OCJENA DOKTORSKE DISERTACIJE

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OCJENA DOKTORSKE DISERTACIJE

1. **Pregled disertacije** (bibliografski podaci o disertaciji i sažetak disertacije)

1. **Pregled disertacije** (bibliografski podaci o disertaciji i sažetak disertacije)

The topic of this dissertation falls in the area of Hypercompositional Algebra, a well established branch of Abstract Algebra, born in 1934 when the French mathematician Frederic Marty introduced the concept of hypergroup during the 8th Congress of Scandinavian Mathematicians. Hypercompositional Algebra deals with structures endowed with multi-valued operations, called hyperoperations or hypercompositions. These are natural generalizations of classical operations with the property that the result of the hyperoperation is a subset of the carrier set, instead of a single element, as it happens in the classical algebraic structures endowed with operations. The algebraic structures endowed with multivalued operations have nowadays wide applications in many areas of mathematics – for example multivalued formal groups have important applications in algebraic topology, multivalued Lie groups in functional equations and integrable systems, join spaces in geometry, etc., but also in physics, chemistry, biology, social sciences. It is worth mentioning here the contributions of Alain Connes, winner of the Fields medal, in the theory of algebraic curves related to the theory of hyperfields.

The aim of this dissertation is to extend the concept of reducibility in hypergroups to the fuzzy case and also to hyperrings. The notion of reducibility in hypergroups was defined for the first time by James Jantosciak in 1990, during the 4th Congress on Algebraic Hyperstructures and Their Applications (by short, AHA Congress), held in Xanthi, Greece. Jantosciak noticed that sometimes the hyperoperation does not distinguish between a pair of elements of a set, because the elements play interchangeble roles with respect to the hyperoperation. In order to explain this property, he defined on a hypergroup three equivalences and called them fundamental relations. These relations are: the operational equivalence, the inseparability and the essential indistinguishability. Then he called a hypergroup to be reduced if the equivalence class of each its element with respect to the essential indistinguishable relation is a singleton. These concepts have been reconsidered after 24 years by Irina Cristea, who presented at the 12th AHA Congress the first approach on the extension of the reducibility concept to the fuzzy case. Her first results have been covered and extended by this thesis.

After an introductory part describing the main topic of the thesis and the state-of-the-art, as well as the detailed structure of the dissertation, the thesis continues with its five chapters and bibliography.

First chapter, the one containing the preliminaries, collects the basic definitions and results related to hypergroups, hyperrings and fuzzy sets. All important notions and results are supported by examples, that help the reader to better understand this topic. The first part of the chapter regards hypergroups, subhypergroups, fundamental relations, and homomorphisms of hypergroups. Several particular types of hypergroups, as B-hypergroups, Corsini hypergroups, join spaces and complee hypergroups are recalled. The second part of the first chapter covers essential results on hyperrings and presents all three types of hyperrings: Krasner hyperrings, multiplicative hyperrings and general hyperrings. This

chapter ends with some notions related to fuzzy setsand with the fundamental construction of a join space using fuzzy sets, defined by Piergiulio Corsini, combining the results published in two articles, one in 1994 and the second one in 2003. First he defined a join space starting from a fuzzy set and then he associated with a hypergroupoid a fuzzy set, called the grade fuzzy set. Iterating these two constructions, a sequence of join spaces and fuzzy sets is obtained, whose length is called the fuzzy grade.

The other approach of the connection between hypercompositional structures and fuzzy sets leads to the so called fuzzy hyperstructures, i.e., structures endowed with fuzzy hyperoperations. The fuzzy hypergroups are briefly recalled at the end of Chapter 1.

The reducibility concept in hypergroups is covered by **Chapter 2,** where first, the main results obtained by Jantosciak are recalled, together with the results concerning the reducibility of hypergroups associated with binary relations. The chapter continues with original results of the author related to the reducibility in canonical hypergroups, in hypergroups with partial scalar identities (known as i.p.s. hypergroups), in some cyclic hypergroups and in complete hypergroups. It is then proved that any i.p.s. hypergroup is reduced (see Theorem 2.2), while any proper complete hypergroup is not reduced (see Theorem 2.3). The second chapter ends with the study of the reducibility in Corsini hypergroups. Necessary and sufficient conditions for Corsini hypergroups to be reduced are determined (see Propositions 2.18, 2.19), as well as for the direct product of hypergroups (see Theorem 2.4).

