it can easily be adapted to deal with multivariate functional data without dramatic increase in computational complexity.

This can be done by considering time-dependent matrices $H_j(t)$ and $G_j(t)$ in (2). The functional data are then encoded into entries of these matrices.

This idea makes it possible to identify clusters over distinct time intervals. In order to extract decisive conclusion over the whole time span, we calculate matrices of pairwise distances between K-L oscillators at several moments.

The method was verified on two data sets that have been studied previously.

The first case study has been implemented on Canadian meteorological data, see Ramsay and Silverman (2005). This data set consists of 35 different locations in Canada characterized by two time-varying attributes (daily temperature and precipitation). We found the following data structure:

First cluster: St. Johns, Halifax, Sydney, Yarmouth, Charlottvi, Fredericton, Quebec, Sherbrooke, Montreal, Ottawa, Toronto, London, Arvida, Thunder Bay, Calgary, Bagottville, Regina, Pr. George, Winnipeg, Edmonton, Kamloops, Pr. Albert, The Pas, Whitehorse.
 Second cluster: Scheffervil, Churchill, Uranium City, Dawson, Yellowknife.
 Third cluster: Vancouver, Victoria.
 Separate clusters: Resolute, Inuvik, Iqaluit, Pr. Rupert.

Fig. 1 Simulation results for the Iris data set: total number of clusters decreases from 150 to 2 (a) and the global order parameter $\det \hat{U}(t)$ increases (b). The coupling strength is set to $K = 10$

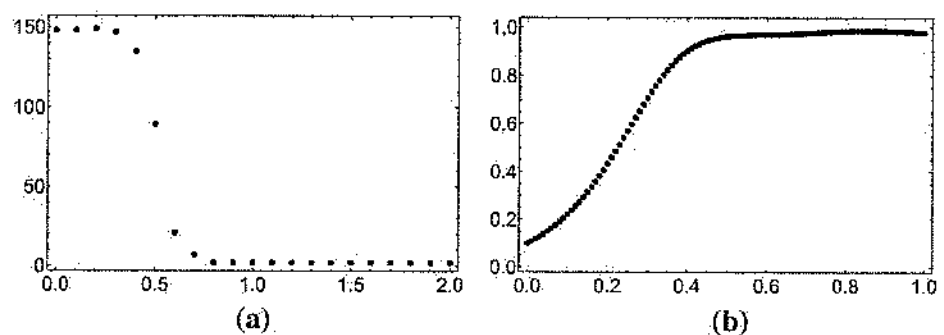
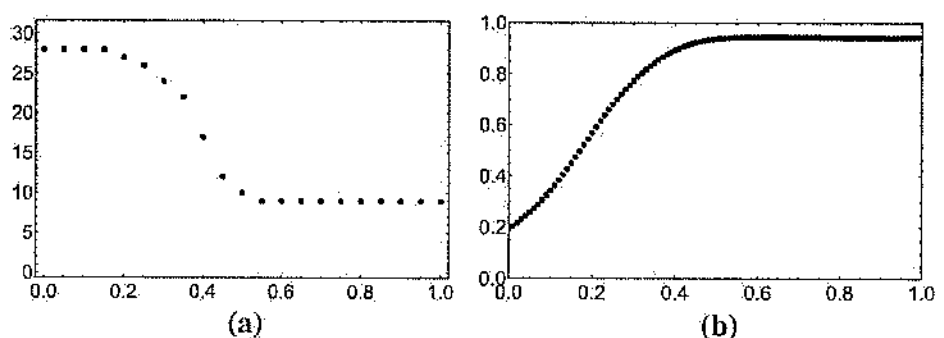


Fig. 2 Simulation results for the data set of 28 countries: total number of clusters decreases from 28 to 9 (a) and the global order parameter $\det \hat{U}(t)$ increases (b). The coupling strength is set to $K = 10$



Notice that our conclusions are almost identical to those of Jacques and Preda (2012, 2014). However, our method does not require to apriori specify number of clusters and therefore it allocates four weather stations in North Canada into separate clusters.

The second data set contains 6 economic indicators for 16 countries over the period 1990–2008. This 6-variate functional data set has been studied by Yamamoto (2012). Our method identifies 5 clusters:

First cluster: Australia, Canada, United States, Sweden, France, Denmark, Finland, Netherlands, United Kingdom, Belgium, Switzerland, New Zealand.
Separate clusters: Norway, Japan, Ireland, Spain.

These results are very similar to those obtained by Yamamoto. However, there are several differences: Switzerland is classified into the great cluster here. On the other hand, Norway and Spain are set into separate clusters. By closer examination of the data one can find the fundamental reasons for such classification. In the case of Norway the distinctive indicator is very high export revenue in the period from 2004. until 2006. When it comes to Spain the reason can be found in high unemployment rate.

5 Conclusion and outlook

In this paper we have proposed a novel algorithm for data clustering which is fully based on quantum synchronization (synchronization in non-Abelian Kuramoto models). We presented the method corresponding to the model on $SU(2)$ assuming that the data are represented by no more than 6 attributes. For the high-dimensional data we suggest to use the model on $SU(n)$. In this case frequency matrices belong to $\mathfrak{su}(n)$, the set of skew-Hermitian trace-zero matrices. Notice that $\mathfrak{su}(n)$ is isomorphic to \mathbb{R}^k , where $k = \frac{n^2+n}{2} - 1$. Since (2) includes two frequency matrices, this means that $SU(n)$ model is sufficient for clustering the data of the dimension $n^2 + n - 2$. Remark that initial conditions for (2) should be taken randomly, thus the algorithm requires generation of random matrices from $SU(n)$.

Underline that this idea is convenient for hierarchical clustering based on the hierarchical nature of synchronization process.

In many cases, the data set consists of objects labeled by vector attributes along with the pairwise relations,

represented by (possibly multilayer) network. Clustering of a such data set can be seen as a community detection problem in the (multilayer) network with (multivariate functional) node attributes. This situation can be studied by replacing the global coupling strength K in (2) by the matrix $K_{ij}(t)$ of (possibly time-varying) pairwise interactions.

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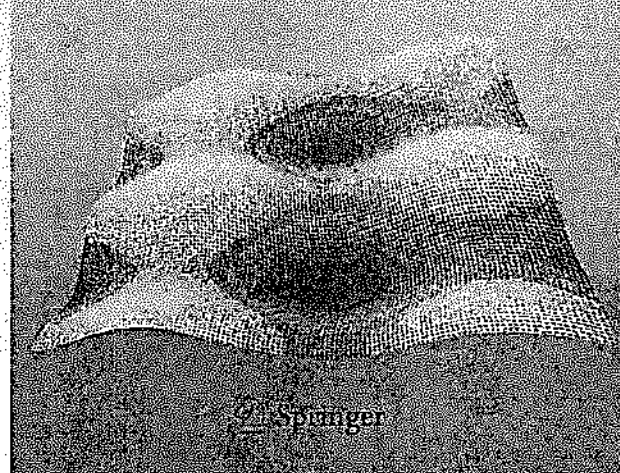
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Consensus and balancing on the three-sphere

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Abstract

We study consensus and anti-consensus on the 3-sphere as the global optimization problems. The corresponding gradient descent algorithm is a dynamical systems on S^3 , that is known in Physics as *non-Abelian Kuramoto model*. This observation opens a slightly different insight into some previous results and also enables us to prove some novel results concerning consensus and balancing over the complete graph. In this way we fill some gaps in the existing theory. In particular, we prove that the anti-consensus algorithm over the complete graph on S^3 converges towards a balanced configuration if a certain mild condition on initial positions of agents is satisfied. The form of this condition indicates an unexpected relation with some important constructions from Complex Analysis.

Keywords Consensus · Balancing · 3-sphere · Synchronization · Non-Abelian Kuramoto models

1 Introduction

Consensus problems play an important role in the broad field of distributed and cooperative control. Various problems of this kind on Euclidean spaces have been studied under different constraints and assumptions, see, for instance [13,14] and references therein. However, in many engineering applications, such as cooperative rigid-body attitude control [2,20], mobile sensing networks [16] or averaging rotations [4], the underlying space is non-Euclidean. Geometric consensus theory [18,22] aims at designing algorithms on certain Riemannian manifolds and elaborating a unified approach to consensus in Euclidean and non-Euclidean setting.

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There are essential differences between these two settings, since convergence properties of the consensus algorithms depend strongly on the geometry of underlying space. In order to get some feeling about this, it is instructive to start with the problem on the circle S^1 .

Consider a *swarm* of N agents whose states are described by points $e^{i\theta_1}, \dots, e^{i\theta_N}$ on S^1 . Suppose that agents communicate to each other through undirected *communication graph* $G = (V, E)$, where $V = \{1, \dots, N\}$ is the set of nodes and $E \subseteq \{(j, k) : j, k \in V, j \neq k\}$ is the set of edges. We say that agents j and k are *neighbors* if there exists an edge between them, i.e. if $(j, k) \in E$.

Based on the information received from his neighbors, each agent continuously adjusts his state. Following an analogy with linear consensus algorithms it is natural to suggest the following continuous-time update of the states of agents [18,21]:

$$\dot{\theta}_k = \frac{\alpha}{N} \sum_{j=1}^N a_{jk} \sin(\theta_j - \theta_k), \quad \alpha > 0, \quad k = 1, \dots, N. \quad (1)$$

here $a_{jk} = 1$ if $(j, k) \in E$ and $a_{jk} = 0$ if $(j, k) \notin E$.

The system (1) defines potential dynamics: it is a gradient system for the *disagreement cost function* on the N -torus $T^N = S^1 \times \dots \times S^1$:

$$\begin{aligned} V_\theta(\theta) &= \frac{1}{2N^2} \sum_{k=1}^N \sum_{j=1}^N a_{jk} \|e^{i\theta_j} - e^{i\theta_k}\|^2 \\ &= \frac{1}{2N^2} \sum_{k=1}^N \sum_{j=1}^N a_{jk} \left(2 \sin \frac{\theta_j - \theta_k}{2} \right)^2. \end{aligned} \quad (2)$$

In other words, (1) is obtained as gradient descent method for minimization problem (2): $\dot{\theta}_k = -\alpha \frac{\partial V_\theta}{\partial \theta_k}$, $k = 1, \dots, N$. Thus, trajectories converge to critical points of V_θ and all strict minima of V_θ are stable equilibria of (1).

Remark 1 The system (1) is well-known as Kuramoto model of coupled oscillators. Notice that, unlike in the original Kuramoto model introduced in [7], oscillators in (1) have identical (zero) frequencies and are coupled through the graph G .

The above Remark demonstrates that the question of convergence of distributed consensus algorithms on some Riemannian manifolds is related to the universal phenomenon of synchronization of coupled oscillators, see [17].

Definition 1 The configuration with $\theta_1 = \dots = \theta_N$ is called *synchronization* of agents.

It is obvious that synchronization corresponds to the global minimum of disagreement cost function V_θ . Therefore, consensus problem can be stated as optimization of the function V_θ . However, there is no guarantee that (1) will converge towards synchronization, since V_θ in general can have critical points different from global minimum. Existence of local minima and other critical points of the cost function depends greatly on the communication graph G .

The graph G is said to be S^1 -*synchronizing*, if (2) does not admit local minima different from synchronization. It is an open problem to characterize (at least approximately) all S^1 -synchronizing graphs. For example, simulations show that the ring graph with 4 nodes is S^1 -synchronizing, while ring graphs containing 5 and 6 nodes are not [18]. One of few universal results on this question states that complete graphs (all-to-all communication) and undirected trees are S^1 -synchronizing, see [18,21].

2 Consensus on S^3 and $SO(3)$ over undirected graphs

As mentioned in Introduction, mathematical formalization of some important engineering problems yields consensus problems on certain higher-dimensional Riemannian manifolds. In order to develop a meaningful geometric consensus theory, it is necessary to impose some conditions on the class of Riemannian manifolds on which the problems are stated. One natural restriction is to work under the assumption that the underlying manifold is a homogeneous space.

In this paper we will focus on consensus (and anti-consensus) on one specific manifold: Lie group $SU(2)$ with the group manifold S^3 .

In order to introduce coordinates on S^3 , we will work with the algebra of unit quaternions. Consider the following system of quaternionic ODE's on S^3 :

$$\dot{q}_j = q_j f_j q_j - \bar{f}_j, \quad j = 1, \dots, N. \quad (3)$$

Here, $q_j(t)$ is a unit quaternion, describing the position of the j -th agent on S^3 and $f_j = f_j(q_1, \dots, q_N)$ are quaternion-valued functions called *coupling functions* or *communication protocols*, depending on context. The notion \bar{a} stands for quaternionic conjugation of the quaternion a .

Notice that Eqs. (3) preserve S^3 ; this is easily verified by checking that $q_j(0) \in S^3$ for all $j = 1, \dots, N$ implies $q_j(t) \in S^3$ for all $t > 0$.

One can study different forms of functions f_j in (3), depending on specific goals. For consensus problems, it makes sense to consider the functions of the following form:

$$f_j = -\frac{\alpha}{2N} \sum_{k=1}^N a_{jk} \bar{q}_k, \quad \text{where } \alpha > 0. \quad (4)$$

The bar denotes quaternionic conjugation as before.

Plugging (4) into (3) yields the following system of quaternion-valued ODE's

$$\dot{q}_j = \frac{\alpha}{N} \sum_{k=1}^N a_{jk} (q_j q_k q_j - \bar{q}_k). \quad (5)$$

Remark 2 The system (5) (written in real variables) appeared in several papers with various interpretations, for instance as a swarm on sphere in [15] and the model of opinion formation on sphere in [1].

