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Extremes of Gaussian fields with a smooth random variance

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ABSTRACT

In this paper we investigate the probabilities of large extremes $\mathbb{P}\left(\sup_{\mathbf{t}\in T}\xi(\mathbf{t})\eta(\mathbf{t}) > u\right)$, as $u \to \infty$, where $\xi(\mathbf{t})$ is a centered homogeneous Gaussian random field, $\eta(\mathbf{t})$ is particular smooth field independent of $\xi(\mathbf{t})$ and $T \subset \mathbf{R}^n$ is a closed Jordan set.

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1. Introduction and main result

Let $\xi(\mathbf{t})$, $\mathbf{t} \in \mathbf{R}^n$, be a homogeneous Gaussian random field and $\eta(\mathbf{t})$ another random field being independent of $\xi(\mathbf{t})$. We will consider the product random field $\xi(\mathbf{t})\eta(\mathbf{t})$ investigating the tail probability of its supremum.

The tail behaviour of random variables has been studied in numerous papers, see e.g., Jessen and Mikosch (2006), Hashorva et al. (2015) and the references therein. The asymptotic theory for large extremes of Gaussian processes and fields is well developed see e.g., Piterbarg (1996, 2015) and Azais and Wschebor (2009). In the recent years several new results show the asymptotic behaviour of extremes of both smooth and non-smooth Gaussian random fields, see e.g., Cheng (2014), Cheng and Schwartzman (2015) and Cheng and Xiao (in press).

Random processes and fields of product type have been extensively investigated in the literature. Recently, Piterbarg and Zhdanov (2015) considered the product of two Gaussian processes. Analysis of extremes of product of random processes is an interesting topic which arises in various contexts of queueing theory Arendarczyk and Dębicki (2011, 2012), insurance mathematics Dębicki et al. (2014), time series analysis Kulik and Soulier (2015). Our investigation in this paper is in the line with those of Hüsler et al. (2011a), where is considered product of stationary Gaussian process $\xi(t)$ and the smooth processes $\eta(t)$, random parabola and more general process which is two times differentiable in the neighbourhood of its essential supremum. Hüsler et al. (2011b) considered the product of the locally stationary Gaussian process and the processes which are less smooth then those of Hüsler et al. (2011a).

In this paper we shall consider extremes of the product of a Gaussian random field and a paraboloid-type field, where the Gaussian random field has a constant variance and a covariance function of (E, α) type. We shall formulate below our assumptions on both the Gaussian field and the multiplier. In order to calculate the exact asymptotic behaviour of the product random field, we will calculate this probability under a fixed multiplier and then average the behaviour over

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all values of the multiplier. For the first part, when the multiplier is fixed, we will use well known asymptotic results for Gaussian fields.

Let the collection $\alpha = (\alpha_1, \ldots, \alpha_k)$ of numbers be given, where $0 < \alpha_i \leqslant 2$, $i = 1, \ldots, k$, as well as the collection $E = (e_1, \ldots, e_k)$ of positive integers such that $\sum_{i=1}^{k} e_i = n$, and let us set $e_0 := 0$. Each pair (E, α) is called a *structure*, Piterbarg (1996). For any vector $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbf{R}^n$ its *structural modulus* is defined by

$$|\mathbf{t}|_{(E,\alpha)} = \sum_{i=1}^{k} \left(\sum_{j=E(i-1)+1}^{E(i)} t_j^2 \right)^{\frac{\alpha_i}{2}},$$

where $E(i) = \sum_{j=0}^{i} e_j$, $i = 1, \ldots, k$. We denote by $\alpha(i)$ the number from the collection α that corresponds to ith coordinate of the vector $\mathbf{t}$, $i = 1, \ldots, n$.