Chapter 3 is dedicated to the study of fuzzy reducibility. For doing this, first, three new equivalences are defined (new in the sense that they are introduced by the author) on a crisp hypergroup endowed with a fuzzy set: the fuzzy operational equivalence, the fuzzy inseparability and the fuzzy essential indistinguishability. Based on them, the concept of fuzzy reducibility is similarly defined as the one of (crisp) reducibility. It is important to know that the fuzzy reducibility depends on the fuzzy set defined on the considered hypergroup. In this thesis it was considered only the grade fuzzy set. It was established also a connection between the reducibility and the fuzzy reducibility of a hypergroup with respect to the grade fuzzy set (see Corollary 3.1). Then the study focusses on the fuzzy reducibility of several types of hypergroups. Theorem 3.1 states that any proper complete hypergroup is not fuzzy reduced with respect to the grade fuzzy set. The same property holds also for I.p.s. hypergroups (see Theorem 3.2). More examples of fuzzy reduced hypergroups are given in Section 3.2: non-complete 1-hypergroups or single power cyclic hypergroups (see Proposition 3.3). The last type of hypergroups for which the fuzzy reducibility is studied in this thesis is the one of Corsini hypergroups. After presenting several new properties of the Corsini hypergroups related to the three fundamental relations, the main result of this subsection is stated in Theorem 3.6: any Corsini hypergroup is not fuzzy reduced with respect to the grade fuzzy set. It is worth mentioning here that Chapter 3 contains a high number of non-trivial examples.

The second extension of the reducibility in hypergroups is proposed in **Chapter 4**, where the reducibility in hyperrings is introduced for the first time. As well explained at the beginning of the chapter, the reducibility has sense to be studied only in general hyperrings, where both addition and multiplication are multi-valued operations. Several relationships between the three fundamental equivalences (that help us to define a reduced hyperring) are presented in particular classes of hyperrings (see Proposition 4.2). Considering the hyperring of formal series R[[X]], it was proved that R[[X]] is reduced if and only if the general hyperring R of coefficients is reduced, too (Proposition 4.3). An example of non-reduced hyperring was given by using a hyperring with P-hyperoperations (see Proposition 4.4). Several examples involving hyperrings with P-hyperoperations and hyperrings constructed with Corsini hypergroups are illustrated.

Similarly to the construction of complete hypergroups, Mario De Salvo proposed a method to obtain a complete hyperring using two rings. Theorem 4.3 states that any complete hyperring is not reduced. Another type of general hyperrings for which the reproducibility property is studied in this dissertation is the one of (H,R)-hyperrings. The main result related to these hyperrings is covered by Proposition 4.6.

The conclusions of this thesis and some proposals to continue the study presented in the dissertation are mentioned in **Chapter 5.** They refer mainly to the reducibility of fuzzy hypergroups and the fuzzy reducibility of general hyperrings.

Bibliography contains 70 items, including publications in English, French and Italian, demonstrating a very good theoretical background of the PhD candidate on this theme. All relevant references are mentioned and also cited in the text.

2. Vrednovanje disertacije

2.1. **Problem** (navesti neriješena i kontraverzna mišljenja o istraživačkom problemu i dosadašnjim pokušajima rješavanja problema, rješenja do kojih su došli drugi autori, ocjenu osnove disertacije u skladu sa radovima i istraživanjima kandidata i način njihove veze sa samom disertacijom)

As already mentioned in the previous section, the concept of reducibility in hypergroups was introduced in 1990 by James Jantosciak. Then it was rediscovered 24 years later by Irina Cristea, who started the first attempt to extend it also to the fuzzy case. In this respect there are two possibilities: to define the fuzzy reducibility of crisp hypergroups, by considering new fundamental relations defined on a hypergroup endowed with a fuzzy set, or to define the reducibility of fuzzy hypergroups. The PhD candidate Milica Kankaraš, in this thesis, continues the previous studies, extending them for severalclasses of hypergroups, but also for hyperrings. The concept of reducibility in hyperrings is firstly proposed by her in the article "The reducibility concept in general hyperrings" jointly published with I. Cristea in the open access journal Mathematics, a Q1 journal indexed by Web of Science. Ms. Kankaraš also published a paper as a single author in "Analele Univ. Constanta-Seria Math.", a Q2 journal indexed by Web of Science, containing her results related to the reducibility in Corsini hypergroups. The third article that she published, jointly with I. Cristea, presents the results of this dissertation concerning the fuzzy reducibility of

hypergroups.