The next step is to notice that (5) displays potential dynamics. The disagreement cost function can be written as follows:

$$\begin{aligned} V_q(q_1, \dots, q_N) &= \frac{1}{N^2} \sum_{j=1}^N \sum_{k=1}^N a_{jk} \left(1 - \frac{1}{2} (\bar{q}_j q_k + \bar{q}_k q_j) \right) \\ &= \frac{1}{N^2} \sum_{j=1}^N \sum_{k=1}^N a_{jk} (1 - \cos \varphi_{jk}), \end{aligned} \quad (6)$$

where φ_{jk} is an angle between the two vectors on S^3 that correspond to quaternions q_j and q_k .

Therefore, the situation is analogous to the case of S^1 that is briefly explained in the previous section: consensus problem on S^3 can be stated as minimization of the function V_q

and the corresponding gradient descent method (5) provides a distributed consensus algorithm on S^3 .

Along with (6) consider also the problem of minimization of the function $V_Q(Q_1, \dots, Q_N)$ defined on $(SO(3))^N$ (see, for instance, [18,19]):

$$V_Q(Q_1, \dots, Q_N) = \frac{1}{2N^2} \sum_{j=1}^N \sum_{k=1}^N a_{jk} \text{Tr}(Q_j^T Q_k). \quad (7)$$

The gradient descent system for (7) reads:

$$Q_j^{-1} \dot{Q}_j = \frac{\alpha}{N} \sum_{k=1}^N a_{jk} (Q_j^{-1} Q_k - Q_k^{-1} Q_j). \quad (8)$$

Remark 3 It has been mentioned in Introduction that the gradient system (1) for consensus problem on S^1 can be interpreted as Kuramoto model of coupled oscillators with identical zero frequencies. There is an analogy in higher dimensions: gradient flow systems of the form (5) and (8) have been introduced in [8] as so-called *non-Abelian Kuramoto models*. This observation points out some parallel developments in Geometric consensus theory and the study of synchronization phenomena. For S^1 consensus is related to classical synchronization, while consensus on S^3 is related to the phenomenon named *quantum synchronization*, see [9].

Definition 2 The configuration with $q_1 = \dots = q_N$ is called *synchronization* on S^3 .

It is obvious that synchronization configurations are the global minima for functions (6) and (7).

Remark 4 There is a certain ambiguity in using terms *consensus* and *synchronization* in the relevant literature. In [11] these two notions are regarded as synonyms. On the other hand, in [18,19,21] the notion of consensus is more general; the swarm can admit many consensus configurations, depending on the graph G and the synchronization is a particular case of consensus.

The exposition so far presented analogous consensus problems and algorithms on spheres S^1 and S^3 and the matrix group $SO(3)$. However, it turns out that the convergence properties of algorithms (1), (5) and (8) are very different due to intrinsic geometric properties of underlying manifolds.

In order to see this, we start by the recent theoretical result, proven in [11]:

Theorem 1 Suppose that the communication graph G is undirected and connected. Then, the set of synchronization configurations is almost globally stable for (5). In other words, the set of all initial conditions for which (5) does not converge towards synchronization has zero Lebesgue measure on $(S^3)^N$.

Remark 5 Theorem 1 provides a theoretical base for consensus algorithms on S^3 : the convergence towards synchronization is *almost guaranteed*. The geometry of the 3-sphere is favorable for synchronization. In fact, Theorem 1 is valid for consensus on any sphere S^n with $n \geq 2$.

At the same time, Theorem 1 does not hold for S^1 . Indeed, as explained in Introduction, there exist many graphs that are not S^1 -synchronizing. The same applies to $SO(3)$, simulations show that the algorithm (8) does not necessarily converges towards synchronization.

The disagreement cost function (7) can have local minima. The structure of the set of local minima of V_Q strongly depends on the communication graph.

Simulation results. In order to illustrate the above point we conducted simulations for the particular case when $N = 5$ agents are coupled into the ring (i.e. communication graph G is the ring with 5 nodes). The initial states of agents have been chosen randomly from the uniform distribution on S^3 . Initial states on $SO(3)$ have been obtained by mapping points from S^3 to $SO(3)$ matrices.

The algorithm (5) on S^3 reached synchronization in 1000 out of 1000 simulations.

The algorithm (8) on $SO(3)$ reached synchronization in 609 out of 1000 simulations.

Notice that the same experiment has already been reported by Markdahl et al. [10,11]; our results are consistent with theirs.

Visualization. Of course, it is problematic to visualize the evolution of points on the sphere in the 4-dimensional space. We will illustrate the evolution of the swarm on S^3 by displaying rotating bodies in 3D. Mathematically, we use the double cover map from S^3 to $SO(3)$ and representation of 3D rotations by unit quaternions. It is very well known that such representation causes some peculiar effects, due to the fact that two antipodal quaternions q and $-q$ correspond to the same matrix in $SO(3)$.

Convergence properties of (5) and (8) are demonstrated in two short videos. The number of agents is $N = 5$ and the communication graph is the ring with 5 nodes. We have randomly chosen initial conditions using uniform distribution on S^3 and the same initial conditions on $SO(3)$ (by mapping unit quaternions into the corresponding $SO(3)$ matrices).

One can see that synchronization is achieved on S^3 (Online Resource 1), while on $SO(3)$ algorithm ends in some local minimum of the disagreement cost function (Online Resource 2). According to the simulation results explained above the probability of such outcome is approximately equal 0.4.

3 Consensus on S^3 over the complete graph

It is known that for the case of complete communication graph algorithms (1), (5) and (8) always converge towards synchronization on S^1 , S^3 and $SO(3)$ respectively, see [18,19]. In other words, if G is the complete graph, synchronization configurations are the only minima of functions V_h , V_q and V_Q .

In this section we discuss consensus problem on S^3 from a different point of view. Throughout this and the next section we assume that the communication graph G is complete. Notice that in some engineering applications such all-to-all topology is very expensive or even not feasible. In the context of coupled oscillators, this situation is named global (all-to-all) or *mean-field* coupling. In this case the system (3) is written as:

$$\dot{q}_j = q_j f q_j - \bar{f}, \quad j = 1, \dots, N, \quad (9)$$

where $\bar{f} = f(q_1, \dots, q_N)$ is a global coupling function.

For consensus over the complete graph, the coupling function is of the form:

$$f = -\frac{\alpha}{2N} \sum_{k=1}^N \bar{q}_k, \quad \text{where } \alpha > 0. \quad (10)$$

Substituting (10) into (9) yields the consensus algorithm on S^3 :

$$\dot{q}_j = \frac{\alpha}{N} \sum_{k=1}^N (q_j q_k q_j - \bar{q}_k), \quad j = 1, \dots, N. \quad (11)$$

Moreover, for the case of the complete graph the disagreement cost function V_q given by (6) has a particularly simple form:

$$\hat{V}_q = 1 - r^2, \quad (12)$$

where $r = \|\frac{1}{N} \sum_{j=1}^N q_j\|$. In order to clarify this, consider a swarm of agents whose states are given by $q_1, \dots, q_N \in S^3$ and introduce $\langle q \rangle = \frac{1}{N} \sum_{j=1}^N q_j$. Obviously, $\langle q \rangle$ is a point in the ball B^4 (interior of S^3), that is centroid (center of mass) of the set of points q_1, \dots, q_N . Further, $r = \|\langle q \rangle\|$ is the norm of the corresponding 4D vector. In the context of coupled oscillators, the real number $r \in [0, 1]$ is called the *order parameter* of the swarm of agents (i.e. of the population of oscillators). The situation $r = 1$ (fully coherent state) corresponds to synchronization. Opposite situation, when $r = 0$, is called fully incoherent state in Statistical Physics. In Systems Theory, the later is referred to as *balanced* configuration.

To resume, in the case of the complete graph, the problem of minimization of the disagreement cost function appears to be simply maximization of the order parameter (coherence degree) of the swarm. The gradient descent system for (12) is (11).

Proposition 1 [18,19,21] *The function (12) does not have local minima different from synchronization. Synchronization configurations are global minima of (12) and the only asymptotically stable equilibria of (11).*

Hence, the problem that we discuss in this section is fully understood from the point of view of optimization theory. However, here we take a different point of view in order to point out some relations with Physics and Hyperbolic geometry and extract some specific novel results.

As mentioned above, system (9) is in fact the non-Abelian Kuramoto model on S^3 with the global (all-to-all) coupling. Recently, the authors have studied some symmetries of this system in [5].

In order to explain this we start with some notations. Denote by \mathbb{H} the algebra of quaternions. The set of linear fractional (Möbius) transformations acting on the extended quaternionic space $\mathbb{H} \cup \{\infty\}$ is the group $GL(2, \mathbb{H})$. Consider the subgroup $G_{\mathbb{H}}$ of all Möbius transformations that preserve S^3 .

Theorem 2 [5] *Let a swarm evolves by (9) from their initial positions $q_1(0), \dots, q_N(0)$. Then in $G_{\mathbb{H}}$ there exists an one-parametric family g_t of Möbius transformations, such that*

$$q_j(t) = g_t(q_j(0)) \quad \text{for all } t > 0 \text{ and } j = 1, \dots, N.$$

Theorem 2 states that at any instant t the positions of agents on S^3 are obtained from their initial positions by some Möbius transformation belonging to $G_{\mathbb{H}}$. Notice that an analogous fact for the swarm on S^1 has been established earlier in [12].

Using Theorem 2 and general results of group theory we can claim that the distribution of agents on S^3 belongs to the orbits of the group $G_{\mathbb{H}}$. This yields the following:

Corollary 1 *Suppose that the swarm evolves by (9). Then the distribution of agents evolves on a certain 10-dimensional invariant submanifold lying in the infinite-dimensional manifold of all distributions on S^3 .*

Proof For the proof it suffices to check that the dimension of orbits of the group $G_{\mathbb{H}}$ is 10. In order to see this, notice that the general Möbius transformation that preserves S^3 can be written in the following form:

$$w(q) = p(1 - q\bar{a})^{-1}(q - a)r, \text{ for } q \in S^3,$$

where p and r are unit quaternions and $a \in \mathbb{H}$, $|a| < 1$.

The two unit quaternions p and r generate rotations on S^3 and a corresponds to the action on the center of the sphere. Rotations (or unit quaternions) yield $2 \times 3 = 6$ independent real parameters, while a is a point in B^4 (4 real parameters). Now, the Corollary follows from the general results of group theory. \square

There is a unique way to extend the Möbius transformation g_t from S^3 to the whole unit ball B^4 . Consider this extension and denote by $P(t)$ the image of the center of S^3 under the action of g_t , that is $P(t) = g_t(0)$.

Proposition 2 [5] *Let a swarm evolves by (9) and let g_t be a family of Möbius transformations that is defined in Theorem 2. Then the point $P(t) = g_t(0)$ evolves in B^4 by the following quaternionic ODE:*

$$\frac{dP}{dt} = PfP - \bar{f}. \quad (13)$$

In the remaining part of this Section and throughout the next Section we will need the notion of the *conformal barycenter* [3] of the probability measure μ on the sphere. Assume that the probability measure μ does not contain atoms of the weight $\geq \frac{1}{2}$.

Following Douady and Earle [3], introduce the function h_μ on B^4 :

$$h_\mu(a) = \frac{1}{2} \int_{S^3} \log \frac{1 - |a|^2}{|a - u|^2} d\mu(u), \quad a \in B^4$$

and denote by ξ_μ the gradient of h_μ in the Poincaré (hyperbolic) geometry of B^4 . Then ξ_μ is a vector field in B^4 and Douady and Earle have proven that there is a unique point $B(\mu)$ in B^4 where the field ξ_μ vanishes. That point is called the conformal barycenter of μ .

Remark 6 Let μ be a probability measure on S^3 and w a Möbius transformation that preserves B^4 . Consider the measure $\nu = w(\mu)$ that is obtained as Möbius transformation of μ . It holds that $w(B(\mu)) = B(\nu)$, where $B(\mu)$ and $B(\nu)$ are conformal barycenters of the two measures.

This Remark emphasizes one special property of conformal barycenter: if the measure is transformed by a Möbius transformation, then the conformal barycenter is transformed by the same Möbius transformation. Centroid does not have this unique property. If we denote by $C(\mu)$ and $C(\nu)$ centroids (mean values) of the measures μ and ν , then, in general, $w(C(\mu)) \neq C(\nu)$.

Lemma 1 *Consider the swarm (9) and suppose the number of agents is large, $N \rightarrow \infty$. Assume that the initial distribution of agents is uniform on S^3 . Then, conformal barycenter of the swarm coincides with the centroid at each moment t . Moreover, the distribution ρ of agents at any moment t is given by the following density function S^3 :*

$$\rho(y; P(t)) = \frac{1}{2\pi^2} \left(\frac{1 - \|P(t)\|^2}{\|y - P(t)\|^2} \right)^3, \quad y \in S^3, P(t) \in B^4. \quad (14)$$

Proof The distribution (14) is called the *Poisson kernel* on S^3 . It is known (see [6]) that Poisson kernels on S^3 (as well as on S^1) arise as Möbius transformations of the uniform measure. The centroid of the Poisson kernel (harmonic measure) is the image of zero under the corresponding Möbius transformation, i.e. point $P(t)$ from Proposition 2. In other words, Poisson kernels are very special distributions on S^3 for which conformal barycenter and centroid coincide. \square

Lemma 2 Consider the swarm that evolves by the consensus algorithm (11) with $\alpha > 0$. Suppose that $N \rightarrow \infty$ and the initial distribution of agents is uniform on S^3 . Then the order parameter $r(t)$ satisfies the following simple real-valued ODE:

$$\frac{dr}{dt} = \frac{\alpha}{2}(r - r^3). \quad (15)$$

Proof Underline that in this Lemma we consider the consensus algorithm (11).

Observe the evolution of conformal barycenter $P(t)$ of the swarm. Due to Lemma 1, $P(t)$ is also the centroid of the swarm. Since $P(t) \in B^4$, we write $P(t) = ru(t)$, where $u(t)$ is a unit quaternion and $r(t) = |P(t)|$ is an order parameter.