Let $T \subset \mathbf{R}^n$ be a closed Jordan set such that $\mathbf{0} \in T$. In this paper we assume that $\xi(\mathbf{t})$, $\mathbf{t} \in T$, is a homogeneous Gaussian random field with the expectation of zero and covariance function $r(\mathbf{t})$ that satisfies

$$r(\mathbf{t}) = 1 - |\mathbf{t}|_{(E,\alpha)} + o\left(|\mathbf{t}|_{(E,\alpha)}\right),$$

as $\mathbf{t} \to \mathbf{0}$, for some structure (E, α), $E = (e_1, \ldots, e_k)$, $\alpha = (\alpha_1, \ldots, \alpha_k)$.

Now, let us introduce paraboloid-type field

$$\eta(\mathbf{t}) := \lambda - \frac{1}{2}\zeta \, \|\mathbf{t}\|^2,$$

where λ and ζ are random variables independent of $\xi(\cdot)$ ($\|\cdot\|$ denotes the Euclidean norm). We will assume that $\lambda \geqslant \varepsilon$ for some $\varepsilon > 0$ and ζ is positive almost surely. With the notation

$$\sigma(G) := \operatorname{ess\,sup}(G)$$

for any random element G, we further assume that $\sigma(\lambda)$, $\sigma(\zeta)$, $\sigma\left(\frac{\zeta}{\lambda}\right)$ are finite.

We want to establish asymptotic exact result for

$$\mathbb{P}\left(\sup_{\mathbf{t}\in T} \xi(\mathbf{t})\eta(\mathbf{t}) > u\right), \quad \text{as } u \to \infty.$$

Note that

$$\eta(\mathbf{t}) = \lambda - \frac{1}{2}\zeta \, |\mathbf{t}|_{(E',\beta)},$$

where $E' = (1, \ldots, 1)$ and $\beta = (2, \ldots, 2)$.

The random variance of $\xi(\mathbf{t})\eta(\mathbf{t})$ conditioned on η is

$$\mathsf{Var}(\xi(\mathbf{t})\eta(\mathbf{t}) \mid \eta(\mathbf{t})) = \eta^2(\mathbf{t}),$$

so motivated by the geometry of $\eta(\mathbf{t})$ and Hüsler et al. (2011a) we will call the product $\xi(\mathbf{t})\eta(\mathbf{t})$ Gaussian field with a smooth random variance.

Also note that the random covariance of $\xi \cdot \eta$ conditioned on η is

$$\mathsf{cov}(\xi(\mathbf{t})\eta(\mathbf{t}), \xi(\mathbf{s})\eta(\mathbf{s}) \mid \eta) = \eta(\mathbf{s})\,\eta(\mathbf{t})\,r(\mathbf{t}-\mathbf{s}),$$

the correlation is

$$\mathsf{Cr}(\xi(\mathbf{t})\eta(\mathbf{t}), \xi(\mathbf{s})\eta(\mathbf{s}) \mid \eta) = r(\mathbf{t}-\mathbf{s}),$$

and

$$\mathsf{E}\left((\xi(\mathbf{t})\eta(\mathbf{t}) - \xi(\mathbf{s})\eta(\mathbf{s}))^2 \mid \eta\right) = \eta^2(\mathbf{t}) - 2\eta(\mathbf{t})\eta(\mathbf{s})r(\mathbf{t}-\mathbf{s}) + \eta^2(\mathbf{s}).$$

Write

$$\Psi(x) = \frac{1}{\sqrt{2\pi}x} \exp\left\{-\frac{x^2}{2}\right\},$$

and from now on H_α denotes Pickands' constant (see Piterbarg (1996)).

Our main result is the next theorem.

Theorem 1. *Let $\xi(\mathbf{t})$ and $\eta(\mathbf{t})$, $\mathbf{t} \in T$, be above introduced fields and let*

$$\sup_{x\in T} \|x\| < \min\left(\frac{1}{\sigma(\zeta)}, \sqrt{\frac{1}{\sigma\left(\frac{\zeta}{\lambda}\right)}}\right).$$

Let the density function $f_\lambda(x)$ of the random variable λ be r times continuously differentiable in the neighbourhood of $\sigma := \sigma(\lambda)$, with $f_\lambda^{(i)}(\sigma) = 0$, $i = 0, \ldots, r-1$, and $f_\lambda^{(r)}(\sigma) \neq 0$ for some $r \in \mathbf{Z}^+$.