2.2. Ciljevi i hipoteze disertacije

The main objectives of this dissertation are the following ones:

1. To present an overview on the existing literature related to the concept of reducibility.
2. To study the reducibility of several classes of hypergroups: Corsini hypergroups, B-hypergroups, cyclic hypergroups, the direct product of hypergroups.
3. To extend the concept of reducibility of the hypergroups to the fuzzy case, by introducing and studying the concept of fuzzy reduced hypergroups.
4. To extend the concept of reducibility to the class of general hyperrings.

2.3. Bitne metode koje su primijenjene u disertaciji i njihovu primjerenost.

Ako je primijenjena nova ili dopunjena metoda, opišite šta je novo

The results obtained in this dissertation are based on classical methods used in solving problems in combinatorics and abstract algebra; in particular, I would mention here the method of induction or the method of reasoning by absurdity. All the results are clearly presented and all the basic involved notions are recalled in preliminaries, making the thesis self-contained. Several examples and counter-examples support the statements discussed in the thesis.

2.4. Rezultati disertacije i njihovo tumačenje

The main and new results of this thesis are covered in Chapters 2,3 and 4 and they bring a new contribution in the area of hypercompositional structures. On one hand, they offer concrete examples of hypergroups and hyperrings that are or not reduced, while on the other hand they open new lines of research in Hypercompositional Algebra.

Chapter 2 deals with the study of the particular types of hypergroups. It was proved that any i.p.s. hypergroup is reduced (Theorem 2.2), while any proper complete hypergroup is not reduced (Theorem 2.3). Furthermore, the reducibility concept was studied also for Corsini hypergroups and necessary and sufficient conditions for Corsini hypergroups to be reduced are determined in Propositions 2.18 and 2.19. The reducibility property is studied also for the direct product of B-hypergroups, proving that it is always a reduced hypergroup (Proposition 2.21). The results concerning the fuzzy reducibility are gathered in Chapter 3. A relationship between the reducibility of a hypergroup and its fuzzy reducibility with respect to the grade fuzzy set $\tilde{\mu}$is stated by corollary 3.1, saying that a non reduced hypergroup is not fuzzy reduced with respect to$\tilde{\mu}$.Theorem 3.1 proves the not fuzzy reducibility of proper complete hypergroups, as stated in theorem 3.2. Examples of non-complete 1-hypergroups that are reduced and not reduced are presented in section 3.2, as well as examples of cyclic hypergroups. The fuzzy reducibility of Corsini hypergroups is discussed in Section 3.3, where among several properties of these hypergroups related to fuzzy reducibility it is shown that any Corsini hypergroup is not fuzzy reduced with respect to the grade fuzzy set $\tilde{\mu}$(Theorem 3.6). Chapter 3 ends with some results related with the direct product of hypergroups. Proposition 3.14 proves that the direct product of two non fuzzy reduced hypergroups with at least two elements is non fuzzy reduced, too, with respect to the grade fuzzy set $\tilde{\mu}$.

Chapter 4 contains all new results obtained on the reducibility of general hyperrings, in particular of some classes of hyperrings. Proposition 4.3 states that the hyperring of formal series R[[X]] with coefficientsin a general commutative hyperring R is reduced if and only if R is reduced. Then in Proposition 4.4 an example of non-reduced general hyperring with P-

hyperoperations is given and other examples of reduced and non- reduced general hyperrings with P-hyperoperations are illustrated. Another class of hyperrings considered here is the one obtained with Corsini hypergroups. A general hyperring having the additive part a B-hypergroup and the multiplicative one a Corsini hypergroup is always reduced (Proposition 4.5). The chapter continues with the study of the reducibility property of complete hyperrings and Theorem 4.3 proves that any complete hyperring is not reduced. The last class of investigated hyperrings is the one of (H,R)-hyperrings. Necessary and sufficient conditions to get a non-reduced (H,R)-hyperring are expressed in Proposition 4.6.

2.5. **Zaključci** (usaglašenost sa rezultatima i logično izvedeno tumačenje)

In the conclussive chapter of the dissertation the main results are emphasized and new lines of research are proposed, in order to continue this study. All the results are mathematically correct, very well explained, with all necessary steps, and supported by numerous examples. They show the maturity of the PhD candidate and her strong preparation on this topic.

3. Konačna ocjena disertacije

3.1. Usaglašenost sa obrazloženjem teme

The dissertation of the doctoral candidate Milica Kankaraš is fully compliant with the theme, approved by the Senat of the University of Montenegro. The goals, hypothesis, and problems were all addressed with a rigorous style and correct mathematical methods.