Then the coupling function (10) reads

$$f = -\frac{\alpha}{2N} \sum_{k=1}^N \hat{q}_k = -\frac{\alpha}{2} \tilde{P}(t) = -\frac{\alpha}{2} r \bar{u}(t), \quad \alpha > 0.$$

Substituting in (13) we obtain:

$$\dot{r}u + r\dot{p} = -\frac{\alpha}{2} r u \bar{u} r u + \frac{\alpha}{2} r \dot{u} = -\frac{\alpha}{2} r^3 u + \frac{\alpha}{2} r \dot{u}.$$

In the last equality we have used that r is a real number and hence commutes with any quaternion.

Multiplication by u^{-1} from the left yields

$$\dot{r} + r \dot{u} u^{-1} = \frac{\alpha}{2}(r - r^3). \quad (16)$$

The last equation is almost what we want to obtain, but it contains an extra term on the left hand side. We notice that (16) is, strictly speaking, a quaternion-valued ODE (the system of 4 real-valued ODE's) and we are interested in ODE for the real part. In order to evaluate the real part of the expression $\dot{u} u^{-1}$, we write the quaternion u in the Cayley–Dickson form:

$$u = z_1 + z_2 j = r_1 e^{i\varphi_1} + r_2 e^{i\varphi_2} j.$$

Since u is a unit quaternion we know that $r_1^2 + r_2^2 \equiv 1$.

Simple calculations yield expressions for u^{-1} and \dot{u} :

$$\begin{aligned} u^{-1} &= \bar{u} = r_1 e^{-i\varphi_1} - r_2 e^{i\varphi_2} j; \\ \dot{u} &= (\dot{r}_1 + i r_1 \dot{\varphi}_1) e^{i\varphi_1} + (\dot{r}_2 + i r_2 \dot{\varphi}_2) e^{i\varphi_2} j. \end{aligned}$$

Using the last two equations it is easy to check that:

$$\begin{aligned} \operatorname{Re}(\dot{u} u^{-1}) &= \dot{r}_1 r_1 + \dot{r}_2 r_2 = \frac{1}{2} \left(\frac{d}{dt} r_1^2 + \frac{d}{dt} r_2^2 \right) \\ &= \frac{1}{2} \frac{d}{dt} (r_1^2 + r_2^2) = \frac{1}{2} \frac{d}{dt} (1) = 0. \end{aligned}$$

Hence, extracting the real part of the quaternion-valued ODE (16) yields real-valued ODE (15). \square

Remark 7 As Lemma 2 requires the uniform initial distribution on S^3 , the initial condition for (15) is $r(0) = 0$. For consensus algorithm $\alpha > 0$ and hence $r = 0$ is unstable equilibrium of (15). Hence, the swarm (11) with $\alpha > 0$ will converge towards synchronization (i.e. $r \rightarrow 1$).

4 Anti-consensus and balancing on S^3

In this section we discuss the anti-consensus problem over the complete graph. This problem can be stated as minimization of the function $-\hat{V}_q$ defined by (12). The gradient descent system is then (11) with $\alpha < 0$.

Definition 3 The configuration of agents is called *balanced*, if their centroid is zero quaternion.

We do not impose any assumptions on the number of agents. The swarm can consist of the finite number N of agents, as well as of continuum of agents. In the later case, the distribution of agents at any moment t is given by the density function $\eta(q, t)$, where $q \in S^3$. The notion of balanced configuration naturally extends to the later case as well.

As explained in the previous section, consensus algorithm over complete graph converges globally towards synchronization. In analogy, one might expect that anti-consensus algorithm converges towards a certain balanced configuration. The situation turns out to be more difficult for this problem. Still, the following theorem claims that all balanced configurations are anti-consensus configurations over complete graph.

Proposition 3 [18,21] All balanced configurations are local minima of $-\hat{V}_q$.

The opposite statement is not proven. Below, we will state a new result claiming that the gradient system (11) converges to a balanced configuration under certain mild conditions.

We assume that the initial distributions of agents is given by the set of points $\eta = \{q_1(0), \dots, q_N(0)\} \subset S^3$ or by density function $\eta(q, 0)$ where $q \in S^3$.

Remark 8 We assign a probability measure to any given distribution on S^3 (regardless if this distribution is discrete or absolutely continuous). In this way we will talk about conformal barycenter of the distribution of agents on S^3 .

Definition 4 We say that the distribution on S^3 contains a *majority cluster*, if the corresponding probability measure contains an atom of the weight $\geq 1/2$. For the case when the distribution η is concentrated at the finite number N of points on S^3 , the existence of majority cluster means that there are $\geq N/2$ coinciding points in η .

Consider the anti-consensus algorithm (11) with $\alpha < 0$. Due to Corollary 1, the distribution of agents will evolve on an invariant submanifold M_η of real dimension 10. This invariant submanifold is determined by the initial distribution η .

Lemma 3 Suppose that the initial distribution $\eta(0)$ of agents does not contain a majority cluster. Then there is a unique (up to rotation on S^3) balanced configuration in invariant submanifold $M_{\eta(0)}$.

Proof Manifold $M_{\eta(0)}$ consists of all distributions on S^3 that can be obtained as Möbius transformations of $\eta(0)$. Then $\eta(t) \in M_{\eta(0)}$ is the distribution of agents at the moment t .

Due to Theorem 2, the swarm converges by the action of the group of Möbius transformations, meaning that there exists an one-parametric family of Möbius transformations g_t , such that $\eta(t) = g_t(\eta(0))$.

Then, conformal barycenter $B(\eta(t))$ also evolves by the action of g_t , $B(\eta(t)) = g_t(B(\eta(0)))$, see Remark 6.

Further, notice that for balanced configuration both conformal barycenter and centroid are at zero (i.e. at the center of S^3). In order to extract our conclusion, it remains to recall that there exists exactly one (up to a rotation on S^3) Möbius transformation that maps the point $B(\eta(0))$ into zero. \square

Theorem 3 Suppose that the initial distribution η of agents does not contain a majority cluster. Then the balanced configuration lying in \mathcal{M}_η is globally stable for the system (11) with $\alpha < 0$.

Proof First, notice that the system (11) can be written in the following form (see [15]):

$$\dot{q}_j(t) = \alpha(q_j \mu(t) q_j - \bar{\mu}(t)), \quad j = 1, \dots, N, \quad (17)$$

where $\mu(t) = \langle q \rangle(t) = \frac{1}{N} \sum q_j \in B^4$ is a centroid of the swarm.

The form (17) unveils the *mean-field* character of (11): it acts like agents are not connected to each other, but at each instant of time coupled to a common direction on S^3 . This common direction represents the mean field and, of course, it depends on configuration of agents and evolves in time.

The function $p = 1 - r^2 \geq 0$ can be taken as Lyapunov function for the system (17). The minimum of p is achieved at the value $r = 1$ that corresponds to synchronization. Differentiation of p yields [15]:

$$\begin{aligned} \dot{p} &= \frac{d}{dt} |\mu|^2 = -2(\bar{\mu} \dot{\mu} + \dot{\bar{\mu}} \mu) = -\frac{2}{N^2} \sum_{j,k} \bar{q}_j \dot{q}_k - \frac{2}{N^2} \sum_{j,k} \dot{\bar{q}}_k q_j \\ &= -\frac{2\alpha}{N^2} \sum_{j,k} \bar{q}_j (q_k \mu q_k - \bar{\mu}) - \frac{2\alpha}{N^2} \sum_{j,k} (\bar{q}_k \bar{\mu} \bar{q}_k - \mu) q_j \\ &= \frac{2\alpha}{N} \sum_j (\bar{q}_j \bar{\mu} + \mu q_j) - \frac{2\alpha}{N^2} \sum_{j,k} (\bar{q}_j q_k \mu q_k + \bar{q}_k \bar{\mu} \bar{q}_k q_j) \\ &= \frac{2\alpha}{N} \sum_j (\bar{q}_j \bar{\mu} + \mu q_j) - \frac{2\alpha}{N} \sum_k (\bar{q}_k \bar{\mu} + \mu q_k) + \frac{2\alpha}{N} \sum_j (\bar{q}_j \mu \bar{q}_j \bar{\mu} + q_j \bar{\mu} q_j \bar{\mu}) \\ &= \frac{2\alpha}{N} \sum_j (\bar{q}_j \mu \bar{q}_j \bar{\mu} + q_j \bar{\mu} q_j \bar{\mu}) = -\frac{2\alpha}{N} \sum_j (1 - \cos^2(\psi_j)) \geq 0, \end{aligned}$$

where ψ_j is an angle between the point q_j and the mean field direction $\langle q \rangle$. The last inequality follows from $\alpha < 0$ and implies that p is non-decreasing. Accordingly, the order parameter r is monotonically non-increasing.

Further, it is easy to observe that any configuration with $1 > r > 0$ can not be stable for the system (17). Indeed, suppose that certain configuration with $r > 0$ is an equilibrium for (17). Obviously, any perturbation of this configuration that decreases r corresponds to a unstable direction of the system (17). Hence, such equilibria can not be stable.

Finally, due to Lemma 2, the system converges towards a unique balanced configuration that lies in an invariant submanifold \mathcal{M}_η . This balanced configuration is determined by the initial distribution of agents: $\eta = \{q_1(0), \dots, q_N(0)\}$. \square

5 Conclusion

The natural setting for geometric consensus theory is the class of homogeneous spaces. In the previous studies a universal conceptual approach to consensus problems on homogeneous spaces has been developed. However, convergence properties of algorithms depend greatly on geometry of the underlying spaces, as we have explained here for examples of S^1 , S^3 and $SO(3)$. Hence, there are few universal results regarding convergence of (anti-)consensus algorithms. Sarlette and Sepulchre have proven convergence of consensus and anti-consensus algorithms on the wide class of manifolds for the cases when the communication graph is complete or undirected tree. Notice that both topologies can be problematic in applications, since complete graph is often too expensive, while the tree topology is not robust. Propositions 1 and 3 from the present paper are partial cases of their results. Recently, Markdahl et al. have established almost global convergence of consensus algorithms on spheres S^n for $n \geq 2$ over arbitrary connected and undirected graph (Theorem 1 in the present paper).

In this paper we pointed out the fact that gradient consensus algorithms are essentially non-Abelian Kuramoto model with zero intrinsic frequencies. This observation offers a new insight into consensus algorithms. Sections 1 and 2 contain existing results from a slightly different point of view. Some novel results are presented in sections 3 and 4.

Throughout the paper we have investigated the particular case of the sphere S^3 . Notice however that analogous results can also be stated for the swarm on S^1 with the complete communication graph. The reason behind this is that these are the only spheres that are Lie groups.

In Section 4 we have proven Theorem 3, stating that the swarm on S^3 under anti-consensus protocol will converge towards a balanced configuration. Moreover, this balanced configuration is uniquely (up to rotation on S^3) determined by the initial distribution of agents. The crucial assumption is the absence of a majority cluster in the initial distribution of agents.

In order to illustrate this, mention that Sepulchre et al. in [23] have characterized all anti-consensus configurations on the circle for the complete graph. The only anti-consensus configuration that is not balanced appears for N odd and consists of $(N + 1)/2$ agents at one point on the circle and $(N - 1)/2$ agents at the antipodal point. Our Theorem 3 provides an explanation; this is possible since such configuration contains a majority cluster. This unveils an unexpected relation with the construction of Douady and Earle [3] that is widely investigated and used in the present paper.

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Exploring Complex Networks by Detecting Collective Dynamics of Kuramoto Oscillators

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Abstract

Different models and concepts from Statistical Mechanics are increasingly exploited to study the structure and topology of complex networks. For instance, famous Kuramoto model of coupled oscillators has been successfully applied to analyze complex networks. It has been shown that the gradual process of synchronization reveals essential information about network topology. Here, we propose the method of exploring complex networks by detecting the collective behavior of certain groups of oscillators. The information on collective behavior is extracted from the statistics of Möbius transformations that govern oscillators dynamics on fixed time intervals. Due to rich geometric and algebraic structure of the group $PSL(2; \mathbb{C})$ of Möbius transformations, we can employ simple concepts from projective geometry (such as cross ratio of four points on the unit circle S^1) in order to study collective dynamics.

1 Introduction

In many cases large amount of empirical data can be represented by the complex network of items and interactions between them. This requires efficient algorithms for investigation of large networks. In study of large complex networks there exist a class of algorithms that are based on concepts and objects from Statistical Mechanics. For instance, a classical problem in Graph Theory is *community detection*. *Community* in the complex networks is defined in different ways and the definitions used in various literature usually depends on the method of investigation of the network (for detailed survey see [Fortunato, 2010]). In any case, by community one means the group of the nodes that are densely interconnected, while their connections to the remaining nodes in the network are relatively sparse.

For the problem of community detection several algorithms based on ideas of Statistical Mechanics have been proposed in 2000's. The first method employs the model of coupled phase oscillators (Kuramoto oscillators, [Kuramoto, 1975]) and well-known phenomena of synchronization. Indeed, since XVI century and famous Huyghens' letters ([Huygens, 1665]), it is known that oscillators that are weakly coupled tend to synchronize, this is an universal phenomena observed in many variations in nature and technology. Arenas et al.

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([Arenas et al., 2006]) proposed the algorithm of community detection that relies on observation that the process of gradual synchronization in the network of coupled oscillators unveils the network topology. Consequently, one might expect that the oscillators belonging to the same community will synchronize their oscillations before the synchronization in the whole network takes place. This is the basic idea standing behind this method: observation of the process of gradual synchronization makes it possible to distinguish densely interconnected communities in the network. Notice, that this is only the rough idea, Arenas et al. introduced mathematical instruments to study this process.