1. *Suppose that the function* $m_1(x) = \mathsf{E}\left(\zeta^{-\frac{n}{2}} \mid \lambda = x\right)$ *exists and is continuous at* $x = \sigma$*, with* $m_1(\sigma) > 0$*. If* $\alpha_i < 2$*,* $i = 1, \ldots, k$*, then,*

$$\mathbb{P}\left(\max_{\mathbf{t}\in T} \xi(\mathbf{t})\eta(\mathbf{t}) > u\right) = (1+o(1))(-1)^r \left(\sqrt{2\pi}\right)^n H_\alpha\, m_1(\sigma)\, \sigma^{-\sum_{i=1}^{k}\frac{2e_i}{\alpha_i}+\frac{3n}{2}+3r+3} \times f_\lambda^{(r)}(\sigma)\, u^{\sum_{i=1}^{k}\frac{2e_i}{\alpha_i}-n-2r-2}\, \Psi\left(\frac{u}{\sigma}\right),$$

as $u \to \infty$.

2. *Suppose that the function* $m_2(x) = \mathsf{E}\left(\left(\frac{2\lambda+\zeta}{\zeta}\right)^{\frac{n}{2}} \mid \lambda = x\right)$ *exists and is continuous at* $x = \sigma$*, with* $m_2(\sigma) > 0$*. If* $\alpha_i = 2$*,* $i = 1, \ldots, k$*, then,*

$$\mathbb{P}\left(\max_{\mathbf{t}\in T} \xi(\mathbf{t})\eta(\mathbf{t}) > u\right) = (1+o(1))(-1)^r \sigma^{3r+3}\, m_2(\sigma)\, \Psi\left(\frac{u}{\sigma}\right) u^{-2-2r} f_\lambda^{(r)}(x),$$

as $u \to \infty$.

Directly from the proof of Theorem 1 follows the next corollary.

Corollary 1. *Under the assumptions of Theorem 1 we have*

$$\mathbb{P}\left(\max_{\mathbf{t}\in T} \xi(\mathbf{t})\eta(\mathbf{t}) > u \,\middle|\, \lambda, \zeta\right) = (1+o(1))\left(\frac{2\lambda+\zeta}{\zeta}\right)^{\frac{n}{2}} \Psi\left(\frac{u}{\lambda}\right),$$

as $u \to \infty$.

2. Proof of Theorem 1

Firstly we will consider the case $\alpha_i < 2$, $i = 1, \ldots, k$, with $\lambda = 1$ almost surely.

Comment. If $\lambda = 1$ almost surely, then density does not exist so $m_1(1)$ does not make any sense. So we have to establish the condition $\mathsf{E}(\zeta)^{-n/2}$ (or $\mathsf{E}\left(1+\frac{2}{\zeta}\right)^{n/2}$, in the latter case $\alpha_i = 2$) exists, but we can deal with the condition that $\mathsf{E}(\zeta)^{-1/2}$ ($\mathsf{E}\left(1+\frac{2}{\zeta}\right)^{1/2}$) exists.

Using Theorem 8.2 in Piterbarg (1996) we get

$$\mathbb{P}\left(\max_{\mathbf{t}\in T} \xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right) > u \,|\zeta\right) = \prod_{i=1}^{k} H_i \prod_{j=E(i-1)}^{E(i)-1} u^{\frac{2}{\alpha_i}-1}\Psi(u)(1+o(1)), \quad u\to\infty, \tag{1}$$

where

$$H_i = H_{\alpha_i}\int_{\mathbf{R}^{e_i}\cap CT^\infty} \exp\left(-|AC^{-1}\mathbf{t}|_{(E',\beta)}\big|_{\mathbf{R}^{e_i}}\right) d\mathbf{t}^i,$$

