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U prilogu ovog akta dostavljamo Vam doktorsku disertaciju mr **Milice Kankaraš** pod naslovom "**Reducibilnost u algebarskim hiperstrukturama**" koja je u skladu sa članom 42 stav 3 Pravila doktorskih studija dostavljena **Centralnoj univerzitetskoj biblioteci** 30. 12. 2022. godine, na uvid i ocjenu javnosti.

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To the Council of the Faculty of Science and Mathematics
and the Senate of the University of Montenegro



# Referee's report of the PhD thesis "Reducibility in algebraic hyperstructures" written by PhD student Milica Kankaraš

The undersigned dr. Michal Novak, associated professor at Brno University of Technology, Czech Republic, dr. Svjetlana Terzić, professor at University of Montenegro, dr. Biljana Zeković, professor at the University of Montenegro, dr. Sanja Jančić Rašović, professor at the University of Montenegro, and dr. Irina Cristea, associated professor at the University of Nova Gorica, Slovenia, nominated with the Decision 03-350/4 from 9.11.2021 of the senate of the University of Montenegro as members of the doctoral committee of the public defense of the doctoral thesis "Reducibility in algebraic hyperstructures" elaborated by the doctoral student Milica Kankaraš, present the following referee report.

The topic of this dissertation falls in the area of Hypercompositional Algebra, a well established branch of Abstract Algebra, born in 1934 when the French mathematician Frederic Marty introduced the concept of hypergroup during the 8th Congress of Scandinavian Mathematicians. Hypercompositional Algebra deals with structures endowed with multi-valued operations, called hyperoperations or hypercompositions. These are natural generalizations of classical operations with the property that the result of the hyperoperation is a subset of the carrier set, instead of a single element, as it happens in the classical algebraic structures endowed with operations. The algebraic structures endowed with multivalued operations have nowdays wide applications in many areas of mathematics – for example multivalued formal groups have important applications in algebraic topology, multivalued Lie groups in functional equations and integrable systems, join spaces in geometry, etc., but also in physics, chemistry, biology, social sciences. It is worth mentioning here the contributions of Alain Connes, winner of the Fields medal, in the theory of algebraic curves related to the theory of hyperfields.

The aim of this dissertation is to extend the concept of reducibility in hypergroups to the fuzzy case and also to hyperrings. The notion of reducibility in hypergroups was defined for the first time by James Jantosciak in 1990, during the 4th Congress on Algebraic Hyperstructures and Their Applications (by short, AHA Congress), held in Xanthi, Greece. Jantosciak noticed that sometimes the hyperoperation does not distinguish between a pair of elements of a set, because the elements play interchangeble roles with respect to the hyperoperation. In order to explain this property, he defined on a hypergroup three equivalences and called them fundamental relations. These relations are: the operational equivalence, the inseparability and the essential indistinguishability. Then he called a hypergroup to be reduced if the equivalence class of each its element with respect to the essential indistinguishable relation is a singleton. These concepts have been reconsidered after 24 years by Irina Cristea, who presented at the 12th AHA Congress the first approach on the extension of the reducibility concept to the fuzzy case. Her first results have been published (unfortunately) in one journal that doesn't exist anymore. Thus it is a great advantage that they have been covered and extended by this thesis.

After an introductory part describing the main topic of the thesis and the state-of-the-art, as well as the detailed structure of the dissertation, the thesis continues with its five chapters and bibliography.