Another idea for the same problem relies on phenomena of ferromagnetism and collective spin dynamics. These methods employ the Potts model ([Reichardt & Bornholdt, 2004, Reichardt & Bornholdt, 2006]) or Ising model ([Son et al., 2006]) of ferromagnetism for community detection. This method is based on the choice of suitable Hamiltonian for the network. In this way, the problem of community detection is approached by minimization of the Hamiltonian.

In the past decade various modifications and extensions of the above mentioned methods have been proposed. Algorithms based on Statistical Mechanics have several advantages. For instance, they are typically well-suited for the networks with various kinds of interactions (including weighted graphs, repulsive interactions, delayed or noisy interactions, etc.). The second advantage is that they allow to detect fuzzy or overlapping communities.

This paper is intended to propose new method for investigation of complex networks, that can be based on the models studied in [Arenas et al., 2006], as well as in [Reichardt & Bornholdt, 2004, Reichardt & Bornholdt, 2006]. The difference is that our method is based on detection of collective behavior of nodes in complex networks. This approach is inspired by the result of [Marvel et al., 2009] for the globally coupled population of oscillators. This result enables us to use classical concepts of Projective Geometry and Complex Analysis. In the next section we briefly explain the main result of [Marvel et al., 2009] and some extensions necessary to apply the whole concept to the investigation of complex networks. In Section 3 we explain in detail the application to two typical problems related to complex networks: community detection and identification of influential nodes. In Section 4 we briefly illustrate the method by depicting results for some random networks. Finally, the paper is concluded by the short outlook for the future research and some applications.

2 Coupled Oscillators

In this paper we consider the model of phase oscillators that are coupled through the complex network of pairwise interactions as a paradigm for collective behavior in large systems. This model is written as the following dynamical system:

$$\dot{\varphi}_j = \omega + \frac{1}{N} \sum_{i=1}^N K_{ij} \sin(\varphi_j - \varphi_i), \quad j = 1, \dots, N. \quad (1)$$

Here, $\varphi_j(t)$ is the phase of the j -th oscillator and ω is the frequency common for all oscillators. The coupling network is given by the matrix K_{ij} . Total number of oscillators N is assumed to be sufficiently large (say, $N \geq 500$).

It is known that if the network is connected (i.e. there exists the path in the network between two arbitrary nodes i and j) and all interactions are attractive (i.e. $K_{ij} \geq 0$ for $\forall i, j$) then in certain moment synchronization of all oscillators in the network will occur.

Underline that the model (1) differs from the one that was considered in the seminal paper [Kuramoto, 1975] of Kuramoto. In [Kuramoto, 1975], the global coupling is assumed, that is $K_{ij} \equiv K > 0$ for $\forall i, j$. On the other hand, intrinsic frequencies ω_j are different for different oscillators.

Introduce the new variable $z_j(t) = e^{i\varphi_j(t)}$. Then we can study dynamics on the unit circle since $z_j(t) \in S^1$ for all $t \geq 0$. We call the variable $z_j(t)$ the state of oscillator j at the moment t .

Further, recall that the set of all Möbius transformations in the complex plane form a group. We will work with the subgroup consisting of all Möbius transformations that preserve the unit disc. The general Möbius transformation that preserves the unit disc can be written in the following form:

$$\mathcal{M}(z) = \frac{e^{i\psi}z + \alpha}{1 + \bar{\alpha}e^{i\psi}z}, \quad (2)$$

for some angle $\psi \in [0, \pi]$ and $\alpha \in \mathbb{C}$, $|\alpha| < 1$.

The main result of [Marvel et al., 2009] can be briefly formulated as follows:

Proposition 1 *The state of each oscillator evolves by the action of Möbius group. More precisely, the state $z_j(t)$ at each moment t is given by a certain Möbius transformation \mathcal{M}_t^j of the initial state $z_j(0)$, that is $z_j(t) = \mathcal{M}_t^j(z_j(0))$.*

In addition, the evolution of parameters of the Möbius transformation acting on the oscillator j is given by the following system of ODE's for parameters of (2):

$$\begin{cases} \dot{\alpha}_j = i(f_j(t, \cdot)\alpha_j^2 + \omega\alpha_j + \bar{f}_j(t, \cdot)); \\ \dot{\psi}_j = (f_j(t, \cdot)\alpha_j + 2\omega + \bar{f}_j(t, \cdot)\bar{\alpha}_j). \end{cases} \quad (3)$$

for some coupling function $f(t, \cdot)$ that depends on time and states of all oscillators z_1, \dots, z_N at each moment t .

Remark 1 *Since the function f depends on large number of variables, it is virtually impossible to specify the exact Möbius transformation \mathcal{M}_t^j acting on the oscillator j on time interval $(0, t)$ a priori.*

Remark 2 *The geometric meaning of the variable α_j is quite transparent: it turns out that $\alpha_j(t)$ is the image of the zero (center of the disc) under the action of corresponding Möbius transformation. This can be simply written as:*

$$\alpha_j(t) = \mathcal{M}_t^j(0).$$

Remark 3 *Notice that in [Marvel et al., 2009] only the case of global coupling has been considered, i.e. $K_{ij} \equiv K$ for $\forall i, j$. In this case the Möbius transformation acting on each oscillator is the same, i.e. the whole population evolves by the action of the same Möbius transformation at the fixed time interval $(0, t)$.*

3 Algorithm

Consider the system (1). We know that $z_j(t) = \mathcal{M}_t^j(z_j(0))$, for some unknown disc-preserving Möbius transformation \mathcal{M}_t^j .

However, one can identify the exact Möbius transformation acting on the j -th oscillator using the classical concept from Projective Geometry: cross ratio. For that, it is not enough to measure only the state of oscillator j , but at least three more suitably chosen oscillators. This is based on the fact that Möbius transformation preserves cross ratio of four points, see [Needham, 1999].

Definition 1 [Jučimović & Crnković, 2017]

1. We say that four oscillators i, j, k, l agree, if for all $t \geq 0$ there exists Möbius transformation \mathcal{M}_t such that $z_i(t) = \mathcal{M}_t(z_i(0))$, $z_j(t) = \mathcal{M}_t(z_j(0))$, $z_k(t) = \mathcal{M}_t(z_k(0))$, $z_l(t) = \mathcal{M}_t(z_l(0))$.
2. Coherence of the network is the probability that four randomly chosen oscillators agree.

In other words, four oscillators agree if they evolve by the action of the same one-parametric family of Möbius transformations. Our algorithm is based on the above definition. It can be roughly explained in the following steps:

1. Assume that the network with N nodes (oscillators) is given. Pick randomly four oscillators i, j, k, l from the population 1, dots, N .
2. Check if i, j, k, l (approximately) agree. If they do not agree, go to the step 1.
3. If i, j, k, l agree, find the corresponding Möbius transformation, i.e. find parameters ψ and α in (2).
4. Parameter α is represented by the point in the unit disc.
5. Repeat the steps 1-4 until M points in the unit disc is found.
6. Obtain the "cloud" of points in the unit disc. Using some of existing algorithms divide this "cloud" into clusters.
7. Each point corresponds to quadruple of nodes (oscillators). Find which nodes appear dominantly in which cluster. In whole, this yields clusterization of the network.

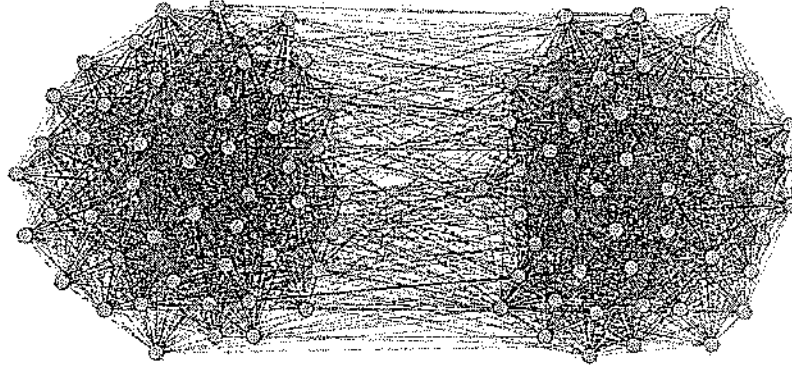


Figure 1: Random graph with two communities.

We also briefly introduce one more concept characterizing the position of the single node in the network. Fix the node i in the network and pick randomly 3 nodes j, k, l different from i . Denote by r the coherence of the network. Denote by p_i the probability that four oscillators i, j, k, l (approximately) preserves cross ratio at time interval $(0, t)$.

Definition 2 [Jaćimović & Crnković, 2017] *Correspondence level of the node i in the network is $\frac{p_i}{r}$.*

Notice that the concept of *correspondence level* is statistical and can be approximately computed using Monte Carlo method. From the above definition it is clear that the average correspondence level in the network equals 1.

In particular, the concept of correspondence level can be used to identify important (influential) nodes in the network. Indeed, one might expect that influential nodes do not participate in collective behavior and therefore have lower correspondence level than average in the network.

Proposition 2 *In the typical network influential nodes have significantly lower correspondence level than average in the network.*

On the other hand, marginal nodes have the same property, which means that nodes with low correspondence level are not necessarily influential ones, this requires additional verification. This will be illustrated in the next section.

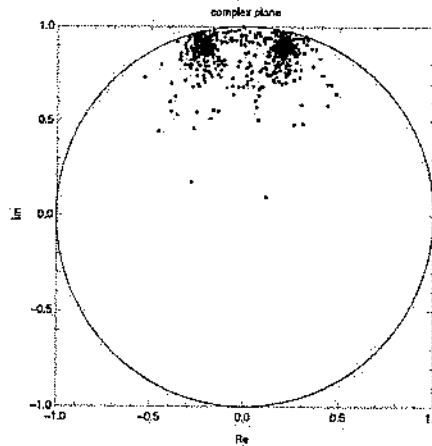


Figure 2: Points in the unit disc obtained by applying our algorithm to the random graph depicted in Fig. 1. Each point corresponds to 4 nodes. Two clusters of points corresponding to two communities are clearly visible.

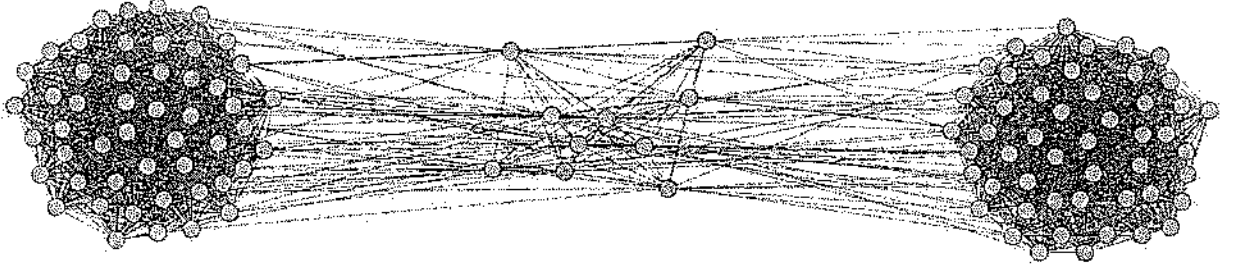


Figure 3: Random graph with three communities. The central community is smaller and mediates between the remaining two.

4 Examples

For the sake of brevity in this section we only consider two illustrative examples of the random networks. These examples are chosen in such way to demonstrate use of our method to two classical problems in study of complex networks: community detection and identification of influential (important, vital) nodes in the network.

As the first example, consider the random network consisting of two communities: each community is Erdős-Rényi graph where each pair of nodes is connected with the probability 0.9, while the nodes belonging to different communities are connected with the probability of only 0.1, see Figure 1.

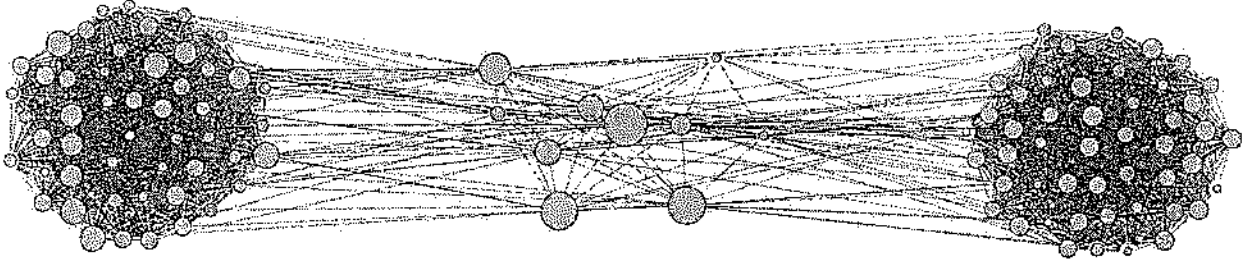


Figure 4: Nodes in the network with three communities are represented by circles. The area of each circle is inverse proportional to the correspondence level of the node. It is visible that nodes from the central community have smaller correspondence levels.

In the Figure 2 we depict the points that correspond to Möbius transformations that are found using the steps 1-5 algorithm explained in the previous section. The presence of two communities is clearly visible. Notice that each point corresponds to the quadruple of nodes. For the remaining steps 6 and 7 it suffices to use one of known algorithms.

We also consider one more example to illustrate our method of identification of influential nodes. Consider the network consisting of three communities, see Figure 3. Inside each community nodes are densely connected with the probability 0.9. However, nodes belonging to communities *A* and *C* are directly connected only with the nodes from the middle community *B* with the probability 0.1. In addition, communities *A* and *C* contain 250 nodes each and are significantly larger than *B*, which contains 50 nodes only. Clearly, one might say that nodes belonging to community *B* are influential in the network as they are essentially small group of *mediators* in the network.

Table 1: Correspondence levels of randomly chosen nodes in the network consisting of three communities. It is clear that nodes belonging to central community (community *B*) have smaller correspondence levels.