$$T^\infty = \mathbf{R}^n, \qquad A = \sqrt{\frac{\zeta}{2}}\cdot I, \qquad C = I,$$

and with I we denote the identity $n \times n$ matrix. So,

$$H_i = H_{\alpha_i}\left(\sqrt{2\pi}\,\zeta^{-1/2}\right)^{e_i}, \quad i = 1, \ldots, k,$$

and

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(\max_{\mathbf{t}\in T} \xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right) > u\,|\zeta\right)}{(\sqrt{2\pi})^n u^{-n}\prod_{i=1}^{k} H_{\alpha_i} u^{\frac{2e_i}{\alpha_i}}\Psi(u)} = \zeta^{-\frac{n}{2}}. \tag{2}$$

We will use a similar idea to that in paper Hüsler et al. (2011a) for the application of the dominating convergence theorem. We denote $\delta := u^{-1}\ln u$, $\pi_\delta := [-\delta, \delta]^n$. Using Lemma 8.1 in Piterbarg (1996) we have

$$\mathbb{P}\left(\max_{\mathbf{t}\in T} \xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right) > u\,|\zeta\right) = \mathbb{P}\left(\max_{\mathbf{t}\in T\cap\pi_\delta} \xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right) > u\,|\zeta\right)(1+o(1)),$$

as $u \to \infty$.

Let $a_i \in (\alpha(i), 2)$, $i = 1, \ldots, n$, and $\Delta_\ell := \prod_{i=1}^n \left[\ell_i u^{-\frac{2}{a_i}}, (\ell_i+1)u^{-\frac{2}{a_i}}\right]$, $\ell = (\ell_1, \ldots, \ell_n) \in \mathbf{Z}^n$.

Let us define $\mathcal{H} \subset \mathbf{Z^+}^n$ ($\mathbf{Z}^+$ is a set of non-negative integers) with

$$\ell \in \mathcal{H} \iff \Delta_\ell \cap (T \cap \pi_\delta) \neq \varnothing.$$

Let g_u, $u > 0$, be a linear transformation of $\mathbf{R}^n$ defined by

$$g_u \mathbf{t} := \left(u^{-\frac{2}{a_1}} t_1, u^{-\frac{2}{a_2}} t_2, \ldots, u^{-\frac{2}{a_n}} t_n\right).$$

Let $\ell \in \mathcal{H}$. By using Lemma 7.1 in Piterbarg (1996) with given ζ we get the upper bound.

$$\mathbb{P}\left(\max_{\mathbf{t}\in\Delta_\ell} \xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right) > u \,\middle|\, \zeta\right) \leqslant \mathbb{P}\left(\max_{\mathbf{t}\in\Delta_\ell} \xi(\mathbf{t}) > \frac{u}{1-\frac{\zeta}{2}\|g_u\ell\|^2} \mid \zeta\right)$$
$$\leqslant (1+\gamma(u))\frac{H_\alpha}{\sqrt{2\pi}}\operatorname{mes}(\Delta_\ell)\prod_{i=1}^k\left(\frac{u}{1-\frac{\zeta}{2}\|g_u\ell\|^2}\right)^{\frac{2e_i}{\alpha_i}} \cdot \left(\frac{u}{1-\frac{\zeta}{2}\|g_u\ell\|^2}\right)^{-1}\exp\left(-\frac{u^2}{2}\left(1+\zeta\|g_u\ell\|^2\right)\right),$$

where $\gamma(u) \downarrow 0$ as $u \to \infty$, not depending on ℓ and ζ.

