HOMOLOGY OF POLYOMINO TILINGS ON FLAT SURFACES

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The homology group of a tiling introduced by M. Reid is studied for certain topological tilings. As in the planar case, for finite square grids on topological surfaces, the method of homology groups, namely the non-triviality of some specific element in the group allows a 'coloring proof' of impossibility of a tiling. Several results about the non-existence of polyomino tilings on certain square-tiled surfaces are proved in the paper.

1. INTRODUCTION

Recreational mathematics comprises various subjects including combinatorial games, puzzles, card tricks, art, etc. Its problems are typically easily understood by a general audience, yet their solution often requires rigorous research. Indeed, a significant number of mathematical disciplines have been grounded on ideas sparked by challenges from recreational mathematics. For example, graph theory has its roots in the solution of the problem of The Seven Bridges of Königsberg, and magic squares contributed to the foundations of combinatorial designs.

A *polyomino* is a planar geometric figure formed by joining one or more identical squares edge-to-edge. It may also be regarded as a finite subset of the regular square grid with a connected interior. A polyomino consisting of exactly n cells is called an *n-omino*. Polyomino shapes for $n \leq 5$ are illustrated in Figures 1, 2 and 3. Some polyominoes were named after letters of the alphabet closely resembling them, as can be seen in Figures 2 and 3. They were popularized by Salmon Golomb who wrote the first monograph on polyominoes [**8**], and by Martin

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Gardner in his Scientific American columns "Mathematical Games", see [**6**]. In fact, the word polyomino was coined by Golomb in [**7**]. Today they are one of the most popular subjects of recreational mathematics, being of great interest to not only mathematicians but physicists, biologists, and computer scientists as well. For more information, we refer the reader to surveys [**2**] and [**3**].
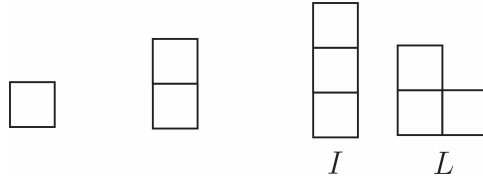

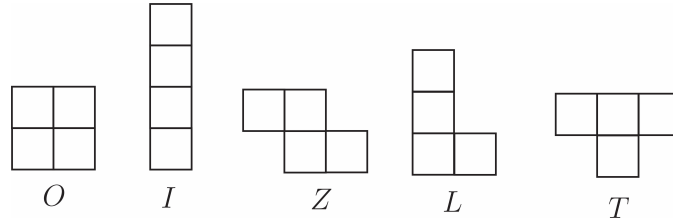Figure 1: Monomino, domino and trominoes

Figure 2: Tetrominoes
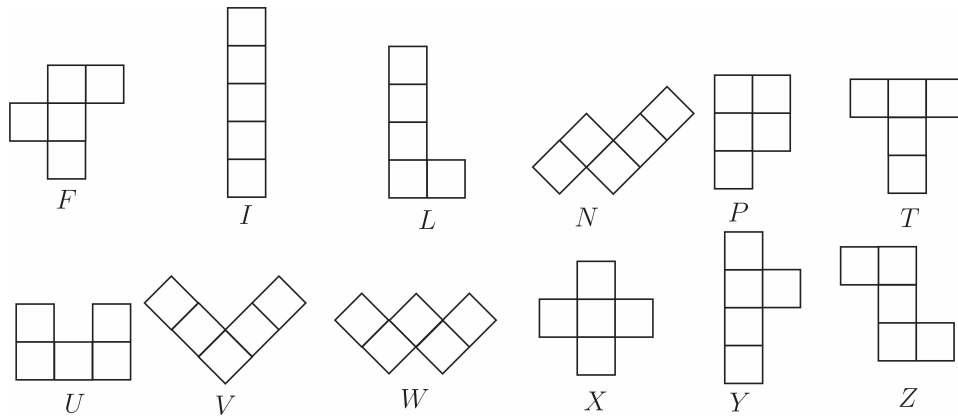

Figure 3: Pentominoes

The polyomino tiling problem asks whether it is possible to properly tessellate a finite region of cells, say M, with polyomino shapes from a given set $\mathcal{T}$. There are numerous generalizations of this question for symmetric and asymmetric tilings, higher dimensional analogs, polyomino type problems on other regular lattice grids

(triangular, hexagonal), etc. However, the problem is NP-hard in general, and we can give definite answers only in a limited number of cases.

This enthralling problem from recreational mathematics has attracted attention of both mathematicians and non-experts. There were many results establishing criteria for proper tilings by some specific polyomino shapes (see [**9**], [**10**], [**11**], [**18**] and [**19**]). Conway and Lagarias developed in [**5**] the so-called 'boundary-word method' for addressing this question. Their ideas were further developed by Reid in [**17**] who assigned to each set of tiles $\mathcal{T}$ *the homology and the homotopy group of tilings* and formulated a necessary condition for existence of a proper tiling of a finite region M in a plane.

Reid's powerful idea allows natural generalization to a much bigger class of combinatorial tilings. Instead of considering planar regions, we study regions which are obtained by identifying parts of the boundary of a planar region resulting in a flat Riemann surface. The only flat compact Riemann surfaces are the torus and Klein bottle, but one can give higher-genus surfaces a flat metric everywhere except at certain cone points, and then remove neighborhoods of the singular points to get a flat surface with boundary. Surfaces with a flat metric obtained by pairwise identification of sides of a collection of plane polygons via translations of their sides, are called translation surfaces. Translation surfaces can also be defined as Riemann surfaces with a holomorphic 1-form. In particular, we are interested in a subclass of translation surfaces called a square-tiled surface. A square-tiled surface is any translation surface obtained from a polygon P which is itself obtained by putting a collection of copies of the unit square side by side. In general, the total angle around a corner of a square of a square-tiled surface S is a non-trivial multiple of 2π. Any such point is called a conical singularity of S. In this paper, we study the problem of tiling a surface S subdivided into a finite 'combinatorial' grid by a finite set of polyomino shapes T and define the homology group $H_S(\mathcal{T})$.

Square-tiled and translation surfaces arise in dynamical systems, where they can be used to model billiards, and in Teichmüller theory. They have a rich mathematical structure and may be studied from multiple points of view (flat geometry, algebraic geometry, combinatorial group theory, etc.). We present some new results and illustrate examples explaining the application of the homology group of generalized polyomino type tilings in the combinatorial and the topological context.

In Section 2, we introduce the homology tiling group for finite square grids on surfaces with boundaries based on [**17**]. Several results about the impossibility of tiling certain concrete square-tiled surfaces are proved using the homology group of the tiling in Section 3. Our main novelty lies in Theorem 12 which establishes a general result connecting the I-polyomino shape with the genus of the surface.