	1	2	3	4	5	6	7	8	9	10	Mean
A	1.4825	1.2707	0.9177	1.4119	1.1295	0.7765	1.1295	1.4119	1.3413	1.0589	1.193
B	1.5039	0.5013	0.2507	0.7519	0.2507	1.2533	1.0027	1.5039	0	0.7519	0.777
C	1.1295	1.0589	1.553	0.9883	1.0589	1.2001	1.3413	1.1295	1.2001	1.411	1.2071

In Table 1 we list correspondence levels of ten randomly chosen oscillators from each community. It is clear that nodes from B have significantly lower correspondence level. This is depicted in Figure 4, where the nodes are represented by the circles. The area of the circles is inverse proportional to the correspondence levels of corresponding nodes. In other words, nodes with lower correspondence level are represented by larger circles.

5 Outlook

We have presented the method of investigation of complex networks based on detecting collective dynamics. Our exposition is based on paradigmatic model of coupled oscillators (Kuramoto oscillators), however it could be reinterpreted in terms of magnetic fields and spin dynamics as well. In whole, our method relies on objects and ideas of Statistical Mechanics, but the approach presented here differs essentially from the previous ones (see Introduction). The base for our method is the result of [Marvel et al., 2009] that explains dynamics of coupled oscillators in algebraic and geometric terms.

Furthermore, our method is essentially statistical and works well for large networks, consisting at least of several hundreds nodes. For the real-life data it can be used in combination with other methods.

One advantage of this method is that it is applicable to different kinds of networks. For instance, one might use it to study the network with noisy or delayed interactions.

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Slušanje		Čitanje		Govorna interakcija		Govorna produkcija			
B2	Samostalni korisnik	B2	Samostalni korisnik	B2	Samostalni korisnik	B2	Samostalni korisnik	B2	Samostalni korisnik
A2	Temeljni korisnik	A2	Temeljni korisnik	A2	Temeljni korisnik	A2	Temeljni korisnik	A2	Temeljni korisnik

(*) Zajednički evropski referentni okvir za jezike

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Programske jezike: Wolfram Mathematica, Latex, C++, Matlab, R.

Vozačka dozvola B

Dodaci Popis objavljenih naučnih radova:

1. Vladimir Jaćimović & Aladin Crnkčić, "Characterizing complex networks through statistics of Möbius transformations", Physica D: Nonlinear Phenomena 345, 56–61 (2017).
2. Vladimir Jaćimović & Aladin Crnkčić, "Collective motions of globally coupled oscillators and some probability distributions on circle", Physics Letters A 381, 1989–1994 (2017).
3. Aladin Crnkčić & Vladimir Jaćimović, "Exploring complex networks by detecting collective dynamics of Kuramoto oscillators, VIII International Conference on Optimization and Applications (OPTIMA-2017), Péterovác, 2017.

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- Vladimir Jaćimović and Aladin Crnkić: *Low-dimensional behavior in non-Abelian Kuramoto model on the 3-sphere*, Chaos: An Interdisciplinary Journal of Nonlinear Science, vol. 28 (8), 083105 (2018).
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- Vladimir Jaćimović and Aladin Crnkić: *Measuring mean fields and their coherence in networks of coupled oscillators*, Journal of Geometry and Physics, vol. 124, pp. 241-248, (2018).
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- Aladin Crnković and Vladimir Jaćimović: *Data clustering based on quantum synchronization*, Natural Computing, <https://doi.org/10.1007/s11047-018-9720-z> (2018).
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- Aram Arutyunov and Vladimir Jaćimović: *2-normal Processes in Controlled Dynamical Systems*, Differential Equations (Russian), vol. 38 (8), pp. 1081-1094 (2002).

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- Aladin Crnkčić and Vladimir Jaćimović: *Exploring complex networks by detecting collective dynamics of Kuramoto oscillators*, Proceedings of the VIII International Conference on Optimization and Applications, Petrovac, (2017).
- Vladimir Jaćimović: *On Geometry of extended Hopf bifurcation*, in: Differential and Difference Equations with Applications, Springer Proceedings in Mathematics and Statistics, vol. 47 (2013).
- Aram Arutyunov, Vladimir Jaćimović and Fernando Pereira: *Second Order Necessary Conditions of Optimality for Impulsive Control Systems*, Decision and Control, Proceedings of the 41th IEEE Conference, pp. 1576-1581 (2002).

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Jezici: engleski, ruski (tečno), italijanski (osnovni).



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Rođen sam 12 decembra 1966. godine u Podgorici, Crna Gora, gdje sam završio osnovnu školu i gimnaziju. Nakon odsluženja vojnog roka u JNA upisao sam 1986. godine studije fizike na Prirodno-matematičkom fakultetu Univerziteta Crne Gore. Studije sam završio u redovnom roku oktobra 1990. godine sa prosječnom ocjenom 9,63. Proglašen sam za najboljeg diplomiranog studenta Univerziteta Crne Gore za školsku 1989/90 i dobio Plaketu Univerziteta Crne Gore.

Odmah nakon završetka studija sam angažovan kao asistent-pripravnik na Univerzitetu Crne Gore. Upisao sam postdiplomske magistarske studije na Fizičkom fakultetu u Beogradu, i paralelno izvodio vježbe na Odsjeku za fiziku u Podgorici. Magistarski rad pod nazivom "NMR linije u visokotemperaturnim superprovodnicima" sam odbranio u oktobru 1993. godine. Kako bih se potpuno posvetio izradi doktorske disertacije u periodu 1993-1995 godine sam bio radno angažovan na Institutu za Fiziku u Zemunu. Doktorsku disertaciju "Magneto-elastični efekti u anizotropnim superprovodnicima" pod mentorstvom Ljiljane Dobrosavljević-Grujić sam odbranio u martu 1996. godine na Fizičkom fakultetu u Beogradu. Rezultati iz doktorske disertacije su objavljeni u dva rada u časopisu Američkog fizičkog društva, Physical Review B. Na osnovu toga sam nagrađen od strane Crnogorske akademije nauka i umjetnosti nagradom iz fonda Petra Vukčevića za uspjeh u istraživačkom radu u periodu 1993-1995. Nakon odbrane doktorske disertacije, u junu 1996. godine sam izabran u zvanje docent za predmet *Kvantna Mehanika* na Univerzitetu Crne Gore.

Tokom ljeta 1996. godine sam započeo saradnju sa Vladimirom G. Koganom na problemu raspodjele magnetnog polja unutar borokarbidnih superprovodnika, a rad sam dovršio za vrijeme studijskih boravaka u Ames National Laboratory, Iowa, USA, tokom decembra 1996. i aprila 1997. godine. Dva rada, koja su proizašla tokom boravaka u Ajovi, su naišli na veoma dobar prijem kod međunarodne naučne javnosti i citirani su preko 100 puta. Na poziv profesora Kenza Mije u periodu jun 1997 – jun 1998 godine sam boravio na Univerzitetu u Tokiju, u zvanju *lecturer*. Nakon povratka iz Japana, boravio sam dva mjeseca na Oak Ridge National Laboratory, Tennessee, USA kod profesora D. K. Christena. Tamo sam dovršio rad na temu „Flux lattice symmetry in V3Si: Nonlocal effects in a high-kappa superconductor” koji je po ocjeni Editora zavrijedio da se predstavi u Physical Review Focus (izdanje od 24. juna 1999. godine). Nakon povratka iz USA radio sam na Institutu za Fiziku u Zemunu u zvanju naučni saradnik. Na Univerzitet Crne Gore sam se vratio u ljeto 1999. godine. U zvanje vanredni profesor za predmet Kvantna fizika sam izabran 2001. godine. Na konkursu Japanskog društva za promociju nauke sam dobio postdoktorsku stipendiju koja mi je omogućila boravak na Univerzitetu u Okajami kod profesora Kazušige Mačide u periodu 2001-2003. Za vrijeme specijalizacije sam uspio da razvijem posebnu metodu za numeričko rješavanje mikroskopskih jednačina superprovodnosti koja je omogućila mnogo lakšu analizu velikog broja problema koji se tiču termodinamičkih i transportnih osobina superprovodnika. Plodnu saradnju sa Univerzitetom u Okajami sam nastavio i nakon povratka na Univerzitet Crne Gore, tako da svake godine boravim mjesec dana na pomenutom Univerzitetu.

Nakon povratka iz Japana sam se, pored nauke i nastave, posvetio i drugim aktivnostima. U periodu 2004-2006 godine sam bio predsjednik Društva fizičara Srbije i Crne Gore. U oktobru 2006. godine sam izabran u zvanje redovni profesor. Vijeće Prirodno-matematičkog fakulteta me je izabralo za člana Nastavno-naučnog vijeća, odnosno Senata Univerziteta 2003 godine. Član sam Nacionalnog savjeta za naučno-istraživačku djelatnost. Od marta 2007. godine obavljam funkciju prorektora za nastavu na Univerzitetu Crne Gore.

Školovanje:

BSc Fizika, Univerzitet Crne Gore, Prirodno-matematički fakultet

1990-1993 MSc Fizika, Univerzitet u Beogradu, Fizički fakultet

1993-1996 PhD Fizika, Univerzitet u Beogradu, Fizički fakultet

Doktorska disertacija Magneto-elastični efekti u anizotropnim superprovodnicima

Mentor: Ljiljana-Dobrosavljević-Grujić

Profesionalne pozicije:

1996-2001 Docent na Univerzitetu Crne Gore

1997-1998 Lecturer na Univerzitetu u Tokiju, Japan

1997-1999 Naučni saradnik, Institut za Fiziku, Zemun, Srbija

2001-2003 Stipendista Japanskog društva za promociju nauke na Univerzitetu u Okayami, Japan

2006- Redovni profesor na Univerzitetu Crne Gore

2004-2006 Predsjednik Društva fizičara Srbije i Crne Gore

2003- 2014 Član Senata Univerziteta Crne Gore

2007- 2008 Prorektor za nastavu Univerziteta Crne Gore

2007- 2010 Član Nacionalnog savjeta za naučno-istraživačku djelatnost.

2008-2014 Rektor Univerziteta Crne Gore

2008- Vanredni član CANU

Nagrade i stipendije:

Plaketa Univerziteta Crne Gore kao najbolji diplomirani student Univerziteta za školsku 1989/90.

Nagrada CANU iz fonda Petra Vukčevića za uspjeh u istraživačkom radu u periodu 1993-1995.

Postdoktorska stipendija stipendija Japanskog društva za promociju nauke za period 2001-2003

Studijski boravci:

1996 (Decembar) Ames National Laboratory, Ames, Iowa, USA (prof. V. G. Kogan)

1997 (April) Ames National Laboratory, Ames, Iowa, USA (prof. V. G. Kogan)

1997-1998 Faculty of engineering, University of Tokyo, Japan (prof. K. Miya)

1998 (Avgust) Oak Ridge National Laboratory, Tennessee, USA (prof. D. K. Christen)

2001-2003 Department of Physics, Okayama University, Japan, (prof. K. Machida)

2004 (Januar) Department of Physics, Okayama University, Japan, (prof. K. Machida)

2006 (Januar) Department of Physics, Okayama University, Japan, (prof. K. Machida)

2007 (Januar) Department of Physics, Okayama University, Japan, (prof. K. Machida)

2008 (Januar) Department of Physics, Okayama University, Japan, (prof. K. Machida)

Publikacije:

Koautor sam u 38 radova objavljenih u renomiranim naučnim časopisima (koji su na SCI listi) kao što su časopisi Američkog fizičkog društva (Physical Review Letters, Physical Review B) i Japanskog fizičkog društva (Journal of Physical Society of Japan).

1. Title: Field-angle-dependent specific heat measurements and gap determination of a heavy fermion superconductor URu₂Si₂
 Author(s): Yano, K.; Sakakibara, T.; Tayama, T.; Yokoyama, M.; Amitsuka, H.; Homma, Y.; Miranović, P.; Ichioka, M.; Tsutsumi, Y.; Machida, K.
 Source: PHYSICAL REVIEW LETTERS Volume: 1 Issue: 1 Article Number: 017004 Published: 2008
2. Title: Low energy excitations in the mixed state of the anisotropic s-wave superconductor CeRu₂
 Author(s): Yamada, Atsushi; Sakakibara, Toshiro; Custers, Jeroen; Hedo, Masato; Ōnuki, Yoshichika; Miranović, Predrag; Machida, Kazushige
 Source: JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN Volume: 76 Issue: 12 Article Number: 123704 Published: 2007
3. Title: Electronic thermal conductivity in a superconducting vortex state
 Author(s): Adachi, H.; Miranovic, P.; Ichioka, M.; Machida, K.
 Source: PHYSICA C-SUPERCONDUCTIVITY AND ITS APPLICATIONS Volume: 463 Pages: 36-39 Published: 2007
4. Title: Quasiclassical calculation of the quasiparticle thermal conductivity in a mixed state
 Author(s): Adachi, Hiroto; Miranović, Predrag; Ichioka, Masanori; Machida, Kazushige
 Source: JOURNAL OF MAGNETISM AND MAGNETIC MATERIALS Volume: 310 Issue: 2 Pages: 640-642 Part: Part 1 Published: 2007
5. Title: Quasi-classical calculation of the mixed-state thermal conductivity in s- and d-wave superconductors
 Author(s): Adachi, Hiroto; Miranović, Predrag; Ichioka, Masanori; Machida, Kazushige Source: JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN Volume: 76 Issue: 6 Article Number: 064708 Published: 2007
6. Title: Effect of field-dependent core size on reversible magnetization of high-kappa superconductors
 Author(s): Kogan, V. G.; Prozorov, R.; Bud'Ko, S. L.; Canfield, P. C.; Thompson, J. R.; Karpinski, J.; Zhigadlo, N. D.; Miranović, P.
 Source: PHYSICAL REVIEW B Volume: 74 Issue: 18 Article Number: 184521 Published: 2006
7. Title: Ubiquitous V-shape density of states in a mixed state of clean limit type II superconductors
 Author(s): Nakai, N.; Miranović, P.; Ichioka, M.; Hess, H. F.; Uchiyama, K.; Nishimori, H.; Kaneko, S.; Nishida, N.; Machida, K.
 Source: PHYSICAL REVIEW LETTERS Volume: 97 Issue: 14 Article Number: 147001 Published: 2006
8. Title: Basal-plane magnetic anisotropies of high-k d-wave superconductors in a mixed state: A quasiclassical approach
 Author(s): Adachi, Hiroto; Miranović, Predrag; Ichioka, Masanori; Machida, Kazushige
 Source: JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN Volume: 75 Issue: 8 Article Number: 084716 Published: 2006
9. Title: Specific heat and low-lying excitations in the mixed state for a type-II superconductor
 Author(s): Nakai, N.; Miranović, P.; Ichioka, M.; Machida, K.
 Source: PHYSICAL REVIEW B Volume: 73 Issue: 17 Article Number: 172501 Published: 2006
10. Title: Theory of gap-node detection by angle-resolved specific heat measurement
 Author(s): Miranovic, P.; Ichioka, M.; Machida, K.; Nakai, N.