**First chapter,** the one containing the preliminaries, collects the basic definitions and results related to hypergroups, hyperrings and fuzzy sets. All important notions and results are supported by

examples, that help the reader to better understand this topic. The first part of the chapter regards hypergroups, subhypergroups, fundamental relations, and homomorphisms of hypergroups. Several particular types of hypergroups, as B-hypergroups, Corsini hypergroups, join spaces and complee hypergroups are recalled. The second part of the first chapter covers essential results on hyperrings and presents all three types of hyperrings: Krasner hyperrings, multiplicative hyperrings and general hyperrings. This chapter ends with some notions related to fuzzy setsand with the fundamental construction of a join space using fuzzy sets, defined by Piergiulio Corsini, combining the results published in two articles, one in 1994 and the second one in 2003. First he defined a join space starting from a fuzzy set and then he associated with a hypergroupoid a fuzzy set, called the grade fuzzy set $\tilde{\mu}$. Iterating these two constructions, a sequence of join spaces and fuzzy sets is obtained, whose length is called the fuzzy grade.
The other approach of the connection between hypercompositional structures and fuzzy sets leads to the so called fuzzy hyperstructures, i.e., structures endowed with fuzzy hyperoperations. The fuzzy hypergroups are briefly recalled at the end of Chapter 1.

The reducibility concept in hypergroups is covered by **Chapter 2,** where first, the main results obtained by Jantosciak are recalled, together with the results concerning the reducibility of hypergroups associated with binary relations. The chapter continues with original results of the author related to the reducibility in canonical hypergroups, in hypergroups with partial scalar identities (known as I.p.s. hypergroups), in some cyclic hypergroups and in complete hypergroups. It is then proved that any I.p.s. hypergroup is reduced (see Theorem 2.2), while any proper complete hypergroup is not reduced (see Theorem 2.3). The second chapter ends with the study of the reducibility in Corsini hypergroups. Necessary and sufficient conditions for Corsini hypergroups to be reduced are determined (see Propositions 2.18, 2.19), as well as for the direct product of hypergroups (see Theorem 2.4).

**Chapter 3** is dedicated to the study of fuzzy reducibility. For doing this, first, three new equivalences are defined (new in the sense that they are introduced by the author) on a crisp hypergroup endowed with a fuzzy set: the fuzzy operational equivalence, the fuzzy inseparability and the fuzzy essential indistinguishability. Based on them, the concept of fuzzy reducibility is similarly defined as the one of (crisp) reducibility. It is important to know that the fuzzy reducibility depends on the fuzzy set defined on the considered hypergroup. In this thesis it was considered only the grade fuzzy set $\tilde{\mu}$.It was established also a connection between the reducibility and the fuzzy reducibility of a hypergroup with respect to the grade fuzzy set (see Corollary 3.1). Then the study focusses on the fuzzy reducibility of several types of hypergroups. Theorem 3.1 states that any proper complete hypergroup is not fuzzy reduced with respect to the grade fuzzy set. The same property holds also for I.p.s. hypergroups (see Theorem 3.2). More examples of fuzzy reduced hypergroups are given in Section 3.2: non-complete 1-hypergroups or single power cyclic hypergroups (see Proposition 3.3). The last type of hypergroups for which the fuzzy reducibility is studied in this thesis is the one of Corsini hypergroups. After presenting several new properties of the Corsini hypergroups related to the three fundamental relations, the main result of this subsection is stated in Theorem 3.6: any Corsini hypergroup is not fuzzy reduced with respect to the grade fuzzy set. It is worth mentioning here that Chapter 3 contains a high number of non-trivial examples.

The second extension of the reducibility in hypergroups is proposed in **Chapter 4,** where the reducibility in hyperrings is introduced for the first time. As well explained at the beginning of the chapter, the reducibility has sense to be studied only in general hyperrings, where both addition and multiplication are multi-valued operations. Several relationships between the three fundamental equivalences (that help us to define a reduced hyperring) are presented in particular classes of hyperrings (see Proposition 4.2). Considering the hyperring of formal series R[[X]], it was proved that R[[X]] is reduced if and only if the general hyperring R of coefficients is reduced, too

(Proposition 4.3). An example of non-reduced hyperring was given by using a hyperring with P-hyperoperations (see Proposition 4.4). Several examples involving hyperrings with P-hyperoperations and hyperrings constructed with Corsini hypergroups are illustrated.
Similarly to the construction of complete hypergroups, Mario De Salvo proposed a method to obtain a complete hyperring using two rings. Theorem 4.3 states that any complete hyperring is not reduced. Another type of general hyperrings for which the reproducibility property is studied in this dissertation is the one of (H,R)-hyperrings. The main result related to these hyperrings is covered by Proposition 4.6.