2. TILING PROBLEM ON SURFACES

The standard square grid in the plane is characterized by the property that exactly four edges meet at each vertex, and each vertex is shared by four squares

in the grid. We assume that every edge in a combinatorial grid on a surface is shared by exactly two squares, unless it is on the boundary. This local property allows us to define a polyomino tiling on a topological surface in the same way as in the planar case, and we will refer to such a structure as *the square grid on a surface*. For example, identification of parallel edges of the boundary of $m \times n$ grid in the same directions provides such a grid on torus. Identification of two pairs of parallel sides of an $m \times m$, $m \geq 3$ square, but in the opposite direction in one of the pairs, provides examples of square grids on Klein bottle, see Figure 4. However, if the surface has no boundary, then each vertex is shared by four squares, so the number of vertices equals the number of squares. Likewise, each edge is shared by two squares, so the number of edges is twice the number of squares. This makes the Euler characteristic

$$\chi(M) = V - E + F = F - 2F + F = 0,$$

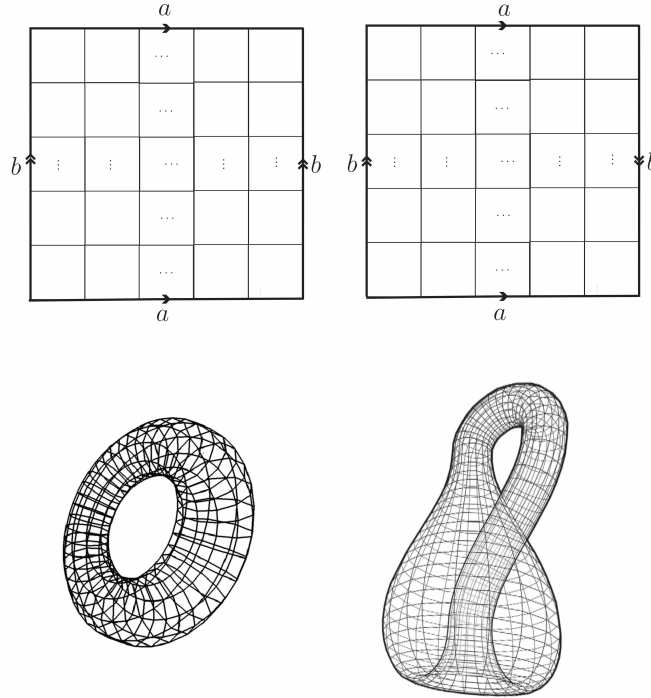so M is either the torus or the Klein bottle.

Figure 4: Square grids on a torus and a Klein bottle

On topological surfaces with boundaries, square grids are not rare structures. One way to obtain them is by identification of certain faces of a finite region in a

planar square grid. Identification of faces allows the additional possibility for placing a polyomino tile, so we have to develop means to treat tiling problems. Surfaces obtained by gluing sides of a polygon are extensively studied in mathematics and this is an interesting research topic in itself (see [**1**], [**12**], [**13**] and [**21**]).

Actually, above mentioned combinatorial structures are directly related to mathematical concepts known as *translation surfaces*. Combinatorially, a translation surface may be defined in the following way. Let $P_1, \ldots, P_m$ be a collection of polygons in the Euclidean plane and suppose that for every side s_i of any P_k there is a side s_j of some P_l with $j \neq i$ and $s_j = s_i + \vec{v}_i$ for some nonzero vector $\vec{v}_i$ and so that $\vec{v}_j = -\vec{v}_i$. The space obtained by identifying all s_i with their corresponding s_j through the map $x \mapsto x + \vec{v}_i$ is a translation surface.

A particular class of translation surfaces known as square-tiled surfaces is of wide interest for mathematics. A square-tiled surface is an orientable connected surface obtained from a finite collection of unit squares in a plane after identifications of pairs of parallel sides via adequate translations. In general, the total angle around a corner of a square of a square-tiled surface M is a non-trivial multiple of 2π and any such point is called a conical singularity of M. In our considerations we will consider flat surfaces with cone points with cone angle a multiple of $\frac{\pi}{2}$.

The tiling problem for a finite subset of the regular planar square grid by a finite set of polyomino prototiles has been studied extensively in the past few decades. However, there exist many other topological 2-manifolds which admit subdivision into a finite number of squares which preserves the structure of regular square grid and for which the tiling problem is also defined. One natural way to obtain such structures is by gluing some of the faces of a finite subset of regular square lattice in the plane, and some results and examples of polyomino tiling problems in this context are known in literature under the notion of topological tilings. Special cases of cylinders, torus, Möbius strip, Klein bottle and projective plane with a 2-disk removed were studied in [**8**], [**20**] and [**14**].

Several techniques for finding obstructions to tiling are known, and one of the most charming is that of a 'generalized chessboard coloring'. This method rests on the fact that the chessboard with two opposite square corners removed cannot be tiled by dominoes, as the difference between the number of white and black squares is two, see [**7**]. The general idea is to use several colors and color the squares of the considered region in a special pattern 'sensitive' to the given set of polyominoes. In other words, the coloring imposes some number theoretical condition which serves as an obstruction to a tiling. However, it is not easy to find a coloring argument for proving nonexistence of a tiling. Michael Reid introduced in [**17**] the so-called homology group of a tiling and showed that proof of nontriviality of a special element in this group assigned to the finite subset of regular square lattice produces a generalized chessboard coloring argument. His homology tiling group method is therefore at least as powerful as the coloring argument. In the same paper, Reid gave many examples where the tiling homology group is inefficient for proving non-existence of a tiling.

The problem of polyomino tilings was studied by Conway and Lagarias in [**5**]

where they introduced a new technique using boundary word invariants to formulate necessary conditions for the existence of tilings. Based on their ideas, Reid presented in [17] a new strategy for treating tiling problems, working with the so-called homotopy group of tiling. Reid's *homotopy tiling group method* was so far the most successful in establishing necessary criteria for existence of tilings.

Our main observation is that Reid's tiling homology group method can be applied to studying topological tilings. A standard model for obtaining topological surfaces is identification of sides of a polygon and as clearly presented in [13] and [21].

Let M be a topological surface with boundary obtained by gluing of sides of some finite subset R of the regular square grid in the plane and let $\mathcal{T}$ be a finite set of polyomino tiles. Gluing of faces provides more ways for placement of tiles from $\mathcal{T}$ on M then in the case of R, so M may be tiled even if R does not admit a tiling by tiles from $\mathcal{T}$. We introduce the tiling homology group $H(M, \mathcal{T})$ in the same fashion as Michael Reid.

Let A be the free abelian group generated by the set of cells of M. We assume that all cells of M preserve labeling by (i, j) from R. The generator of A corresponding to the cell (i, j) is denoted by $a_{i,j}$. Let $B(M, \mathcal{T})$ be the subgroup generated by elements corresponding to all possible placements of tiles in $\mathcal{T}$, i.e. by the sums of elements assigned to cells of M that can be covered by a tile from $\mathcal{T}$.

Definition 1. *The tiling homology group of $(M, \mathcal{T})$ is the quotient group*

$$H(M, \mathcal{T}) = A/B(M, \mathcal{T}).$$

Let us denote by $\bar{a}_{i,j}$ the image of $a_{i,j}$ in $H(M, \mathcal{T})$. As in the planar case, there is an element $\Theta \in H(M, \mathcal{T})$ assigned to M

$$\Theta := \sum_{(i,j) \in M} \bar{a}_{i,j}$$

which is clearly zero when there is a tiling of M by polyominoes from $\mathcal{T}$. Thus, Θ is an obstruction to tiling. Recall that Reid considered in his paper the so-called *signed tiling*, where he allowed polyomino tiles to have positive and negative signs. Clearly, the signed tiling of M by $\mathcal{T}$ exists if and only if Θ is trivial in $H(M, \mathcal{T})$.

Reid's [17, Proposition 2.10] also holds for topological tilings by polyominoes. It states that nontrivial Θ produces special numbering of cells in M that yields a generalized chessboard coloring argument. We adapt his proof to the case of topological tilings.