Source: JOURNAL OF PHYSICS-CONDENSED MATTER Volume: 17 Issue: 50 Pages: 7971-7980
Published: 2005

11. Title: Theoretical study on the field dependence of the zero energy density of states in an anisotropic gap superconductors

Author(s): Nakai, N.; Miranović, P.; Ichioka, M.; Machida, K.

Source: JOURNAL OF PHYSICS AND CHEMISTRY OF SOLIDS Volume: 66 Issue: 8-9 Pages: 1362-1364 Published: 2005

12. Title: Field-angle-dependent specific heat in the unconventional heavy-fermion superconductor CeCoIn₅

Author(s): Aoki, H.; Sakakibara, T.; Shishido, H.; Settai, R.; Ōnuki, Y.; Miranović, P.; Machida, K.

Source: PHYSICA B-CONDENSED MATTER Volume: 359 Pages: 410-412 Published: 2005

13. Title: Anisotropic diamagnetic response in type-II superconductors with gap and Fermi-surface anisotropies

Author(s): Adachi, H.; Miranović, P.; Ichioka, M.; Machida, K.

Source: PHYSICAL REVIEW LETTERS Volume: 94 Issue: 6 Article Number: 067007 Published: 2005

14. Title: Electronic state around vortex in a two-band superconductor

Author(s): Ichioka, Masanori; Machida, Kazushige; Nakai, Noriyuki; Miranović, Predrag

Source: PHYSICAL REVIEW B Volume: 70 Issue: 14 Article Number: 144508 Published: 2004

15. Title: Field dependence of the zero-energy density of states around vortices in an anisotropic-gap superconductor

Author(s): Nakai, N.; Miranović, P.; Ichioka, M.; Machida, K.

Source: PHYSICAL REVIEW B Volume: 70 Issue: 10 Article Number: 100503 Published: 2004

16. Title: Effects of nonmagnetic scatterers on the local density of states around a vortex in s-wave superconductors

Author(s): Miranović, P.; Ichioka, M.; Machida, K.

Source: PHYSICAL REVIEW B Volume: 70 Issue: 10 Article Number: 104510 Published: 2004

17. Title: Field-angle dependence of the zero-energy density of states in the unconventional heavy-fermion superconductor CeCoIn₅

Author(s): Aoki, H.; Sakakibara, T.; Shishido, H.; Settai, R.; Onuki, Y.; Miranović, P.; Machida, K.

Source: JOURNAL OF PHYSICS-CONDENSED MATTER Volume: 16 Issue: 3 Pages: L13-L19
Published: 2004

18. Title: Low temperature specific heat in anisotropic superconductors

Author(s): Dobrosavljević-Grujić, L.; Miranović, P.

Source: PHYSICA C-SUPERCONDUCTIVITY AND ITS APPLICATIONS Volume: 397 Issue: 3-4
Pages: 117-122 Published: 2003

19. Title: Orientational field dependence of low-lying excitations in the mixed state of unconventional superconductors

Author(s): Miranović, P.; Nakai, N.; Ichioka, M.; Machida, K.

Source: PHYSICAL REVIEW B Volume: 68 Issue: 5 Article Number: 052501 Published: 2003

20. Title: Microscopic study of low-kappa type-II superconductors

Author(s): Miranović, P.; Nakai, N.; Ichioka, M.; Machida, K.

Source: PHYSICA B-CONDENSED MATTER Volume: 329 Pages: 1382-1383 Part: Part 2 Published: 2003

21. Title: Theoretical study on vortex lattices in tetragonal superconductors

Author(s): Nakai, Noriyuki; Miranović, Predrag; Ichioka, Masanori; Machida, Kazushige

Source: PHYSICA C-SUPERCONDUCTIVITY AND ITS APPLICATIONS Volume: 388 Pages: 677-678 Published: 2003

22. Title: Thermodynamics and magnetic field profiles in low-kappa type-II superconductors

Author(s): Miranović, P.; Machida, K.

Source: PHYSICAL REVIEW B Volume: 67 Issue: 9 Article Number: 092506 Published: 2003

23. Title: Theoretical studies on vortices in unconventional and conventional superconductors

Author(s): Machida K, Ichioka M, Miranovic P, et al.

Source: ACTA PHYSICA POLONICA B Volume: 34 Issue: 2 Pages: 545-548 Published: 2003

24. Title: Anisotropy of the upper critical field in superconductors with anisotropic gaps: Anisotropy parameters of MgB₂

Author(s): Miranović, Predrag; Machida, Kazushige; Kogan, Vladimir G.

Source: JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN Volume: 72 Issue: 2 Pages: 221-224 Published: 2003

25. Title: Anisotropy of the superconducting state properties and phase diagram of MgB₂ by torque magnetometry on single crystals

Author(s): Angst, M.; Puzniak, R.; Wisniewski, A.; Roos, J.; Keller, H.; Miranović, P.; Jun, J.; Kazakov, S. M.; Karpinski, J.

Source: PHYSICA C-SUPERCONDUCTIVITY AND ITS APPLICATIONS Volume: 385 Issue: 1-2 Pages: 143-153 Published: 2003

26. Title: Reentrant vortex lattice transformation in fourfold symmetric superconductors

Author(s): Nakai, N.; Miranović, P.; Ichioka, M.; Machida, K.

Source: PHYSICAL REVIEW LETTERS Volume: 89 Issue: 23 Article Number: 237004 Published: 2002

27. Title: Elastic moduli of vortex lattices within nonlocal London model

Author(s): Miranović, P.; Kogan, V. G.

Source: PHYSICAL REVIEW LETTERS Volume: 87 Issue: 13 Article Number: 137002 Published: 2001

28. Title: Nonlocal effects in angular dependence of in-plane magnetization of tetragonal superconductors

Author(s): Kogan, V. G.; Bud'ko, S. L.; Canfield, P. C.; Miranović, P.

Source: PHYSICAL REVIEW B Volume: 60 Issue: 18 Pages: R12577-R12580 Published: 1999

29. Title: Flux lattice symmetry in V₃Si: Nonlocal effects in a high-kappa superconductor

Author(s): Yethiraj, M.; Christen, D. K.; Paul, D. Mck.; Miranovic, P.; Thompson, J. R.

Source: PHYSICAL REVIEW LETTERS Volume: 82 Issue: 25 Pages: 5112-5115 Published: 1999

30. Title: Irreversibility field analysis for Bi₂Sr₂Ca₂Cu₃O_x tapes by using axial probe

Author(s): Rábara, M.; Yoshida, Y.; Takeuchi, T.; Miranović, P.; Miya, K.

Source: PHYSICA C-SUPERCONDUCTIVITY AND ITS APPLICATIONS Volume: 305 Issue: 3-4 Pages: 285-292 Published: 1998

31. Title: Resistivity and magnetic susceptibility of single-crystal $\text{Lu}(\text{Ni}_{1-x}\text{Co}_x)(2)\text{B}_2\text{C}$ ($x=0.0-0.09$)
Author(s): Cheon, K. O.; Fisher, I. R.; Kogan, V. G.; Canfield, P. C.; Miranović, P.; Gammel, P. L.
Source: PHYSICAL REVIEW B Volume: 58 Issue: 10 Pages: 6463-6467 Published: 1998
32. Title: Vortex lattices in cubic superconductors
Author(s): Kogan, V. G.; Miranović, P.; Dobrosavljević-Grujić, Lj.; Pickett, W. E.; Christen, D. K.
Source: PHYSICAL REVIEW LETTERS Volume: 79 Issue: 4 Pages: 741-744 Published: 1997
33. Title: Vortex lattice transitions in borocarbides
Author(s): Kogan, V. G.; Bullock, M.; Harmon, B.; Miranović, P.; Dobrosavljević; Grujić, Lj.; Gammel, P. L.; Bishop, D. J.
Source: PHYSICAL REVIEW B Volume: 55 Issue: 14 Pages: R8693-R8696 Published: 1997
34. Title: GINZBURG-LANDAU THEORY OF VORTEX LATTICE STRUCTURE IN DEFORMABLE ANISOTROPIC SUPERCONDUCTORS
Author(s): Miranović, P.; Dobrosavljević-Grujić, Lj.; Kogan, V. G.
Source: PHYSICAL REVIEW B Volume: 52 Issue: 17 Pages: 12852-12857 Published: 1995
35. Title: ON THE STRAIN-INDUCED VORTEX MASS IN ANISOTROPIC SUPERCONDUCTORS
Author(s): Miranović, P.; Dobrosavljević-Grujić, Lj.
Source: PHYSICS LETTERS A Volume: 207 Issue: 3-4 Pages: 225-229 Published: 1995
36. Title: COMMENT ON THE TRANSCENDENTAL METHOD IN THE THEORY OF NEUTRON SLOWING-DOWN
Author(s): Miranovic, P.
Source: JOURNAL OF PHYSICS A-MATHEMATICAL AND GENERAL Volume: 28 Issue: 14 Pages: 4189-4190 Published: 1995
37. Title: VORTEX-INDUCED STRAIN AND FLUX LATTICES IN ANISOTROPIC SUPERCONDUCTORS
Author(s): Kogan, V. G.; Bulaevskii, L. N.; Miranović, P.; Dobrosavljević-Grujić, L.
Source: PHYSICAL REVIEW B Volume: 51 Issue: 21 Pages: 15344-15350 Published: 1995

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Датум, 25.10.2012 г.

Ref: _____
Date, _____

УНИВЕРЗИТЕТ ЦРНЕ ГОРЕ
Природно-математички факултет
Број 2556
Подгорица, 01.11.2012 год.

Na osnovu člana 75 stav 2 Zakona o visokom obrazovanju (Sl.list RCG, br. 60/03 i Sl.list CG, br. 45/10 i 47/11) i člana 18 stav 1 tačka 3 Statuta Univerziteta Crne Gore, Senat Univerziteta Crne Gore, na sjednici održanoj 25.10.2012. godine, donio je

ODLUKU O IZBORU U ZVANJE

Dr DAVID KALAJ bira se u akademsko zvanje redovni profesor Univerziteta Crne Gore za predmete: Kompleksna analiza 2 (studijski program Matematika), Analiza 3 (studijski program Računarske nauke) i Analiza 3 (studijski program Fizika) na Prirodno-matematičkom fakultetu.



REKTOR

Prof. dr Predrag Miranović

Biografija dr. Davida Kalaj

Lični podaci

Ime i prezime: David Kalaj

Državljanstvo: Crna Gora

Datum i mjesto rođenja: 11. 12. 1971, Podgorica

Institucija

Univerzitet Crne Gore,

- Prirodno-matematički fakultet, UCG

Zvanje: Redovni profesor

- Samostalni studijski program za obrazovanje učitelja na albanskom jeziku, UCG

Funkcija: rukovodilac

A) OBRAZOVANJE

1991: Maturirao Srednju školu „25. Maj“, Tuzi, Titograd, sa odličnim uspjehom.

1995: Diplomirao na Prirodno-Matematičkom Fakultetu, Univerziteta Crne Gore sa prosječnom ocjenom 9.52. (Za postignuti uspjeh tokom školovanja je 1995. godine dobio Studentsku nagradu 19. decembar (Nagrada opštine Podgorice))

1995: Magistrirao na Matematičkom fakultetu, Univerziteta u Beogradu, smjer matematička analiza, sa temom *Harmonijske funkcije i kvazikonformna preslikavanja*, pošto je položio sve predviđene ispite sa ocjenom 10.

2002: Odbranio doktorsku disertaciju pod nazivom *Harmonijske funkcije i kvazikonformne harmonijske funkcije između konveksnih domena* na Matematičkom fakultetu, Univerziteta u Beogradu.

B) NASTAVNA ISKUSTVA

1995 – 1997 Saradnik u nastavi, Prirodno-Matematičkog fakulteta UCG

1998 – 2002 Asistent, Prirodno-Matematičkog fakulteta UCG

2002 – 2007 Docent, Prirodno-Matematičkog fakulteta UCG

Kompleksna Analiza, Matematička Analiza 3 (PMF), Matematike 1,2,3,4
(Studijski programi za obrazovanje učitelja na albanskom jeziku)

2007- 2012 Vanredni profesor Prirodno-Matematičkog fakulteta UCG

Kompleksna Analiza, Matematička Analiza 3 (PMF),

Matematike 1,2,3,4 (Studijski programi za obrazovanje učitelja na albanskom jeziku)

Realna i kompleksna analiza (kurs na posdiplomskim studijama PMF-a)

Viša analiza (kurs na doktorskim studijama PMF i Matematičkog fakulteta u

Beogradu)

2012 - Redovni profesor Prirodno-Matematičkog fakulteta UCG

- Kompleksna Analiza II, Matematička Analiza 3, Matematička analiza - (PMF),

- Matematike 1,2,3,4 (Studijski programi za obrazovanje učitelja na albanskom jeziku)

- Realna i kompleksna analiza (kurs na posdiplomskim studijama PMF)
- Viša analiza (kurs na doktorskim studijama PMF)
- Harmonijske funkcije, doktorski kurs, Prirodno-Matematički fakultet, Sarajevo (2014).