3.2. Mogućnost ponovljivosti

It is very difficult to think of a repetition of a theoretical mathematical dissertation, as it is the thesis under discussion. Here a new basis was settled up and some types of hypergroups and general hyperrings have been considered. A similar study could be conducted also for other types of algebraic hypercompositional structures, but they are not considered as a repetition of the results of this thesis.

3.3. Buduća istraživanja

The reducibility concept belongs to the area of algebraic structures endowed with multivalued operations. Thuis it can be studied not only for hypergroups or general hyperrings, as in this thesis, but also for hyperfields or different types of ordered hyperalgebras, as BCK-hyperalgebras, MV-hyperalgebras, etc. The other direction of study suggested by this diddertation is the one offered by the fuzzy case. It was explained in the thesis that this can happen in two directions: considering the fuzzy reducibility, or the reducibility of fuzzy structures. The second direction has not been investigated by now and could offer new lines of research.

3.4. Ograničenja disertacije i njihov uticaj na vrijednost disertacije

Since the topic of this dissertation is represented by a theoretical study in Algebra, the limitations of the dissertation are very few, or better they cannot be considered as limitations, but could represent new research lines. It is clear that a doctoral thesis cannot cover all types of hypergroups or general hyperrings, but here the candidate chose the ones that are more representative in this area. Besides, it was very well explained in the thesis that the concept of fuzzy reducibility depends on the fuzzy set that the hypergroup/hyperring is endowed with. That is why the studies presented in this thesis regard only the grade fuzzy set.

Orginalni naučni doprinos

(dati pojašnjenje: originalnost (originalnost (sasvim nova saznanja, dopuna/proširenje postojećeg znanja ili pobijanje postojećeg znanja), uticaj rezultata disertacije na napredak naučne oblasti, uticaj rezultata na struku (direktno, indirektno))

Without any doubt, this thesis is original. In the first part, the one related to hypergroups, the concept of reducibility and fuzzy reducibility is investigated, for the first time, for some particular important classes of hypergroups, as Corsini hypergroups, B-hypergroups, cyclic hypergroups. The second part is based on the original idea to define, again for the first time, the concept of reducibility for general hyperrings. Besides, most of the examples presented in the thesis are the original work of the candidate, examples being one powerful tool in abstract algebra to illustrate the use of the new theorems.
With these results, a new step forward in the theory of algebraic hypercompositional structures has been done, first by preparing a complete survey on the theory of reducibility of hypergroups and hyperrings, and secondly by opening new lines of research in this area.

Mišljenje i prijedlog komisije

(dati mišljenje i prijedlog)

The PhD thesis elaborated by Milica Kankaraš brings new and significant results in the theory of algebraic hypercompositional structures, by deepening the study of the reducibility property in hypergroups and extending it into two different directions: the first one regards the fuzzy case of the property, while the second one, proposed for the first time by the PhD candidate, is in the class of general hyperrings. The thesis is well structured, the style of the presentation is clear and pleasant, the results are not banal and they are mathematically correct, being strongly motivated and supported by numerous examples. The results obtained by the author (both in amount and quality) are sufficient for the PhD level. They have a good potential for being developed in some further research, as mentioned in the conclusive part of the thesis. The stated aims of the dissertation have been fulfilled.
The bibliographic list shows that the candidate consulted all relevant publications related to the topic of the dissertation, that offered her a solid background for research. The original results are included in 3 articles, 1 of them as single author, published in recognized international journals, indexed by Web of Science in the first two categories. These articles are:

1. Fuzzy reduced hypergroups, Kankaraš M., Cristea I., Mathematics, 2020, 8(2), 263

2. Reducibility in Corsini hypergroups, Kankaraš M., Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica, 2021, 29(1), 99-109

3. The reducibility concept in general hyperrings, Cristea I., Kankaraš M., Mathematics, 2021, 9(17), 2037.

Concluding, we consider that this thesis fulfills the conditions requested by a doctoral thesis, thus it can be publicly discussed and **we warmly propose to grant the title of doctor in Mathematics to the doctoral student Milica Kankaraš.**

Izdvojeno mišljenje	
(popuniti ukoliko neki član komisije ima izdvojeno mišljenje)	Ime i prezime
Napomena	
(popuniti po potrebi)	
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Datum i ovjera (pečat i potpis odgovorne osobe)	
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