Proposition 2. *Let M be a topological surface with boundary with a finite square grid and finite set of polyominoes $\mathcal{T}$ such that Θ is nontrivial in $H(M, \mathcal{T})$. Then there is the numbering of the cells in M by rational numbers such that*

   i) for any placement of a tile from $\mathcal{T}$, the total sum of covered numbers is an integer, and

ii) the total covered by the cells of M is not an integer.

Proof. Consider the cyclic subgroup $\langle \Theta \rangle \subset H(\mathcal{T})$ generated by Θ. We define a homomorphism $\varphi : \langle \Theta \rangle \to \mathbb{Q}/\mathbb{Z}$ with $\varphi(\Theta) \neq 0$. If Θ has infinite order we set $\varphi(\Theta) = \frac{1}{2} \mod \mathbb{Z}$, while if Θ has finite order $n > 1$, then we define $\varphi(\Theta) = \frac{1}{n} \mod \mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group, the homomorphism φ extends to a homomorphism $H(M, \mathcal{T}) \to \mathbb{Q}/\mathbb{Z}$, also called φ. Here we used the familiar fact about equivalence of the notions of injective group and divisible group for abelian groups [**4**, Proposition 6.2]. Since A is a free abelian group, the composite map

$$A \longrightarrow\!\!\!\!\!\!\rightarrow A/B(M, \mathcal{T}) = H(M, \mathcal{T}) \xrightarrow{\ \varphi\ } \mathbb{Q}/\mathbb{Z}$$

lifts to a homomorphism $\psi : A \to \mathbb{Q}$, such that the following diagram commutes

$$
\begin{array}{ccc}
A & \xrightarrow{\ \psi\ } & \mathbb{Q} \\
\downarrow & & \downarrow \\
H(M, \mathcal{T}) & \xrightarrow{\ \varphi\ } & \mathbb{Q}/\mathbb{Z}
\end{array}
$$

where the vertical surjections are the quotient maps. Desired numbering of the cells is defined by ψ, and since $B(M, \mathcal{T})$ is in the kernel of $A \to \mathbb{Q}/\mathbb{Z}$, every tile placement covers an integral total. But, $\varphi(\Theta) \neq 0$ and total of the cells in M is not an integer. $\square$

Reid's tiling homology group was systematically studied using Gröbner bases in the works of Muzika-Dizdarević, Timotijević and Živaljević, see [**15**] and [**16**].

3. NONEXISTENCE OF POLYOMINO TILINGS ON SURFACES

In this section we prove several results on nonexistence of tilings on surfaces of different genus with boundaries by some given polyomino sets as an illustration of the homology method.

First we formulate three results for polyomino tilings on a torus square grid. Such cases were also studied in the past [**20**] as they are close to the planar case.

Theorem 3. *A square torus grid of dimension $(4m + 2) \times (4n + 2)$ cannot be tiled by I-tetrominoes, see Figure 2.*

Proof. Consider a $(4m + 2) \times (4n + 2)$ square torus grid model in a plane with cells labelled as in Figure 3.1.

Investigate all possible placements of a tile in the given model. To each placement one can assign one of two types of relations:

$$\bar{a}_{i,j} + \bar{a}_{i,j+1} + \bar{a}_{i,j+2} + \bar{a}_{i,j+3} = 0 \quad \text{and} \quad \bar{a}_{i,j} + \bar{a}_{i+1,j} + \bar{a}_{i+2,j} + \bar{a}_{i+3,j} = 0$$
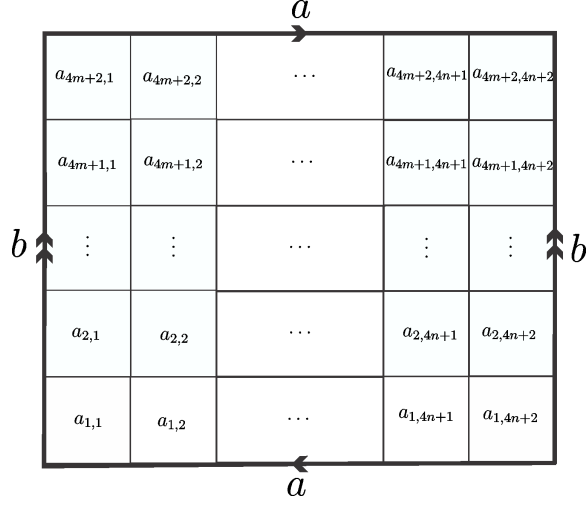
Figure 3.1: Torus grid of dimension $(4m + 2) \times (4n + 2)$

where $i = 1, \ldots, 4m + 2$ labels a row, and $j = 1, \ldots, 4n + 2$ labels a column on the given torus grid. We assume that indices of rows is modulo $4m + 2$ and modulo $4n + 2$ for columns in the relations above. Considering the relation

$$\bar{a}_{i,j+1} + \bar{a}_{i,j+2} + \bar{a}_{i,j+3} + \bar{a}_{i,j+4} = 0$$

we obtain that in the homology group of this tiling it holds that

$$\bar{a}_{i,j} = \bar{a}_{i,j+4}$$

for all $i, j \in \{1, 2, \ldots, 4k + 2\}$. Analogously, $\bar{a}_{i,j} = \bar{a}_{i+4,j}$.

From the relations corresponding to placements over the identified faces of the rectangle representing our torus grid, we obtain additional cells of the grid whose corresponding generators in the homology group of tiling are equal. Using

$$\bar{a}_{i,4m-1} + \bar{a}_{i,4m} + \bar{a}_{i,4m+1} + \bar{a}_{i,4m+2} = 0 \quad \text{and}$$
$$\bar{a}_{i,4m} + \bar{a}_{i,4m+1} + \bar{a}_{i,4m+2} + \bar{a}_{i,1} = 0$$

we conclude that $\bar{a}_{i,1} = \bar{a}_{i,4m-1}$. In the same fashion we deduce that $\bar{a}_{i,2} = \bar{a}_{i,4m}$, $\bar{a}_{1,i} = \bar{a}_{4n-1,i}$ and $\bar{a}_{2,i} = \bar{a}_{4n,i}$ for all i. Combining the equalities above, we obtain

$$\bar{a}_{i,j} = \begin{cases} \bar{a}_{1,1}, & \text{if} \quad i \equiv 1 \pmod 2, j \equiv 1 \pmod 2, \\ \bar{a}_{1,2}, & \text{if} \quad i \equiv 1 \pmod 2, j \equiv 0 \pmod 2, \\ \bar{a}_{2,1}, & \text{if} \quad i \equiv 0 \pmod 2, j \equiv 1 \pmod 2, \\ \bar{a}_{2,2}, & \text{if} \quad i \equiv 0 \pmod 2, j \equiv 0 \pmod 2. \end{cases}$$
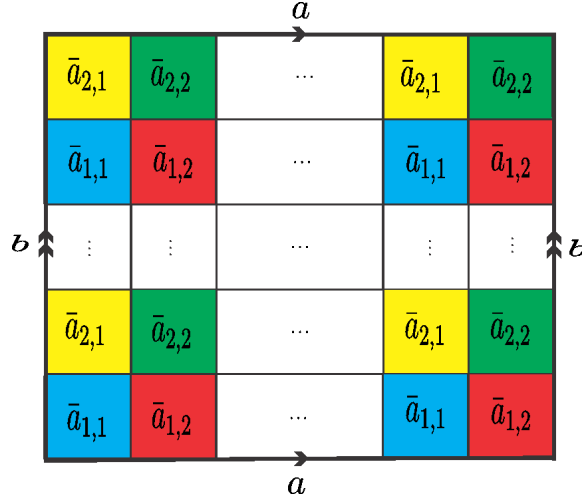
as depicted in Figure 3.2.