Mentorstva na doktorskim disertacijama

2013. Marijan Marković (Beogradski univerzitet).

2014. Djordjije Vujadinović (Beogradski univerzitet)

Mentorska na magistarskim tezama

2010. Djordjije Vujadinović (UCG)

C) NAUČNO-ISTRAŽIVAČKI INTERES

Geometrijska teorija funkcija: Harmonijske funkcije, Kvazikonformna preslikavanja, Holomorne funkcije, Funkcionalni prostori: Hardijevi i Bergmanovi prostori, Parcijalne diferencijalne jednačine: Poissonova, Laplaceova, Eliptičke PDE, Diferencijalna geometrija: Harmonijske površi, Minimalne površi, Izoperimetrija nejednakost itd.

• Upravljanje projektima

1. Rukovodilac nacionalnog projekta **Analiza na mnogostrukosti i primjene (2011-2015)**, koga finansira Ministarstvo nauke Republike Crne Gore. Projektni tim čine renomirani matematičari iz Crne Gore. Pri tome je projekat pri evaluaciji osvojio maksimalan broj poena od strane međunarodnih eksperata. (Nagrada Ministarstva nauke za najbolji naučni projekat za 2013 godinu)

3. Trenutno je rukovodilac dva bilateralna projekta jednog sa Kinom i drugog sa Hrvatskom.

2. Bio je Rukovodilac uspješnog nacionalnog projekta **Analiza na mnogostrukosti (2008-2011)**.

• Izvod iz bibliografije

Publikovani (ukupno 65 radova), između ostalog, u sledećim vrhunskim matematičkim časopisima: *Advances in Mathematics*, *Transactions of American Mathematical Society*, *Calculus of Variations and PDE*, *International mathematics research notices*, *Proceedings of American Mathematical Society*, *Journal D'Analyse Math.*, *Israel Journal of Math*, *Mathematische Zeitschrift*, *Annali della Scuola Normale Superiore di Pisa- Classe di Scienze*, *Annales Academiæ Scientiarum Fennicæ Mathematica*, *Annali di Matematica Pura ed Applicata*, *Pacific Journal of Mathematics*.
U pripremi ima još 6 radova koji se nalaze na arxiv.org serveru.

Ukupan broj radova publikovanih na žurnalima koji pripadaju SCI listi je 60. Svoje radove je izložio na više od 25 naučnih konferencija i seminara i to u sljedeće države SAD, Rusija, Japan, Kina, Južna Koreja, Njemačka, Francuska, Finska, Rumunija, Srbija, Norveška, Češka, Poljska itd. Njegovi koautori su između ostalog: Noam Elkies, Eero Saksman, Matti Vuorinen, Miroslav Pavlović, Miodrag Mateljević koji su dali svojevrstan pečat modernoj matematici. Kao dokaz ove teze je činjenica da je Noam Elkies svojevremeno postao najmladji profesor u historiji Harvard univerziteta (http://en.wikipedia.org/wiki/Noam_Elkies), dok je Eero Saksman urednik Acta Mathematica), koji je najprestižniji svjetski matematički časopis. (<http://www.springer.com/mathematics/journal/11511?detailsPage=editorialBoard> Kalajevi radovi su citirani više od 550 puta (www.google.com)). (Spisak radova i konferencija su dati u prilogu).

D) UDŽBENICI

1. D. Kalaj: Zbirka zadataka iz kompleksne analize, Univerzitet Crne Gore, 2006, 219 str.
2. M. Jačinović, D. Kalaj: Uvod u kompleksnu analizu, Univerzitet Crne Gore, 2009, 347 str.

Prevodi i adaptacija školskih udžbenika

Prevod i adaptacija udžbenika iz matematike za ukupno 8 razreda za osnovnu i srednju školu sa srpskog (crnogorskog) na albanski jezik u izdanju izdavačke kuće "Zavod za udžbenike i nastavna sredstva" u periodu 2008-2010 i 2014.

E) UREDNIŠTVA

Urednik je sljedećih matematičkih časopisa:

1. World scientific journal
(<http://www.hindawi.com/journals/tswj/editors/mathematical.analysis/>)
2. Abstract and applied mathematics,
<http://www.hindawi.com/journals/aaa/psi/>
3. Bulletin of mathematical analysis and application,
<http://91.187.98.171/bmathaa/>
4. Albanian journal of mathematics

F) RECENZIJE I EKSPERTIZE:

Recenzija radova za renomirane časopise:

Transaction of AMS, Indiana journal of mathematics, Proceeding of AMS, Annales Academiæ Scientiarum Fennicæ Mathematica, Applied Mathematics Letters, Abstract and Applied Analysis, Applied Mathematics and Computation, Complex variables and Elliptic equations, Filomat, Publications d'Institut mathématique, Bulletin of the Malaysian Mathematical Sciences Society, Journal of mathematical analysis and application, Bulletin of mathematical analysis and application, Publicationes Mathematicae Debrecen, World scientific journal, Acta Mathematica

Sinica, Turkish journal of math, Mathematica slovaca, Bulletin of London math society,
Journal of Indian Academy of Mathematics etc.

Međunarodne ekspertize za projekte:

- - Evaluator za projekte iz oblasti matematika koje je raspisalo Israel science fondation države Israel na period 2008-2012.
- - Evaluator za projekte iz oblasti matematika koje je raspisalo Israel science fondation države Israel na period 2012-2015.
- - Evaluator za projekte iz oblasti matematika koje je raspisalo Israel science fondation države Israel na period 2015-2018.
- - Evaluator za projekte iz oblasti matematika koje je raspisalo Ministarstvo prosvjete i nauke Republike Srbije na period 2011-2014.
- - Evaluator za projekte FONDECYT, Chile 2014.

DAVID KALAJ – CURRICULUM VITAE

Mart 2015

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EDUCATION

- March 2002 University of Belgrade, Faculty of Mathematics
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- 1995 – 1998 University of Belgrade, Faculty of Mathematics
M.Sc. Program. Thesis title: "Harmonic diffeomorphisms and
quasiconformal mappings"
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- 1991 - 1995 University of Montenegro, Faculty of Sciences
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FELLOWSHIPS AND AWARDS

- 1993 Annual fellowship of the Ministry of Education of the Republic of
Montenegro
- 1994 "Decembarska nagrada grada Podgorice" (the Award of the
Municipality of Podgorica for distinctive results achieved as a
student)
- 2012 The award for the best project funded by the Ministry of
science of Montenegro

TEACHING EXPERIENCE

- 1995 – 1997 Teach. assistant, University of Montenegro, Faculty of Sciences
Mathematical Analysis 2, undergraduate course
Differential Calculus, undergraduate course
- 1998 – 2002 Teach. assistant, University of Montenegro, Faculty of Sciences
Mathematical Analysis 2, undergraduate course
Complex Analysis, undergraduate course
- 2002 – 2007 Assist. professor, University of Montenegro, Faculty of Sciences
Complex Analysis, undergraduate course
Mathematical Analysis 3, undergraduate course
- 2007 – 2012 Associate professor, University of Montenegro
Complex Analysis, Mathematics 1, Mathematics 2.

Mathematics 3, Mathematics 4, (Study programme for teachers in Albanian Language) undergraduate course
 Mathematical Analysis 3, undergraduate course
 Real and Complex Analysis, graduate course
 2012 – Full professor, University of Montenegro

Mentorstva na doktorskim disertacijama

2013, Marijan Markovic (Beogradski univerzitet)
 2014, Djordjije Vujadinovic (Beogradski univerzitet)

Mentorska na magistarskim tezama

2010, Djordjije Vujadinovic

DODATNE INFORMACIJE

Born December 11, 1971; Podgorica, Yugoslavia
 Citizenship Montenegrin
 Languages Albanian (native command), Serbian (native command), English (fluent), Russian (passive), Italian (passive).
 Computer skills Latex, C++, Mathematica software

Projects: a) Establishment and management of Study programme for teacher education at Albanian since 2004.

b) PI of the national project Analysis on manifolds (2008-2011).

c) PI of the national project Analysis on manifolds and applications (2012-2015).

RADOVI

1. D. Kalaj, *Univalent harmonic mappings between Jordan domains*, Publ. Inst. Math., Nouv. Ser. 69(83), 108-112 (2001).
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8. D. Kalaj, *On the Nitsche conjecture for harmonic mappings in R^2 and in R^3* , Israel J. Math. 150 (2005) 241-251.
9. D. Kalaj, M. Matešević, *Inner estimate and quasiconformal harmonic maps between smooth domains*, J. Anal. Math. 100 (2006), 117-132.
10. Sh. Najafzadeh, S. R. Kulkarni and D. Kalaj *Application of convolution and Cauchy integral operators on analytic and p-valent functions*, Filomat 20:2 (2006), 115-124.
11. D. Kalaj, *On the solution of PDE $\Delta u = f$ between spherically symmetric domains*, Math. Anal. Appl. Volume 327, Issue 1, Pages 1-11 (2007).
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21. R. Meštrović, D. Kalaj, *A converse of Minkowski's type inequalities*, Journal of inequalities and applications, Volume 2010 (2010), Article ID 461215, 9 pages doi:10.1155/2010/461215.
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24. D. Kalaj, *Harmonic maps between annuli on Riemann surfaces*, Israel J. Math. 182 (2011), 123-147.
25. D. Kalaj, M. Mateljević, *On quasiconformal harmonic surfaces with rectifiable boundary*, Complex Complex Anal. Oper. Theory 5, No. 3, 633-646 (2011).
26. D. Kalaj, M. Mateljević, *On certain nonlinear elliptic PDE and quasiconformal maps between Euclidean surfaces*, Potential analysis, Volume 34, Number 1, 13-22, DOI: 10.1007/s11118-010-9177-x (10 pages).
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36. D. Kalaj, M. Mateljević, *On conformal, harmonic mappings and Dirichlet integral*, Filomat, Volume 25, Number 2, June 2011, 91-108.
37. D. Kalaj, *On boundary correspondence of q.c. harmonic mappings between Jordan domains*, Math. Nachr. 285, No. 2-3, 283-294 (2012).
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60. D. Kalaj, *On quasiconformal harmonic maps between surfaces*, *International Math. Research Notices* 2015, No. 2, 355-380 (2015).
61. D. Kalaj, *On J. C. C. Nitsche's type inequality for hyperbolic space \mathbb{H}^n* , *Potential Anal.* 41, No. 3, 931-943 (2014).
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63. David Kalaj, *Quasiconformal harmonic mappings between domains in \mathbb{C}^n* , *To appear in Pacific journal of mathematics (2015)*.
64. David Kalaj, Djordjije Vujadinović, *Norm of the Bergman projection in \mathbb{H}^n space. To appear in Journal of operator theory (2015)*.

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72. D. Kalaj, *Muckenhoupt weights and Lindelöf theorem for harmonic mappings*, arXiv:1410.5478, Revision sent in *Advances in mathematics*.
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RADOVI NA KONFERENCIJAMA

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KONFERENCIJE I SEMINARI

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77. D. Kalaj: *On Quasiconformal harmonic function of the unit disk onto a convex domain*, Rom-Finn.Seminar, 2001, Braşov, Rumunija
78. D Kalaj: *5 International symposium of mathematical analysis and its applications*, MAA5, Niška Banja, October 2-6, 2002
79. D Kalaj: *On the Nitsche's conjecture for harmonic mappings in \mathbb{R}^2 and in \mathbb{R}^3* 934 TH AMS Meeting, Courant Institute New York, April 12-13, 2003 page 48-49
80. D Kalaj: M Pavlovic: *Boundary correspondence under harmonic quasiconformal mapping of the halfplane*, The book of abstract of XI Congress of Yugoslav Mathematicians, page 32, Petrovac, October 2004.
81. D. Kalaj: *On the first and on the radial derivative of harmonic function defined on the unit ball*, Proceedings of the Workshop devoted to 25 anniversary of the Faculty of Natural Sciences and mathematics, University of Montenegro, September 2005.