We get (by summing on ℓ)

$$\begin{aligned}
\mathbb{P}\left(\max_{\mathbf{t}\in T\cap\pi_\delta} \xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right) > u \,\middle|\, \zeta\right) &\leqslant 2^n \sum_{\ell\in\mathcal{H}} \mathbb{P}\left(\max_{\mathbf{t}\in\Delta_\ell}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right) > u \,\middle|\, \zeta\right)\\
&\leqslant (1+\gamma(u))\frac{2^n H_\alpha}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \prod_{i=1}^n u^{-\frac{2}{a_i}}\\
&\quad \times \sum_{\ell\in\mathcal{H}}\left(\prod_{i=1}^k\left(\frac{u}{1-\frac{\zeta}{2}\|g_u\ell\|^2}\right)^{\frac{2e_i}{\alpha_i}} \cdot \left(\frac{u}{1-\frac{\zeta}{2}\|g_u\ell\|^2}\right)^{-1} \cdot e^{-\frac{\zeta}{2}\|g_u\ell\|^2 u^2}\right)\\
&\leqslant C_1 2^n H_\alpha \Psi(u)\, u^{-n} \prod_{i=1}^k u^{\frac{2e_i}{\alpha_i}} \cdot \prod_{i=1}^n u^{-\frac{2}{a_i}+1} \cdot \sum_{\ell\in\mathcal{H}} e^{-\frac{\zeta}{2}\|u\cdot g_u\ell\|^2}\\
&\leqslant C_2 2^n H_\alpha \Psi(u)\, u^{-n} \prod_{i=1}^k u^{\frac{2e_i}{\alpha_i}} \cdot \left(\int_0^{+\infty} e^{-\frac{1}{2}\zeta x^2}\,dx\right)^n\\
&= C_2 H_\alpha \Psi(u)\, u^{-n+\sum_{i=1}^k \frac{2e_i}{\alpha_i}} \left(\frac{2\pi}{\zeta}\right)^{\frac{n}{2}},
\end{aligned}$$

for some positive constants C_1 and $C_2 (> 1)$. Here, the last inequality follows by using the fact that for all sufficiently large u expression $\prod_{i=1}^n u^{-\frac{2}{a_i}+1} \cdot \sum_{\ell\in\mathcal{H}} e^{-\frac{\zeta}{2}\|u\cdot g_u\ell\|^2}$ is not larger than

$$\int_{\mathbf{R^+}^n} \exp\left(-\frac{\zeta}{2}(x_1^2+\cdots+x_n^2)\right) dx_1 \ldots dx_n.$$

Using dominating convergence theorem it follows that

$$\mathbb{P}\left(\max_{\mathbf{t}\in T}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right) > u\right) = \left(\sqrt{2\pi}\right)^n H_\alpha\, u^{-n+\sum_{i=1}^k\frac{2e_i}{\alpha_i}}\, \Psi(u)\, \mathsf{E}\left(\zeta^{-\frac{n}{2}}\right)(1+o(1)), \tag{3}$$

as $u \to \infty$.

Using that $\lambda \in [\varepsilon, \sigma]$ almost surely, for sufficiently small $\varepsilon > 0$, and using $\mathbb{P}(A) = \mathsf{E}(\mathbb{P}(A|X))$ we have, by conditioning on λ

$$\begin{aligned}
\mathbb{P}\left(\max_{\mathbf{t}\in T}\xi(\mathbf{t})\left(\lambda-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right) > u\right) &= \mathsf{E}\left(\mathbb{P}\left(\max_{\mathbf{t}\in T}\xi(\mathbf{t})\left(1-\frac{1}{2}\frac{\zeta}{\lambda}\|\mathbf{t}\|^2\right) > \frac{u}{\lambda} \,\middle|\, \lambda\right)\right)\\
&= (1+o(1))\left(\sqrt{2\pi}\right)^n H_\alpha\, \mathsf{E}\left(\left(\frac{u}{\lambda}\right)^{-n+\sum_{i=1}^k\frac{2e_i}{\alpha_i}} \Psi\left(\frac{u}{\lambda}\right) \mathsf{E}\left(\left(\frac{\zeta}{\lambda}\right)^{-\frac{n}{2}} \,\middle|\, \lambda\right)\right)\\
&= (1+o(1))\left(\sqrt{2\pi}\right)^n H_\alpha\, u^{-n+\sum_{i=1}^k\frac{2e_i}{\alpha_i}} \int_\varepsilon^{\sigma(\lambda)} x^{\frac{3n}{2}-\sum_{i=1}^k\frac{2e_i}{\alpha_i}}\, \Psi\left(\frac{u}{x}\right) m_1(x) f_\lambda(x)\,dx. \qquad (4)
\end{aligned}$$