Figure 3.2: Coloring of the equivalent cells of the torus grid

If we put I-tetromino shape on the torus grid with equivalent cells we obtain one of the following relations

$$2\bar{a}_{1,1} + 2\bar{a}_{1,2} = 0, \qquad\qquad 2\bar{a}_{1,1} + 2\bar{a}_{2,1} = 0,$$
$$2\bar{a}_{2,1} + 2\bar{a}_{2,2} = 0, \qquad\qquad 2\bar{a}_{1,2} + 2\bar{a}_{2,2} = 0.$$

Therefore, our homology group is isomorphic to the quotient group of the free abelian group with four generators by the four relations given above. Let us observe that one of these relations can be obtained from the remaining three so we can omit the relation $2\bar{a}_{2,1} + 2\bar{a}_{2,2} = 0$. We can consider the presentation of the group using the following four generators $a = \bar{a}_{1,1}$, $b = \bar{a}_{1,1} + \bar{a}_{1,2}$, $c = \bar{a}_{1,1} + \bar{a}_{2,1}$ and $d = \bar{a}_{2,2} - \bar{a}_{1,1}$. It is clear that $2b = 2c = 0$, and little more effort gives $2d = 0$. Thus, our homology group of tiling is isomorphic to

$$G\langle a, b, c, d | 2b = 2c = 2d = 0\rangle \cong \mathbb{Z} \oplus (\mathbb{Z}_2)^3.$$

It is easily seen that everything but the top two (or bottom two) rows of our grid are easily tiled by vertical I-tetrominoes, and that in the top two rows everything but the right-most two columns are tiled by horizontal I-tetrominoes, so Θ is the sum of elements corresponding to the four upper right cells. Thus,

$$\Theta = \bar{a}_{1,1} + \bar{a}_{1,2} + \bar{a}_{2,1} + \bar{a}_{2,2} = b + c + d$$

is nontrivial in the tiling homology group, so desired tiling is not possible.   □

Remark 4. *We can reach the same conclusion using coloring of the square torus grid as in Figure 3.2. Each tile covers 2 blue and 2 yellow cells, or 2 blue and 2*

red, or 2 yellow and 2 green, or 2 red and 2 green. Since the number of cells of each color is odd and each tile covers an even number of cells of the same color, we conclude that tiling is not possible.

Theorem 5. *A square torus grid of dimension $(4m + 2) \times (4n + 2)$ cannot be tiled with T tetrominoes.*

Proof. Consider the torus grid presented as in Figure 3.1. Consider all possible placements of T tetromino. To each placement we can assign one of the following relations:

$$\begin{aligned}
\bar{a}_{i,j} + \bar{a}_{i,j+1} + \bar{a}_{i,j+2} + \bar{a}_{i+1,j+1} &= 0, \\
\bar{a}_{i,j} + \bar{a}_{i,j+1} + \bar{a}_{i,j+2} + \bar{a}_{i-1,j+1} &= 0, \\
\bar{a}_{i,j} + \bar{a}_{i+1,j} + \bar{a}_{i+2,j} + \bar{a}_{i+1,j+1} &= 0 \quad \text{and} \\
\bar{a}_{i,j} + \bar{a}_{i+1,j} + \bar{a}_{i+2,j} + \bar{a}_{i+1,j-1} &= 0,
\end{aligned}$$

where we use the same labelling as in the proof of Theorem 3. From them we directly deduce that in the homology group of tiling it holds that

$$\bar{a}_{i+2,j} = \bar{a}_{i,j} = \bar{a}_{i,j+2}$$

for all i and j. Therefore,

$$\bar{a}_{i,j} = \begin{cases} \bar{a}_{1,1}, & \text{if } i - j \equiv 0 \pmod{2}, \\ \bar{a}_{1,2}, & \text{if } i - j \equiv 1 \pmod{2}, \end{cases}$$

as it is illustrated in Figure 5.1.

Placing a T-tetromino shape on the torus grid with equivalent cells, we get one of the following two relations

$$\begin{aligned}
3\bar{a}_{1,1} + \bar{a}_{1,2} &= 0 \quad \text{and} \\
3\bar{a}_{1,2} + \bar{a}_{1,1} &= 0.
\end{aligned}$$

Therefore, our homology group is isomorphic to the group

$$G\langle \bar{a}_{1,1} | 8\bar{a}_{1,1} = 0 \rangle \cong \mathbb{Z}_8.$$

Our grid has $2m$ cells $\bar{a}_{1,1}$ and $\bar{a}_{1,2}$, where $k = (2m+1)(2n+1)$. So the element that corresponds to this grid

$$\Theta = 2k\bar{a}_{1,1} + 2k\bar{a}_{1,2} = -4k\bar{a}_{1,1} = 4\bar{a}_{1,1}$$

is nontrivial in the homology group, so desired tiling does not exist.   □

Remark 6. *The same conclusion can be obtained using coloring in Figure 5.1 and parity argument for the total number of cells in the grid.*
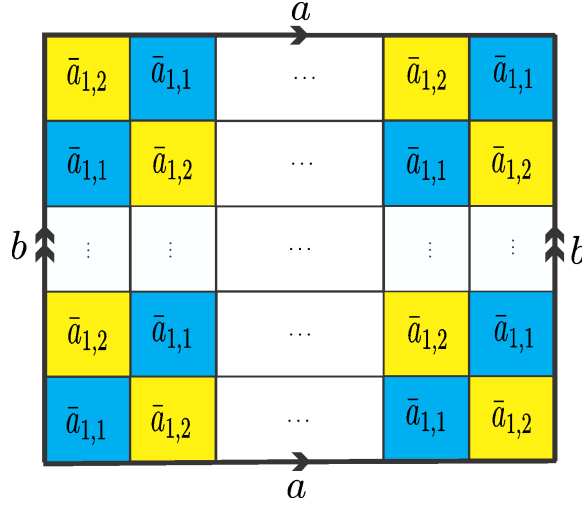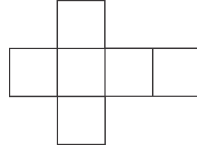
Figure 5.1: Coloring of the equivalent cells of the torus grid

Figure 7.1: X hexomino

Theorem 7. *A square torus grid of dimension* $(4m + 2) \times (4n + 2)$ *cannot be tiled with X hexominoes (Figure 7.1).*

Proof. Consider planar model of torus grid of dimension $(4m + 2) \times (4n + 2)$ as in Figure 3.1. Examine all possible horizontal placements of our tile. Each of them yields a relation

$$(7.1) \qquad \bar{a}_{i,j} + \bar{a}_{i,j+1} + \bar{a}_{i,j+2} + \bar{a}_{i,j+3} + \bar{a}_{i+1,j+1} + \bar{a}_{i-1,j+1} = 0,$$

where where the rows and columns are labelled analogously as in the proof of Theorem 3. From (7.1) we conclude that in the homology group of this tiling it holds $\bar{a}_{i,j} = \bar{a}_{i,j+4}$ for all i and j. Since $\bar{a}_{i,4n-1} = \bar{a}_{i,1}$, $\bar{a}_{i,4n} = \bar{a}_{i,2}$, $\bar{a}_{i,4n+1} = \bar{a}_{i,3}$ and $\bar{a}_{i,4n+2} = \bar{a}_{i,4}$ we further get that for all i and j it also holds $\bar{a}_{i,j} = \bar{a}_{i,j+2}$.