The conclusions of this thesis and some proposals to continue the study presented in the dissertation are mentioned in **Chapter 5.** They refer mainly to the reducibility of fuzzy hypergroups and the fuzzy reducibility of general hyperrings.

**Bibliography** contains 70 items, including publications in English, French and Italian, demonstrating a very good theoretical background of the PhD candidate on this theme. All relevant references are mentioned and also cited in the text.

## Conclusions

The PhD thesis elaborated by Milica Kankaraš brings new and significant results in the theory of algebraic hypercompositional structures, by deepening the study of the reducibility property in hypergroups and extending it into two different directions: the first one regards the fuzzy case of the property, while the second one, proposed for the first time by the PhD candidate, is in the class of general hyperrings. The new results obtained by the candidate are covered in Chapters 2,3,and 4 of this dissertation. One one hand, they offer concrete examples of hypergroups and hyperrings that are or not reduced, while on the other hand they open new lines of research in Hypercompositional Algebra. Among several results, we stress here that it was proved that any i.p.s. hypergroup is reduced (Theorem 2.2), but not fuzzy reduced with respect to to the grade fuzzy set (Theorem 3.2), while any complete hypergroup is neither reduced (Theorem 2.3) or fuzzy reduced (Theorem 3.1). The reducibility and fuzzy reducibility property is studied also for Corsini hypergroups (Propositions 2.18, 2.19, and Theorem 3.6). Chapter 4 contains all new results obtained by the candidate on the reducibility of general hyperrings: Proposition 4.3 states that the hyperring of formal series R[[X]] with coefficients in a general commutative hyperring R is reduced if and only if R is reduced. Proposition 4.4. illustrates a construction of a non-reduced general hyperring with P-hyperoperations. Besides, it was proved in Proposition 4.5 that a general hyperring having the additive part a B-hypergroup and the multiplicative one a Corsini-hypergroup is always reduced. Theorem 4.3. proves that any complete hyperring is not reduced. Finally, necessary and sufficient conditions to get a non-reduced (H,R)-hyperring are expressed in Proposition 4.6. This study opens also new lines of research: the study of the reducibility can be conducted also for fuzzy hypergroups and fuzzy hyperrings, but also the fuzzy reducibility of hyperrings appears an interesting topic in this theory.

The thesis is well structured, the style of the presentation is clear and pleasant, the results are not banal and they are mathematically correct, being strongly motivated and supported by numerous examples. The results obtained by the author (both in amount and quality) are sufficient for the PhD level. They have a good potential for being developed in some further research, as mentioned in the conclusive part of the thesis. The stated aims of the dissertation have been fulfilled.
The bibliographic list shows that the candidate consulted all relevant publications related to the topic of the dissertation, that offered her a solid background for research. The original results are included in 3 articles, 1 of them as single author, published in recognized international journals, indexed by Web of Science in the first two categories. These articles are:

1. Fuzzy reduced hypergroups, Kankaraš M., Cristea I., Mathematics, 2020, 8(2), 263
2. Reducibility in Corsini hypergroups, Kankaraš M., Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica, 2021, 29(1), 99-109
3. The reducibility concept in general hyperrings, Mathematics, 2021, 9(17), 2037.

Concluding, we consider that this thesis fulfills the conditions requested by a doctoral thesis, thus it can be publicly discussed and **we warmly propose to grant the title of doctor in Mathematics to the doctoral student Milica Kankaraš.**

23.12.2021

Committee for the reviewing of the doctoral thesis

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