Applying Proposition 2 of Hüsler et al. (2011a) on (4) with $g_1(x) = x^{\frac{3n}{2}-\sum_{i=1}^{k}\frac{2e_i}{\alpha_i}} m_1(x)$, $g_2(x) = f_\lambda(x)$, we get

$$\mathbb{P}\left(\max_{\mathbf{t}\in T}\xi(\mathbf{t})\left(\lambda-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\right)=(1+o(1))(-1)^r\left(\sqrt{2\pi}\right)^n H_\alpha\, m_1(\sigma)\,\sigma^{-\sum_{i=1}^{k}\frac{2e_i}{\alpha_i}+\frac{3n}{2}+3r+3}$$
$$\times f_\lambda^{(r)}(\sigma)\,u^{\sum_{i=1}^{k}\frac{2e_i}{\alpha_i}-n-2r-2}\,\Psi\left(\frac{u}{\sigma}\right),$$

as $u\to\infty$.

Let us now consider the case $\alpha_i = 2(=\beta_j)$, $i=1,\dots,k$. In this case we will apply Theorem 8.2(ii) of Piterbarg (1996). Here,

$$H_i=\lim_{S\to\infty}H^{\sqrt{\frac{\zeta}{2}}I}_{(E,\alpha),(E',\beta)|_i}\left(\left[-\frac{S}{2},\frac{S}{2}\right]^{e_i}\right),$$

and

$$H^{\sqrt{\frac{\zeta}{2}}I}_{(E,\alpha),(E',\beta)|_i}\left(\left[-\frac{S}{2},\frac{S}{2}\right]^{e_i}\right)=\mathrm{E}\exp\left(\max_{\mathbf{t}\in\left[-\frac{S}{2},\frac{S}{2}\right]^{e_i}}\left(\chi(\mathbf{t})-\frac{\zeta}{2}|\mathbf{t}|_{(E',\beta)}\right)_{|_i}\,\Big|\,\zeta\right),$$

where

$$\chi(\mathbf{t})=\sqrt{2}(U_1t_1+\cdots+U_nt_n)-\|\mathbf{t}\|^2,$$

with independent Gaussian $\mathcal{N}(0,1)$ random variables $U_1,\dots,U_n$, which are independent of ζ. So,

$$\mathrm{E}\exp\left(\max_{\mathbf{t}\in\left[-\frac{S}{2},\frac{S}{2}\right]^{e_i}}\left(\sqrt{2}\sum_{j=E(i-1)+1}^{E(i)}U_jt_j-\left(1+\frac{\zeta}{2}\right)\sum_{j=E(i-1)+1}^{E(i)}t_j^2\right)\Big|\,\zeta\right)\to\mathrm{E}\exp\left(\frac{\sum_{j=E(i-1)+1}^{E(i)}U_j^2}{2+\zeta}\,\Bigg|\,\zeta\right),$$

as $S\to\infty$, by Fatou's Theorem. By using independence, last term is equal $\left(\sqrt{\frac{2+\zeta}{\zeta}}\right)^{e_i}$, and finally

$$\lim_{u\to\infty}\frac{\mathbb{P}\left(\max_{\mathbf{t}\in T}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\,|\,\zeta\right)}{\Psi(u)}=\left(\sqrt{\frac{2+\zeta}{\zeta}}\right)^n. \tag{5}$$