82. D. Kalaj: On the univalent solution of PDE $\Delta u = f$ between spheres, *Abstracts of the book of abstracts of Harmonic Analysis and partial Differential Equations*, June 27-July 1, 2005, Keil, Germany.
83. D. Kalaj: Harmonic and quasiconformal maps, *Extremal Problems in Complex and Real Analysis*, Peoples Friendship University of Russia Moscow, Russia May 22-26, 2007. The book of abstracts.
84. D. Kalaj: Quasiconformal harmonic maps, Seminar: Mathematical Colloquium, Beograd 11. 05. 2007, http://www.mi.sanu.ac.yu/colloquiumis/mathcoll_prog/rams/mathcoll.may2007.htm, Predavanje po pozivu
85. D. Kalaj: On the univalent solution of PDE $\Delta u = f$ between spherical annuli: Seminar: Differential Equations in Theory and Applications 06.06. 2007 www.math.ntnu.no/seminar/delta, Predavanje po pozivu
86. D. Kalaj: On quasiconformal harmonic mappings, Congress in memory of Adrien Douady, Paris, France May, 2008, Poster.
87. D. Kalaj: Boundary correspondence under q.c. harmonic mappings between Jordan domains, Mini conference on quasiconformal harmonic mappings, Beograd, Srbija, 2009, septembar, Predavanje po pozivu.
88. D. Kalaj: On quasiconformal mappings and elliptic PDE in the plane, Helsinki seminar on Analysis, October, 2010, predavanje po pozivu.
89. D. Kalaj: On quasiconformal mappings and elliptic PDE in the plane, Turku seminar on Analysis, October, 2010, predavanje po pozivu.
90. D. Kalaj: Deformation of annuli under smallest mean distortion on Riemann surfaces, Workshop on Quasiconformal mappings and Mappings of finite distortion, Prague, September 2011, Predavanje od 30-minuta.
91. D. Kalaj: Deformation of annuli under smallest mean distortion on Riemann surfaces and generalization of J. C. C. Nitsche Conjecture Workshop on Complex Analysis, Belgrade, February 2012, Predavanje po pozivu.
92. D. Kalaj: Deformations of Annuli on Riemann surfaces and the generalization of Nitsche conjecture and Quasiconformal harmonic mappings, Predavanje po pozivu, International Conference on Complex Analysis and Related Topics, Romania, Ploiesti 2012, <http://imar.ro/RoFinSem2012/conf.php>, Romanian finish seminar.
93. D. Kalaj: Deformations of Annuli on Riemann surfaces and the generalization of Nitsche conjecture, The 6th European Congress of Mathematics, 2012. Poster.
94. David Kalaj: Cauchy transform and Poisson equation, Turku analysis seminar, Finska (4.10. 2012) <http://users.utu.fi/ripekl/seminar/index.html>
95. David Kalaj, Energy-minimal diffeomorphisms between doubly connected Riemann surfaces, "Conference on Riemann surfaces and Kleinian groups", held in Osaka University, Japan, from January 12 to January 14, 2013.
96. David Kalaj, Quasiconformal harmonic mappings between surfaces, Conference of Geometric function theory, October 2013, Beograd.
97. David Kalaj, Muckenhoupt weights and Lindelöf theorems for harmonic mappings, March 2014, Helsinki seminar of analysis, Helsinki 2014.
98. David Kalaj, Quasiconformal harmonic mappings between surfaces. March 2014, Helsinki seminar of analysis, Helsinki 2014.
99. David Kalaj, Muckenhoupt weights and Lindelöf theorem for harmonic mappings, ICM August, 2014, Seoul, Korea, Short communication.
100. David Kalaj, Harmonic and quasiconformal mappings and generalizations, School of Mathematical Science Huaqiao University, P.R.China, January 26, 2015.
101. David Kalaj, On quasi-inversions, School of Mathematical Science Huaqiao University, P.R.China, January 27, 2015.
102. David Kalaj, Energy-minimal diffeomorphisms between doubly connected Riemann surfaces, School of Mathematical Science Huaqiao University, P.R.China, January 27, 2015.

103. David Kalaj, Poisson equation and Cauchy transform, School of Mathematical Science Huaqiao University, P.R.China, January 28, 2015.
104. David Kalaj, Quasiconformal harmonic mappings between surfaces, School of Mathematical Science Huaqiao University, P.R.China, January 28, 2015.
105. David Kalaj, "Norm of the Bergman projection", School of Mathematical Science Huaqiao University, P.R.China, January 25, 2015.
106. David Kalaj, "Schwarz lemma for harmonic functions, School of Mathematical Science Huaqiao University, P.R.China, January 27, 2015.
107. David Kalaj, "The J. C. C. Nitsche conjecture and generalization" School of Mathematical Science Huaqiao University, P.R.China, January 25, 2015.

KNJIGE

1. D. Kalaj: Zbirka zadataka iz kompleksne analize, Univerzitet Crne Gore, 2006, 219 str.
2. M. Jaćimović, D. Kalaj: Uvod u kompleksnu analizu, Univerzitet Crne Gore, 2009, 347 str.

PREVODI

Prevod i adaptacija sljedećih udžbenika iz matematike sa srpskog (crnogorskog) na albanski jezik u izdanju izdavačke kuće "Zavod za udžbenike i nastavna sredstva" u periodu 2008-2010.

- 1) Matematika IV (četvrti razred osnovne škole)
- 2) Matematika V (peti razred osnovne škole)
- 3) Matematika VIII (osmi razred osnovne škole)
- 4) Matematika IX (deveti razred osnovne škole)
- 5) Matematika II (drugi razred srednje škole)
- 6) Matematika III (treći razred srednje škole).
- 7) Algoritmi i programiranje (treći i četvrti razred srednje škole).

Citati: 550 citata (<http://scholar.google.com>).

Република Србија
МИНИСТАРСТВО ПРОСВЕТЕ,
НАУКЕ И ТЕХНОЛОШКОГ РАЗВОЈА
Комисија за стицање научних звања

Број: 660-01-00006/460

27.04.2018. године

Београд

ИНСТИТУТ ЗА ФИЗИКУ			
ПРИМЉЕНО: 11-06-2018			
Рад. јед.	Б р о ј	Арх. шифра	Прилог
88/1	896/1		

На основу члана 22. став 2. члана 70. став 5. Закона о научноистраживачкој делатности ("Службени гласник Републике Србије", број 110/05, 50/06 – исправка, 18/10 и 112/15), члана 3. ст. 1. и 3. и члана 40. Правилника о поступку, начину вредновања и квантитативном исказивању научноистраживачких резултата истраживача ("Службени гласник Републике Србије", број 24/16, 21/17 и 38/17) и захтева који је поднео

Институт за физику у Београду

Комисија за стицање научних звања на седници одржаној 26.04.2018. године, донела је

ОДЛУКУ
О СТИЦАЊУ НАУЧНОГ ЗВАЊА

Др Марија Мишировић Данкулов

стиче научно звање

Виши научни сарадник

у области природно-математичких наука - физика

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Институт за физику у Београду

утврдио је предлог број 986/1 од 18.07.2017. године на седници Научног већа Института и поднео захтев Комисији за стицање научних звања број 1003/1 од 20.07.2017. године за доношење одлуке о испуњености услова за стицање научног звања *Виши научни сарадник*.

Комисија за стицање научних звања је по претходно прибављеном позитивном мишљењу Матичног научног одбора за физику на седници одржаној 26.04.2018. године разматрала захтев и утврдила да именована испуњава услове из члана 70. став 5. Закона о научноистраживачкој делатности ("Службени гласник Републике Србије", број 110/05, 50/06 – исправка, 18/10 и 112/15), члана 3. ст. 1. и 3. и члана 40. Правилника о поступку, начину вредновања и квантитативном исказивању научноистраживачких резултата истраживача ("Службени гласник Републике Србије", број 24/16, 21/17 и 38/17) за стицање научног звања *Виши научни сарадник*, па је одлучила као у изреци ове одлуке.

Доношењем ове одлуке именована стиче сва права која јој на основу ње по закону припадају.

Одлуку доставити подносиоцу захтева, именованој и архиви Министарства просвете, науке и технолошког развоја у Београду.

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2000-2005 Fakultet za fiziku, Univerzitet u Beogradu, BSc in Physics (2005)

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S. Fortunato, A. Chatterjee, M. Mitrovic, R. Ku. Pan, P. Della Briotta Parolo, and F. Becattini:
"Growing Time Lag Threatens Nobels", Nature 508, 186 (2014).

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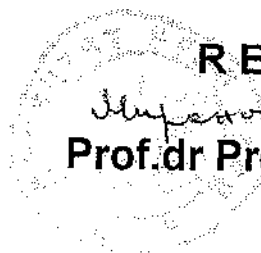
Na osnovu člana 75 stav 2 Zakona o visokom obrazovanju (Sl.list RCG, br. 60/03 i Sl.list CG, br. 45/10) i člana 18 stav 1 tačka 3 Statuta Univerziteta Crne Gore, Senat Univerziteta Crne Gore, na sjednici održanoj 02.06.2011. godine, donio je

ODLUKU O IZBORU U ZVANJE

Dr IGOR ĐUROVIĆ bira se u akademsko zvanje **redovni profesor** Univerziteta Crne Gore za predmete: Programiranje I (osnovne studije, ETF), Programiranje II (osnovne studije, ETR) i Teorija informacija i kodova (osnovne studije, ETR) na **Elektrotehničkom fakultetu**.

UNIVERZITET CRNE GORE
PODGORICA

02/2-767
02.06.2011. g.



REKTOR

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Prof.dr Predrag Miranović

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BIOGRAFIJA

Igor Đurović je rođen 29. 08. 1971. u Cetinju. Osnovnu i srednju školu prirodno-matematičkog smjera završio je u Herceg Novom. Dobitnik je više priznanja na republičkim takmičenjima učenika srednjih škola iz matematike. Diplomirao je na smjeru Elektronika na Elektrotehničkom fakultetu u Podgorici, 1994. godine. Na istom fakultetu je magistrirao („Funkcija jezgra u vremensko-frekvencijskoj analizi i softverski paket za realizaciju distribucija”) i doktorirao („Vremensko-frekvencijske reprezentacije u estimaciji parametara signala sa primjenom u digitalnom watermarking-u”), 1996. i 2000. respektivno. U zvanja docenta, vanrednog profesora i redovnog profesora biran je 2001, 2006. i 2011. godine na Elektrotehničkom fakultetu. Bio je šef Katedre za računare, rukovodilac postdiplomskih studija, rukovodilac doktorskih studija na Fakultetu, član Senata Univerziteta Crne Gore (2011–2014) i član Strukovnog vijeća za prirodne i tehničke nauke Univerziteta. Predavao je na više drugih jedinica Univerziteta Crne Gore a bio je i gostujući nastavnik na Fakultetu za proizvodnju i menadžment, Trebinje, Univerziteta u Istočnom Sarajevu, BiH.

Autor i koautor više univerzitetskih udžbenika kao i više skriptata, praktikuma itd. Bio je mentor na više doktorskih disertacija i magistarskih radova.

Autor je oko 200 radova, od toga preko 100 u vodećim međunarodnim časopisima iz obrade signala i srodnih oblasti. Editor je jedne monografije objavljene u našoj zemlji i autor radova u dvije monografije. Autor je 6 poglavlja u naučnim monografijama izdatim od renomiranih međunarodnih izdavača i koautor jedne knjige objavljene u Njemačkoj. Radovi su vezani za više tema iz obrade signala sa primjenama: estimacija parametara signala; vremensko-frekvencijska analiza sa primjenama; multimedijalni signali i sistemi; obrada signala u telekomunikacijama; primjena u električnim kolima i električnim mjerenjima itd.

Do sada su ovi radovi citirani više od 2200 puta u okviru ISI Web of Knowledge, od kojih samo radovi iz oblasti digitalnog watermarking-a oko 300 puta. Rad dr Igora Đurovića je ostao veoma zapažen pa je recenzent sa više stotina recenzija u više od 40 vodećih međunarodnih časopisa. Pored ovoga, bio je recenzent i više domaćih i regionalnih časopisa. Član editorijalnog odbora ili pridruženi editor bio je u više uglednih međunarodnih časopisa, od kojih se izdvaja jedan od najuglednijih časopisa iz oblasti obrade signala Elsevier „Signal Processing”. Bio je član editorijalnog odbora Journal of Electrical and Computer Engineering, Hindawi kao i časopisa Research Letters in Signal Processing, Hindawi. Bio je vodeći gostujući editor za Eurasip (Evropsko udruženje za obradu signala) Journal on Advances in Signal Processing za specijalni broj „Robust processing of non-stationary signals”. Član je editorijalnih odbora, recenzent i član programskih komiteta nekoliko međunarodnih i regionalnih konferencija. Pored toga, veći broj radova je proglašavan najboljim u sekcijama na domaćim konferencijama. Senior Member IEEE (vodeće svjetsko udruženje inženjera elektrotehnike i elektronike) je od 2006. godine.

Bio je rukovodilac lokalnih timova, partner i učesnik na većem broju nacionalnih, bilateralnih i međunarodnih projekata finansiranih od strane Volkswagen stiftung, FP 7, Tempus, CNRS, JSPS, DoD Canada, WUS Austria, Ministarstva nauke Crne Gore itd. Osmislio je i bio prvi direktor prvog domaćeg Centra izvrsnosti BIO-ICT u periodu 2014–2015.

U periodu od novembra 2001. do novembra 2002. boravio je kao stipendista Japanskog društva za unapređenje nauke (JSPS) na Kyoto Institute of Technology. Bio je na kraćim boravcima na inostranim univerzitetima i to: Univerzitet Aristotel Solun, Artificial Intelligence and Image Analysis Laboratory (Laboratorija za vještačku inteligenciju i analizu slike), Grčka, veći broj univerziteta u SAD u okviru International Visitors Program, Ruhr University Bochum, Signal Theory Group, Njemačka u okviru Volkswagen stiftung projekta, ENSIETA, Brest, Francuska, u okviru PAI Pelikan projekta, GIPSA lab, INP Grenoble, Francuska u okviru CNRS projekta, Nacionalni aerokosmički univerzitet, Kharkov, Ukrajina, Tampere univerzitet za tehnologiju, Tampere, Finska, itd. Bio je član Komisije za odbranu doktorske disertacije na Department for Mathematical Statistics, Lund University, Lund, Švedska.

Dobitnik je Nagrade Crnogorske akademije nauka i umjetnosti iz Fonda Petra Vukčevića 2002. godine i Trinaestojulske nagrade za 2016. godinu. Član je i prvi predsjednik Centra mladih naučnika CANU. Organizovao je više skupova u okviru CANU: „Mobilne i bežične komunikacije: stanje i perspektive“ 2009. godine, „Visoko obrazovanje u Crnoj Gori: stanje i perspektive“, u organizaciji Centra mladih naučnika CANU, sa izlaganjem „Nauka u Crnoj Gori“ i skup „Važnost GEO inicijativa i crnogorski kapaciteti u ovim oblastima“ 2011. godine. Trenutno rukovodi Komisijom CANU za naučne i umjetničke djelatnosti i Odborom CANU za informaciono-komunikacione tehnologije. Organizator je skupa o trendovima u savremenim mobilnim komunikacionim sistemima, koji je održan u CANU 2014. godine.

Za vanrednog člana CANU izabran je 29. novembra 2011. godine.

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