Analogous consideration of vertical placements implies $\bar{a}_{i,j} = \bar{a}_{i+2,j}$ for all i and j. Equivalences of the cells in the grid in the homology group of tiling are depicted in Figure 7.2.

Thus, we deduce that the homology group of tiling is the quotient of the free
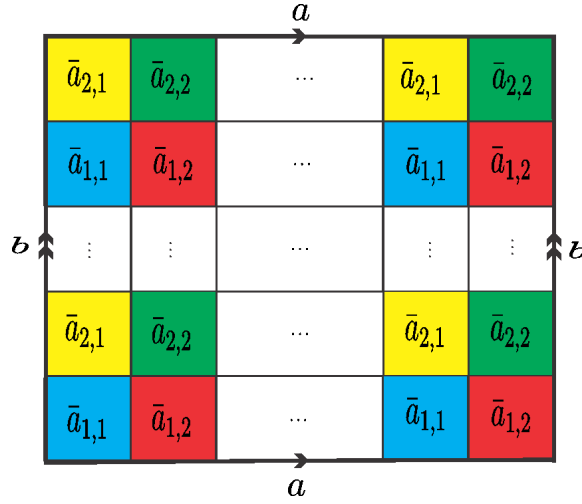
Figure 7.2: Coloring of the equivalent cells of the torus grid

abelian group with four generators $G\langle\bar{a}_{1,1}, \bar{a}_{1,2}, \bar{a}_{2,1}, \bar{a}_{2,2}\rangle$ modulo following relations

$$
\begin{aligned}
2\bar{a}_{1,2} + 2\bar{a}_{2,1} + 2\bar{a}_{2,2} &= 0, \\
2\bar{a}_{1,1} + 2\bar{a}_{2,1} + 2\bar{a}_{2,2} &= 0, \\
2\bar{a}_{1,1} + 2\bar{a}_{1,2} + 2\bar{a}_{2,2} &= 0 \quad \text{and} \\
2\bar{a}_{1,1} + 2\bar{a}_{1,2} + 2\bar{a}_{2,1} &= 0.
\end{aligned}
$$

We consider the presentation of the homology group of tiling using the following generators $x = \bar{a}_{1,2} + \bar{a}_{2,1} + \bar{a}_{2,2}$, $y = \bar{a}_{1,1} + \bar{a}_{2,1} + \bar{a}_{2,2}$, $z = \bar{a}_{1,1} + \bar{a}_{1,2} + \bar{a}_{2,2}$ and $t = \bar{a}_{1,1} + \bar{a}_{1,2} + \bar{a}_{2,1} + \bar{a}_{2,2}$. The upper relations in new generators are

$$2x = 2y = 2z = 6t - 2x - 2y - 2z = 0.$$

Finally, we find that the homology group of tiling is

$$G\langle x, y, z, t | 2c = 2b = 2z = 6t = 0\rangle \cong (\mathbb{Z}_2)^3 \oplus \mathbb{Z}_6.$$

It follows that the element corresponding to this grid

$$
\begin{aligned}
\Theta &= (2k+1)\bar{a}_{1,1} + (2k+1)\bar{a}_{1,2} + (2k+1)\bar{a}_{2,1} + (2k+1)\bar{a}_{2,2} \\
&= (2k+1)(\bar{a}_{1,1} + \bar{a}_{1,2} + \bar{a}_{2,1} + \bar{a}_{2,2}) \\
&= (2k+1)u
\end{aligned}
$$

is a nontrivial element in the homology group as 6 does not divide $2k+1$. Therefore, tiling does not exist. $\qquad\square$

Remark 8. *The same conclusion can be obtained by colouring of torus grid as in Figure 7.2. Each tile covers 2 blue cells, 2 red and 2 green or 2 blue, 2 yellow and 2 green or 2 red, 2 yellow and 2 green or 2 blue, 2 red and 2 green cells. Given that the number of cells of each color is odd, and every tile covers even number of cells of each color, we conclude that tilling is not possible.*

Now we prove some results on surfaces with boundaries. As we will see, topology contributes significantly to the homology group of tiling.

Theorem 9. *A square grid on a non-orientable surface of genus 6 with boundary formed by identifying the sides of a dodecagon consisting of five $4k \times 4k$ squares and removing 20 corner cells around cone point as in Figure 9.1 cannot be tiled with I-tetrominoes and Z-tetrominoes.*
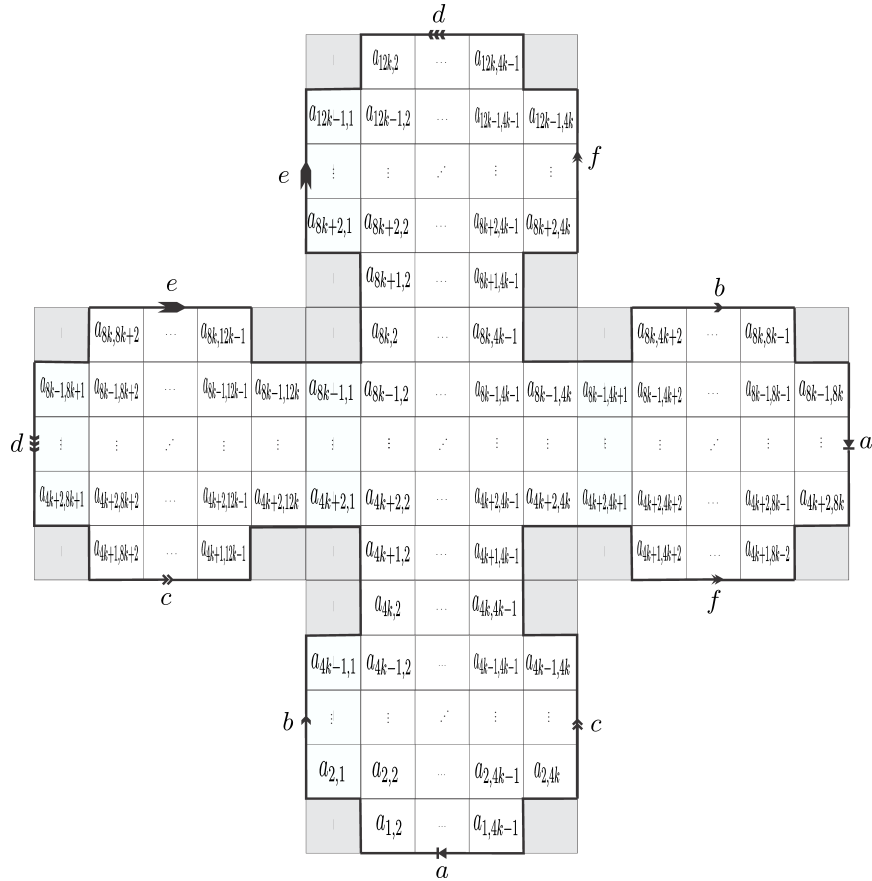

Figure 9.1: Square grid on a non-orientable surface of genus 6 with boundary

Proof. Let us denote the cells of this square grid as in Figure 9.1. Observe that

cells $a_{1,1}$, $a_{1,4k}$, $a_{4k,1}$, $a_{4k,4k}$, $a_{4k+1,8k+1}$, $a_{4k+1,12k}$, $a_{4k+1,1}$, $a_{4k+1,4k}$, $a_{4k+1,4k+1}$, $a_{4k+1,8k}$, $a_{8k,8k+1}$, $a_{8k,12k}$, $a_{8k,1}$, $a_{8k,4k}$, $a_{8k,4k+1}$, $a_{8k,8k}$, $a_{8k+1,1}$, $a_{8k+1,4k}$, $a_{12k,1}$ and $a_{12k,4k}$ are deleted and that, topologically, after gluing their union becomes a disk. Thus, we study a gluing of non-orientable surface of genus 6 with one boundary component.