As in the previous case we will find the upper bound for the probability in (5) so we can apply dominating convergence theorem. The probability

$$\mathbb{P}\left(\max_{\mathbf{t}\in T\cap\pi_\delta}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\,|\,\zeta\right)$$

is not greater than

$$\mathbb{P}\left(\max_{\mathbf{t}\in D_0}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\,|\,\zeta\right)+\sum_{\ell\in\mathcal{H}_1}\mathbb{P}\left(\max_{\mathbf{t}\in D_\ell}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\,|\,\zeta\right),$$

for some $b>0$ and $D_0:=\left[-bu^{-1},bu^{-1}\right]^n$, $D_\ell:=u^{-1}\ell\,b+\left[0,bu^{-1}\right]^n$, $\mathcal{H}_1\subset\mathbf{Z}^n$ such that

$$\ell\in\mathcal{H}_1\iff D_\ell\cap(T\cap\pi_\delta)\neq\varnothing\text{ and }\ell\neq\mathbf{0}.$$

We will obtain the upper bound of the first term using Theorem 8.2(ii) in Piterbarg (1996) and the argument for obtaining (5):

$$\mathbb{P}\left(\max_{\mathbf{t}\in D_0}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\,|\,\zeta\right)\leqslant 2\Psi(u)\left(\frac{2+\zeta}{\zeta}\right)^{\frac{n}{2}}.$$

Now we will find the upper bound of the terms in the sum.

$$\mathbb{P}\left(\max_{\mathbf{t}\in D_\ell}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\,|\,\zeta\right)\leqslant\mathbb{P}\left(\max_{\mathbf{t}\in D_\ell}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|u^{-1}\ell b\|^2\right)>u\,|\,\zeta\right)$$
$$\leqslant\mathbb{P}\left(\max_{\mathbf{t}\in\left[0,bu^{-1}\right]^n}\xi(\mathbf{t})>u\left(1+\frac{\zeta}{2}u^{-2}b^2\|\ell\|^2\right)\,|\zeta\right).$$

If we use Lemma 6.1 in Piterbarg (1996) we will find that the last probability is not greater than

$$(1+\gamma(u))H_\alpha\left(\left[0, bu^{-1}\right]^n\right)\Psi\left(u+\frac{\zeta}{2}u^{-1}b^2\|\ell\|^2\right) \leqslant (1+\gamma(u))H_\alpha\left(\left[0, bu^{-1}\right]^n\right)\Psi(u)\exp\left(-\frac{\zeta}{2}b^2\|\ell\|^2\right).$$

Here $\gamma(u)\downarrow 0$ as $u\to\infty$ and does not depend of ℓ, ζ and b. Hence,

$$\sum_{\ell\in\mathcal{H}_1}\mathbb{P}\left(\max_{\mathbf{t}\in D_\ell}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\,|\,\zeta\right)\leqslant C\,\Psi(u)\left(e^{-\frac{\zeta b^2}{2}}+\int_{x_1>1,\ldots,x_n>1}e^{-\frac{\zeta}{2}\left(x_1^2+\cdots+x_n^2\right)b^2}dx_1\ldots dx_n\right),$$

for some $C>0$. Finally, by using Dominating Convergence Theorem we can take the expectation in (5) to find

$$\mathbb{P}\left(\max_{\mathbf{t}\in T}\xi(\mathbf{t})\left(1-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\right)=(1+o(1))\Psi(u)\mathsf{E}\left(\frac{2+\zeta}{\zeta}\right)^{\frac{n}{2}}, \tag{6}$$

as $u\to\infty$ and $S\to\infty$.

Repeating the argument in the previous case we thus have

$$\begin{aligned}\mathbb{P}\left(\max_{\mathbf{t}\in T}\xi(\mathbf{t})\left(\lambda-\frac{1}{2}\zeta\|\mathbf{t}\|^2\right)>u\right)&=\int_\varepsilon^{\sigma(\lambda)}\mathbb{P}\left(\max_{\mathbf{t}\in T}\xi(\mathbf{t})\left(1-\frac{1}{2}\frac{\zeta}{\lambda}\|\mathbf{t}\|^2\right)>\frac{u}{\lambda}\Big|\,\lambda=x\right)f_\lambda(x)dx\\&=(1+o(1))\int_\varepsilon^{\sigma(\lambda)}\mathsf{E}\left(\left(\frac{2+\frac{\zeta}{x}}{\frac{\zeta}{x}}\right)^{\frac{n}{2}}\Big|\,\lambda=x\right)\Psi\left(\frac{u}{x}\right)f_\lambda(x)\,dx\\&=(1+o(1))(-1)^r\sigma^{3r+3}\,\mathsf{E}\left(\left(\frac{2\sigma+\zeta}{\zeta}\right)^{\frac{n}{2}}\,|\,\lambda=\sigma\right)\Psi\left(\frac{u}{\sigma}\right)u^{-2-2r}f_\lambda^{(r)}(x),\end{aligned}$$

as $u\to\infty$. □

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References

Arendarczyk, M., Dębicki, K., 2011. Asymptotics of supremum distribution of a Gausssian process over a Weibullian time. Bernoulli 17 (1), 194–210. URL: http://dx.doi.org/10.3150/10-BEJ266.
Arendarczyk, M., Dębicki, K., 2012. Exact asymptotics of supremum of a stationary Gaussian process over a random interval. Statist. Probab. Lett. 82, 645–652. URL: http://dx.doi.org/10.1016/j.spl.2011.11.015.
Azais, J.-M., Wschebor, M., 2009. Level Sets and Extrema of Random Processes and Fields, first ed. Wiley, New York.
Cheng, D., 2014. Double extreme on joint sets for Gaussian random fields. Statist. Probab. Lett. 92, 79–82. URL: http://dx.doi.org/10.1016/j.spl.2014.05.001.
Cheng, D., Schwartzman, A., 2015. Distribution of the height of local maxima of Gaussian random fields. Extremes 18 (2), 213–240. URL: http://dx.doi.org/10.1007/s10687-014-0211-z.
Cheng, D., Xiao, Y., 2016. The mean Euler characteristic and excursion probability of Gaussian random fields with stationary increments. Ann. Appl. Probab. in press.
Dębicki, K., Hashorva, E., Ji, L., 2014. Tail asymptotics of supremum of certain Gaussian processes over threshold dependent random intervals. Extremes 17 (3), 411–429. URL: http://dx.doi.org/10.1007/s10687-014-0186-9.
Hashorva, E., Korshunov, D., Piterbarg, V., 2015. Asymptotic expansion of Gaussian chaos via probabilistic approach. Extremes 18, 315–347. URL: http://dx.doi.org/10.1007/s10687-015-0215-3.
Hüsler, J., Piterbarg, V., Rumyantseva, E., 2011a. Extremes of Gaussian processes with a smooth random variance. Stochastic Process. Appl. 121 (11), 2592–2605. URL: http://dx.doi.org/10.1016/j.spa.2011.06.006.
Hüsler, J., Piterbarg, V., Zhang, Y., 2011b. Extremes of Gaussian processes with a random variance. Electron. J. Probab. 16, 1254–1280. URL: http://dx.doi.org/10.1214/EJP.v16-904.
Jessen, A., Mikosch, T., 2006. Regularly varying functions. Publ. Inst. Math., N.S. 80 (94), 171–192. URL: http://dx.doi.org/10.2298/PIM0694171J.
Kulik, R., Soulier, P., 2015. Heavy tailed time series with extremal independence. Extremes 18 (2), 273–299. URL: http://dx.doi.org/10.1007/s10687-014-0213-x.
Piterbarg, V.I., 1996. Asymptotic Methods in the Theory of Gaussian Processes and Fields, first ed., AMS Translations of Mathematical Monographs, vol. 148, Providence, Rhode Island.
Piterbarg, V.I., 2015. Twenty Lectures About Gaussian Processes, first ed. Atlantic Financial Press, London, New York.
Piterbarg, V., Zhdanov, A., 2015. On probability of high extremes for product of two independent Gaussian stationary processes. Extremes 18 (1), 99–108. URL: http://dx.doi.org/10.1007/s10687-014-0205-x.