Using I-tetrominoes it is easy to deduce that in the homology group of tiling it holds that $\bar{a}_{i,j} = \bar{a}_{i+4,j}$ and $\bar{a}_{i,j} = \bar{a}_{i+4,j}$.

A placement of a Z-tetromino yields one of the following two relations

$$\bar{a}_{i,j}+\bar{a}_{i+1,j}+\bar{a}_{i+1,j+1}+\bar{a}_{i+2,j+1} = 0 \quad \text{and} \quad \bar{a}_{i+1,j}+\bar{a}_{i+1,j+1}+\bar{a}_{i+2,j+1}+\bar{a}_{i+2,j+2} = 0.$$

They imply $\bar{a}_{i+2,j+2} = \bar{a}_{i,j}$.

Considering placement of I-tetromino across the edge d it is easy to see that $\bar{a}_{4k+2,8k+1} = \bar{a}_{12k-3,2}$, $\bar{a}_{4k+2,8k+2} = \bar{a}_{12k-2,2}$, $\bar{a}_{4k+2,8k+3} = \bar{a}_{12k-1,2}$ and $\bar{a}_{4k+2,8k+4} = \bar{a}_{12k,2}$. With the relations above we obtain the following equivalences in the homology group of this tiling depicted in Figure 9.2.

Thus, the homology group of tiling is a free abelian group with four generators $\bar{a}_{1,2}, \bar{a}_{1,3}, \bar{a}_{2,3}, \bar{a}_{2,4}$ quotiented by the following relations

$$\bar{a}_{1,2} + \bar{a}_{1,3} + \bar{a}_{2,3} + \bar{a}_{2,4} = 0,$$
$$2\bar{a}_{1,2} + 2\bar{a}_{1,3} = 0,$$
$$2\bar{a}_{1,3} + 2\bar{a}_{2,3} = 0,$$
$$2\bar{a}_{2,3} + 2\bar{a}_{2,4} = 0,$$
$$2\bar{a}_{1,2} + 2\bar{a}_{2,4} = 0.$$

We eliminate generator $\bar{a}_{2,4}$ from its presentation and consider generators $\bar{a}_{1,3}$, $b = \bar{a}_{1,2} + \bar{a}_{1,3}$ and $c = \bar{a}_{1,3} + \bar{a}_{2,3}$. We obtain that our group of homology is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2^2$.
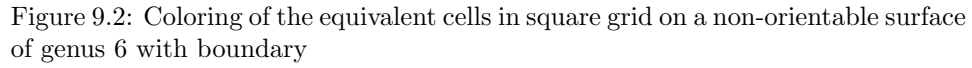
Our square grid contains $20k^2$ cells $\bar{a}_{1,2}$, $5(4k^2 - 1)$ cells $\bar{a}_{1,3}$ and $\bar{a}_{2,4}$, as well as $10(2k^2 - 1)$ cells $\bar{a}_{2,3}$. The element corresponding to this grid

$$\begin{aligned} \Theta &= 20k^2\bar{a}_{1,2} + 5(4k^2 - 1)\bar{a}_{1,3} + 10(2k^2 - 1)\bar{a}_{2,3} + 5(4k^2 - 1)\bar{a}_{2,4} \\ &= 20k^2(\bar{a}_{1,2} + \bar{a}_{1,3} + \bar{a}_{2,3} + \bar{a}_{2,4}) - 5\bar{a}_{1,3} - 10\bar{a}_{2,3} - 5\bar{a}_{2,4} \\ &= 5\bar{a}_{1,2} - 5\bar{a}_{1,3} = b - 10\bar{a}_{1,3} \end{aligned}$$

is a non-trivial element of the homology group and desired tiling is not possible. □

Theorem 10. *A grid on a non-orientable surface of genus 4 with boundary is formed by identifying the sides of a dodecagon consisting of five $4k \times 4k$ squares and with removed* 20 *cells around cone points as in Figure 10.1 cannot be tiled with L-tetrominoes.*

Proof. Model in Figure 10.1 after gluing along marked sides and deletion of 20 corner cells gives a non-orientable surface of genus 4 with three boundary components. Denote the cells in the grid as in the previous example.

Figure 9.2: Coloring of the equivalent cells in square grid on a non-orientable surface of genus 6 with boundary

Placing L-tetromino in the given model in vertical position before taking identification into account will give one of the two relations

$$
\begin{aligned}
(10.1) \qquad \bar{a}_{i,j} + \bar{a}_{i+1,j} + \bar{a}_{i+2,j} + \bar{a}_{i+2,j-1} &= 0 \quad \text{and} \\
(10.2) \qquad \bar{a}_{i,j} + \bar{a}_{i+1,j} + \bar{a}_{i+2,j} + \bar{a}_{i+2,j+1} &= 0
\end{aligned}
$$

in the homology group of tiling. From (10.1) and (10.2) we obtain that in the group of homology of this tiling the cells $\bar{a}_{i,j-1} = \bar{a}_{i,j+1}$ are equivalent. Analogously, it holds that $\bar{a}_{i-1,j} = \bar{a}_{i+1,j}$ are equivalent in the homology group of this tiling.
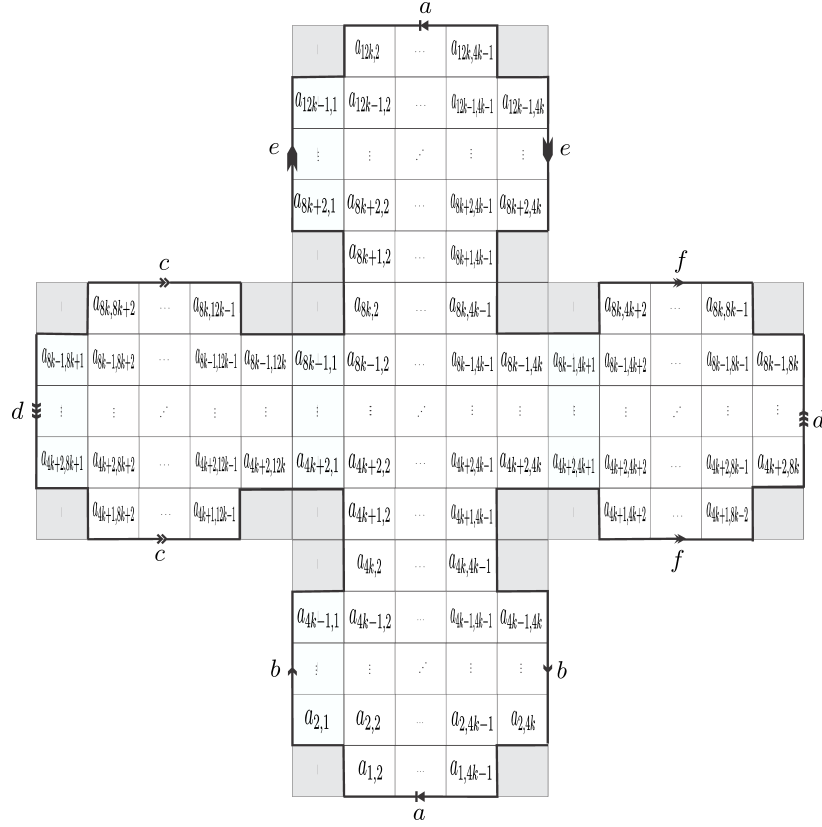
Figure 10.1: Square grid on a non-orientable surface of genus 4 with three boundary components

We summarize all upper equivalences of cells in

$$
\bar{a}_{i,j} = \begin{cases}
\bar{a}_{1,1}, & \text{if} \quad i \equiv 1 \pmod{2}, \; j \equiv 1 \pmod{2}, \\
\bar{a}_{1,2}, & \text{if} \quad i \equiv 1 \pmod{2}, \; j \equiv 0 \pmod{2}, \\
\bar{a}_{2,1}, & \text{if} \quad i \equiv 0 \pmod{2}, \; j \equiv 1 \pmod{2}, \\
\bar{a}_{2,2}, & \text{if} \quad i \equiv 0 \pmod{2}, \; j \equiv 0 \pmod{2}.
\end{cases}
$$

Consider a placement of L-tetromino along edge denoted by e in Figure 10.1 and corresponding equations in the homology group of tiling

$$
\begin{aligned}
\bar{a}_{1,1} + \bar{a}_{1,2} + \bar{a}_{2,1} + \bar{a}_{1,2} &= 0 \quad \text{and} \\
\bar{a}_{2,1} + \bar{a}_{1,2} + \bar{a}_{2,1} + \bar{a}_{1,2} &= 0.
\end{aligned}
$$

From them we deduce that $\bar{a}_{1,1} = \bar{a}_{2,1}$. In a similar way we obtain that $\bar{a}_{1,2} = \bar{a}_{2,2}$. These equivalences are illustrated in Figure 10.2.
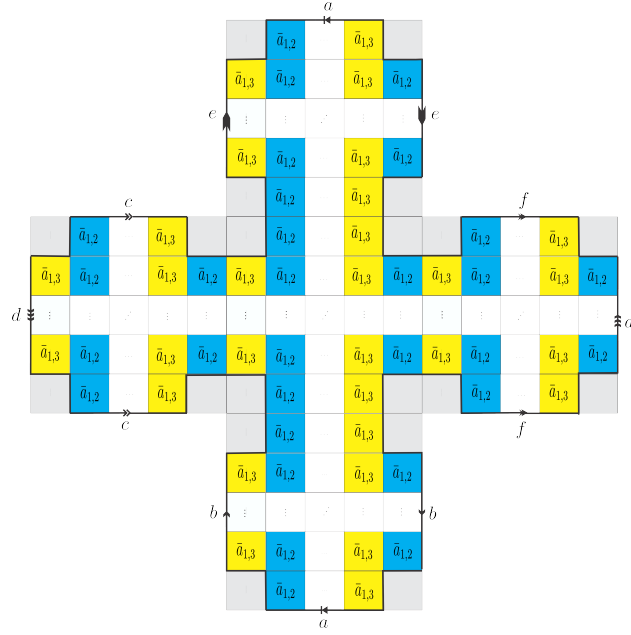
Figure 10.2: Equivalent cells on square grid on a non-orientable surface of genus 4 with boundary

Placement of L tetromino on the grid with equivalent cells, including placements across glued sides, we obtain one of the two relations

$$3\bar{a}_{1,1} + \bar{a}_{1,2} = 0 \quad \text{and}$$
$$3\bar{a}_{1,2} + \bar{a}_{1,1} = 0.$$

Now we conclude that $8\bar{a}_{1,1} = 0$. Therefore, the homology group is isomorphic to the group

$$G\langle \bar{a}_{1,1} | 8\bar{a}_{1,1} = 0 \rangle \cong \mathbb{Z}_8.$$

Our square grid contains $10(4k^2 - 1)$ cells $a_{1,1}$ and $a_{1,2}$, so the element assigned to this grid

$$\Theta = 10(4k^2 - 1)\bar{a}_{1,1} + 10(4k^2 - 1)\bar{a}_{1,2} = 4\bar{a}_{1,1}$$

is a non-trivial element in the homology group of tiling and it is not possible to tile the given grid using L-tetrominoes. $\square$

Theorem 11. *A square grid on an orientable surface of genus 3 with boundary formed by identifying the sides of a dodecagon consisting of five $4k \times 4k$ squares and removing 20 cells meeting in the cone point as in Figure 11.1 cannot be tiled by T-tetrominoes.*
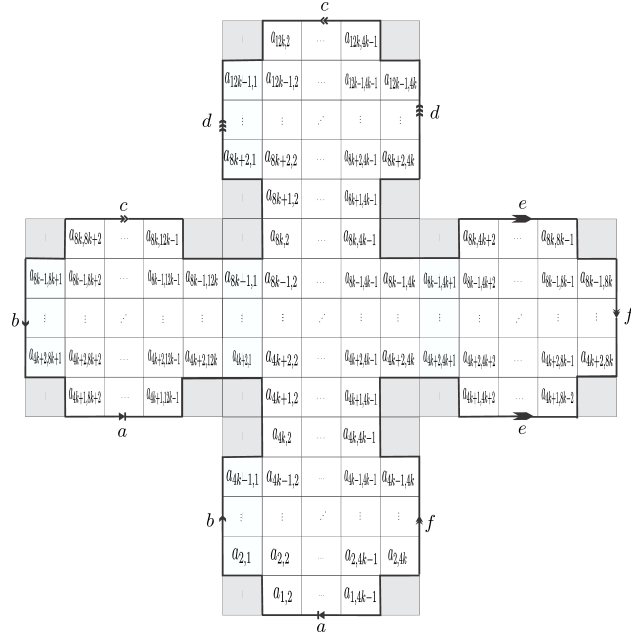
Figure 11.1: Square grid on an orientable genus 3 surface with boundary

Proof.

It is straightforward to check that model in Figure 11.1 after gluing along marked sides and deletion of 20 corner cells gives a genus 3 surface with one boundary component. Denote the cells in the grid as in the previous theorem. The following equality is easily obtained

$$\bar{a}_{i,j} = \begin{cases} \bar{a}_{1,1}, & \text{if} \quad i - j \equiv 0 \pmod{2}, \\ \bar{a}_{1,2}, & \text{if} \quad i - j \equiv 1 \pmod{2}, \end{cases}$$

as it is illustrated in Figure 11.2.

If we put T tetrominoes on the grid with equivalent cells, even placing it across a glued sides, we obtain one of the two relations

$$3\bar{a}_{1,3} + \bar{a}_{1,2} = 0 \quad \text{and}$$
$$3\bar{a}_{1,2} + \bar{a}_{1,3} = 0.$$

Therefore, we get that the homology group of this is isomorphic to the group

$$G\langle \bar{a}_{1,2} | 8\bar{a}_{1,2} = 0 \rangle \cong \mathbb{Z}_8.$$

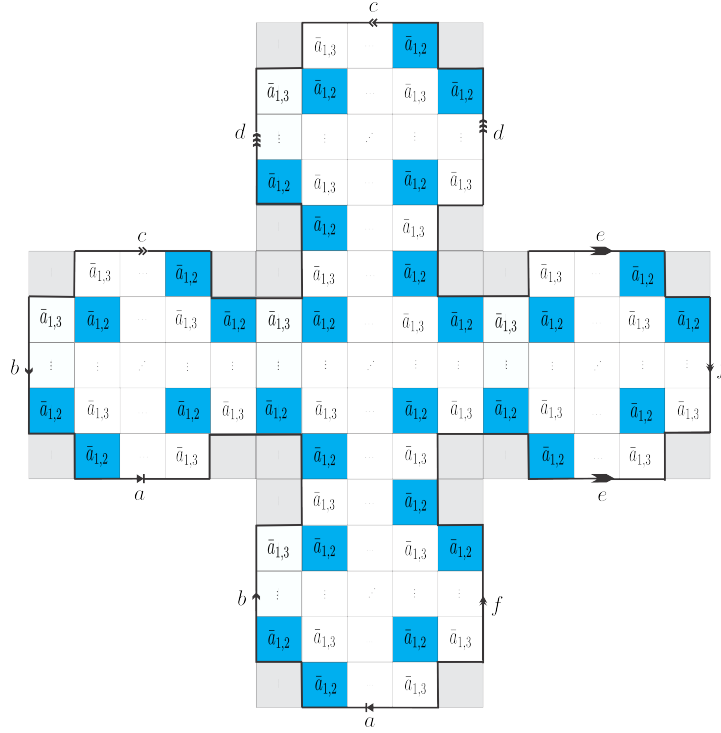Our square grid contains $10(4k^2 - 1)$ cells $\bar{a}_{1,2}$ and $10(4k^2 - 1)$ cells $\bar{a}_{1,3}$, so

Figure 11.2: Equivalent cells on square grid on an orientable genus 3 surface with boundary

the element assigned to this grid is

$$
\begin{aligned}
\Theta &= 10(4k^2 - 1)\bar{a}_{1,2} + 10(4k^2 - 1)\bar{a}_{1,3} \\
&= 40k^2\bar{a}_{1,2} - 10\bar{a}_{1,2} - 120k^2\bar{a}_{1,2} + 30\bar{a}_{1,2} = 4\bar{a}_{1,2}.
\end{aligned}
$$

Θ is a non trivial element in the homology group of tiling, and therefore it is not possible to til the given square grid using T-tetrominoes. $\square$

Theorem 12. *A square grid on an orientable surface of genus $2k-1$ with boundary formed by identifying the sides of a $(8k-4)$-gon consisting of $2k^2 - 2k + 1$ squares of side $(4k-3)d$ where d is a positive integer, without corner cells as in Figure 12.1 can not be tiled with $1 \times (4k-3)$ polyomino.*

Proof. From Figure 12.1 it is clear that the surface is orientable. Label the cells in the grid in standard way. Denote the cell by $a_{i,j}$ in standard way assuming that the bottom left corner cell is $a_{1,1}$. As with other I-minoes it is straightforward to get

$$
\bar{a}_{i,j} = \bar{a}_{i,j+4k-3} = \bar{a}_{i+4k-3,j}.
$$
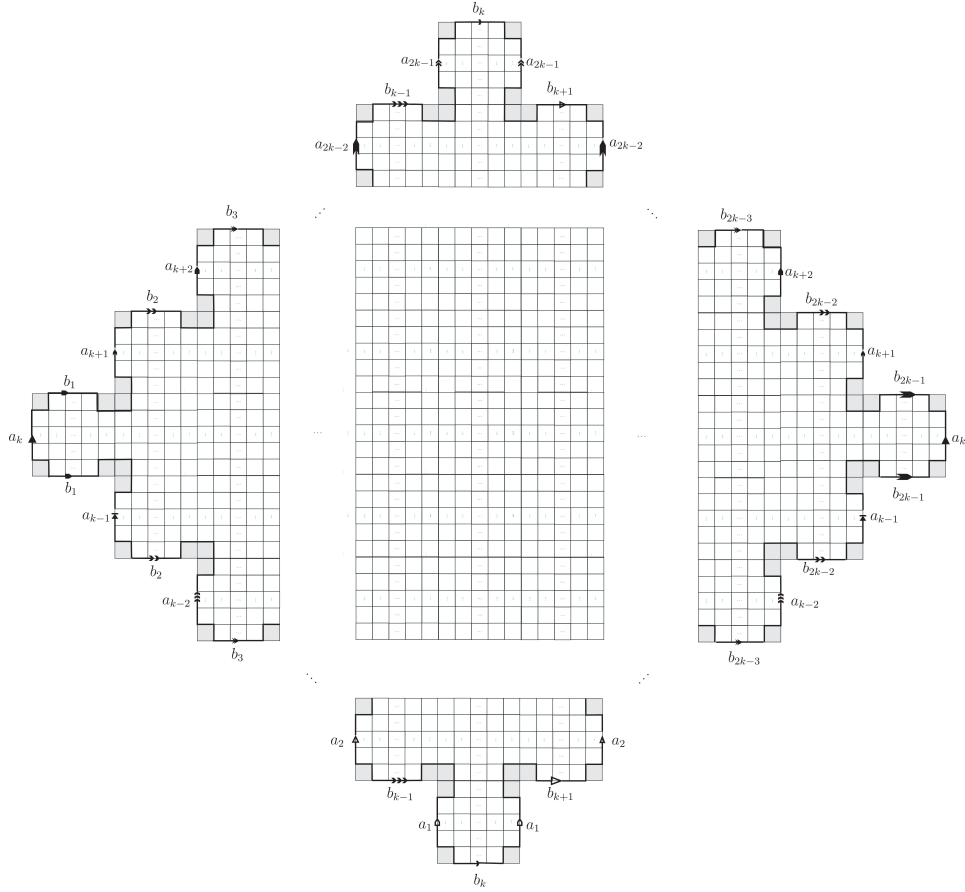
Figure 12.1: Square grid on an orientable genus $2k - 1$ surface with boundary

Using this equivalences we find that there are $(4k-3)^2$ types of the cells $\bar{a}_{i,j}$, $1 \leq i, j \leq 4k - 3$ in the homology group of tiling. We see that there are $8k - 6$ relations

$$\sum_{j=1}^{4k-3} \bar{a}_{i,j} = 0 \quad \text{for } i = 1, \ldots, 4k - 3 \quad \text{and}$$

$$\sum_{i=1}^{4k-3} \bar{a}_{i,j} = 0 \quad \text{for } j = 1, \ldots, 4k - 3$$

assigned to a placement of $1 \times (4k-3)$ polyomino on the board (including placements across gluing sides). Therefore, our homology group of tiling is isomorphic to

$$G\langle a_{i,j} | 1 \leq i, j \leq 4k - 4 \rangle \cong \mathbb{Z}^{16(k-1)^2}.$$

Element Θ assigned to the grid is

$$\Theta = -(2k-1) \sum_{i=2}^{4k-4} \sum_{j=2}^{4k-4} \bar{a}_{i,j}.$$

This is a non-trivial element in the homology group of tiling and the claim is therefore proved. □

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