Crna Gora
UNIVERZITET CRNE GORE
PRIRODNO-MATEMATIČKI FAKULTET
Broj 2027/01-1069
Podgorica, 26.05. 2023 god.

Univerzitet Crne Gore

Vijeću Prirodno-matematičkog fakulteta

Predmet: Predlog komisije za ocjenu doktorske disertacije Antona Gjokaja

Doktorska disertacija *„Granična svojstva kvazikonformnih harmonijskih preslikavanja u prostoru"*, doktoranda Antona Gjokaja predata je na ocjenu. Predlažem komisiju u sljedećem sastavu:

1. Prof. dr Darko Mitrović, redovni profesor Prirodno-matematičkog fakulteta Univerziteta Crne Gore
2. Prof. dr David Kalaj, redovni profesor Prirodno-matematičkog fakulteta Univerziteta Crne Gore
3. Prof. dr Miodrag Mateljević, redovni profesor Matematičkog fakulteta Univerziteta u Beogradu
4. Prof. dr Marijan Marković, vanredni profesor Prirodno-matematičkog fakulteta Univerziteta Crne Gore
5. Prof. dr Đorđije Vujadinović, vanredni profesor Prirodno-matematičkog fakulteta Univerziteta Crne Gore

U Podgorici,

Mentor

24.05.2023. godine

Prof. dr David Kalaj

Univerzitet Crne Gore

Prirodno-matematički fakultet

Na osnovu člana 37 Pravila doktorskih studija Univerziteta Crne Gore dajem sljedeću

SAGLASNOST

Doktorska disertacija pod naslovom *„Granična svojstva kvazikonformnih harmonijskih preslikavanja u prostoru"* kandidata MSc Antona Gjokaja zadovoljava kriterijume propisane Statutom Univerziteta Crne Gore i Pravilima doktorskih studija.

Mentor

Prof. dr David Kalaj

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Podgorica, 26. 05. 2023. god.

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ObrazacD2: Ispunjenost uslova doktoranda

ISPUNJENOST USLOVA DOKTORANDA

OPŠTI PODACI O DOKTORANDU			
Titula, ime, ime roditelja, prezime	MSc Anton, Gjelosh, Gjokaj		
Fakultet	Prirodno-matematički fakultet		
Studijski program	Matematika		
Broj indeksa	2/19		
NAZIV DOKTORSKE DISERTACIJE			
Na službenom jeziku	Granična svojstva kvazikonformnih harmonijskih preslikavanja u prostoru		
Na engleskom jeziku	Boundary behaviour of quasiconformal harmonic mappings in space		
Naučna oblast	Matematika		
MENTOR/MENTORI			
Prvi mentor	Prof. dr David Kalaj	Prirodno-matematički fakultet, Univerzitet Crne Gore, Crna Gora	Matematička analiza
KOMISIJA ZA PREGLED I OCJENU DOKTORSKE DISERTACIJE			
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Datum značajni za ocjenu doktorske disertacije			
Sjednica Senata na kojoj je data saglasnost na ocjenu teme i kandidata	15.04.2022. godine		
Dostavljanja doktorske disertacije organizacionoj jedinici i saglasanost mentora	24.05.2023. godine		
Sjednica Vijeća organizacione jedinice na kojoj je dat prijedlog za imenovanje komisija za pregled i ocjenu doktorske disertacije			
ISPUNJENOST USLOVA DOKTORANDA			

U skladu sa članom 38 pravila doktorskih studija kandidat **je** cjelokupna ili dio sopstvenih istraživanja vezanih za doktorsku disertaciju publikovao u časopisu sa **(SCI/SCIE)/(SSCI/A&HCI)** liste kao prvi autor.

Spisak radova doktoranda iz oblasti doktorskih studija koje je publikovao u časopisima sa (upisati odgovarajuću listu)

1. Anton Gjokaj, Hölder continuity of quasiconformal harmonic mappings from the unit ball to a spatial domain with C^1 boundary, Indagationes Mathematicae, Volume 33, Issue 5, 2022, Pages 1061-1070, ISSN 0019-3577, https://doi.org/10.1016/j.indag.2022.05.003

2. A. Gjokaj, D. Kalaj, Quasiconformal harmonic mappings between the unit Ball and a spatial domain with $C^{1,\alpha}$ Boundary, Potential Analysis, Volume 57, 367-377 (2022). https://doi.org/10.1007/s11118-021-09919-y

Obrazloženje mentora o korišćenju doktorske disertacije u publikovanim radovima

U radu objavljenom u časopisu *Ingadationes Mathematicae* kandidat je dokazao uniformnu Hölder neprekidnost za kvazikonformna harmonijska Bloch preslikavanja f iz jedinične lopte $B \subset R^n$ u prostornu oblast sa C^1 granicom.

U radu objavljenom u časopisu *Potential Analysis* dokazana je Lipschitz neprekidnost kvazikonformnih harmonijskih preslikavanja iz jedinične lopte B u prostornu oblast sa $C^{1,\alpha}$ granicom. U ovom radu su date i dvije verzije Hardy-Littlewood teoreme za prostor, kod koje se daje veza između μ-Hölder koeficijenta ($\mu < 1$) u odnosu na tačku $\eta \in S$, tj.

$$\sup_{\xi \in S, \xi \neq \eta} \frac{\|u(\eta) - u(\xi)\|}{\|\eta - \xi\|^{\mu}}$$

i vrijednosti

$$\sup_{x \in [0,\eta)} (1 - \|x\|)^{1-\mu} \|\nabla u(x)\|.$$

Datum i ovjera (pečat i potpis odgovorne osobe)
U Podgorici, ________________ DEKAN MP ________________________

Prilog dokumenta sadrži:

1. Potvrdu o predaji doktorske disertacije organizacionoj jedinici
2. Odluku o imenovanju komisije za pregled i ocjenu doktorske disertacije
3. Kopiju rada publikovanog u časopisu sa odgovarajuće liste
4. Biografiju i bibliografiju kandidata
5. Biografiju i bibliografiju članova komisije za pregled i ocjenu doktorske disertacije sa potvrdom o izboru u odgovarajuće akademsko zvanje i potvrdom da barem jedan član komisije nije u radnom odnosu na Univerzitetu Crne Gore

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Quasiconformal Harmonic Mappings Between the Unit Ball and a Spatial Domain with $C^{1,\alpha}$ Boundary

Anton Gjokaj[1] · David Kalaj[1]

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Abstract
We prove the following. If f is a harmonic quasiconformal mapping between the unit ball in $\mathbb{R}^n$ and a spatial domain with $C^{1,\alpha}$ boundary, then f is Lipschitz continuous in B. This generalizes some known results for $n = 2$ and improves some others in higher dimensional case.

Keywords Harmonic mappings · Quasiconformal mappings · Hölder continuity · Lipschitz continuity

Mathematics Subject Classification (2010) Primary 30C65 · Secondary 31B05

1 Introduction

For $n > 1$, let $\mathbb{R}^n$ be the standard Euclidean space with the norm $|x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$, where $x = (x_1, \ldots, x_n)$. We denote the unit ball $\{x \in \mathbb{R}^n : |x| < 1\}$ by B, and its boundary, the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ by S.

Let $U \subset \mathbb{R}^n$ be a domain. We say $f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ is a harmonic mapping if the functions f_j are harmonic real mappings, i.e. satisfy the n-dimensional Laplace equation

$$\Delta f_j = \sum_{i=1}^{n} D_{ii} f_j = 0.$$

Let

$$P(x, \xi) = \frac{1 - |x|^2}{|x - \xi|^n}$$

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be the Poisson kernel for B, where $x \in B, \xi \in S$, and

$$P[u](x) = \int_S P(x,\xi)u(\xi)d\sigma(\xi)$$

the Poisson integral of continuous function u on S, where σ denotes the normalized surface-area measure on S. Then $P[u](x)$ is continuous on $\overline{B}$ and harmonic on B. Since we will focus on continuous function u on $\overline{B}$, that are harmonic on B, then we will usually express them using the Poisson integral as

$$u = P[u|_S](x).$$

A homeomorphism $f : U \to V$, where U, V are domains in $\mathbb{R}^n$, will be called K quasiconformal (see [27]) ($K \geq 1$) if f is absolutely continuous on lines (i.e. absolutely continuous in almost every segment parallel to some of the coordinate axes and there exist partial derivatives which are locally L^n integrable in U) and

$$|\nabla f(x)| \leq K l(\nabla f(x)),$$

for all points $x \in U$, where

$$l(\nabla f(x)) = \inf\{|f'(x)h| : |h| = 1\}.$$

A function $\Phi : U \subset \mathbb{R}^n \to \mathbb{R}$ is said to be μ-Hölder continuous, $\Phi \in C^\mu(U)$ if

$$\sup_{x,y\in U, x\neq y} \frac{|\Phi(x) - \Phi(y)|}{|x-y|^\mu} < \infty.$$

Similarly, one defines the class $C^{1,\mu}(U)$ to consist of all functions $\Phi \in C^1(U)$ such that $\nabla\Phi \in C^\mu(U)$. The above two definitions extends in a natural way to the case of vector-valued mappings.

We say that a domain $\Omega \subset \mathbb{R}^n$ has $C^{1,\alpha}$ boundary if there is a $C^{1,\alpha}$ diffeomorphism $G : \overline{B} \to \overline{\Omega}$.

Pavlović in [26] showed that harmonic quasiconformal mappings of the unit disk in $\mathbb{R}^2$ onto itself are bi-Lipschitz mappings. From then, several important results have been obtained regarding harmonic quasiconformal mappings in $\mathbb{R}^2$ and the Lipschitz continuity. The second author in [8] proved that every quasiconformal harmonic mapping between Jordan domains with $C^{1,\alpha}$ boundaries is Lipschitz continuous on the closure of domain. The result in [8] was extended in [9] for Jordan domains with only Dini's smooth boundaries. Lately, in [13] it was proved the Hölder continuity (but in general, Lipschitz continuity does not hold) of a harmonic quasiconformal mapping between two Jordan domains having only C^1 boundaries. Other important results for $n = 2$ with different conditions and settings can be found in [1, 4, 6, 11, 12, 15, 16, 18–20, 23, 24] and in their references.

For higher dimensional case there are some important results also (see e.g. [2, 10, 17, 21]). In [10] it was proven that a quasiconformal mapping of the unit ball onto a domain with C^2 smooth boundary, satisfying Poisson differential inequality, is Lipschitz continuous. This implies that harmonic quasiconformal mappings from unit ball B to Ω with C^2 boundary are Lipschitz continuous. This was also proved by Astala and Manojlovic in [2] using a slight modification of the following statement also proved there: a harmonic K-quasiconformal mapping from B to B is Lipschitz with the Lipschitz constant depending on the value of K, dimension of n and dist$(f(0), S)$.

Our main result generalizes the result in [8] and improves the mentioned corollaries in [2] and [10]. It reads as follow.

Theorem 1.1 *Let $f : B \to \mathbb{R}^n$ be a quasiconformal harmonic (qch) mapping, $f(B) = \Omega$, and $\partial\Omega \in C^{1,\alpha}$. Then f is Lipschitz continuous in B.*

The proof of the corresponding result for 2-dimensional case in [8] uses conformal mappings, however conformal mappings in higher-dimensional setting are very rigid, and this is why we need to find another way to deal with the proof of Theorem 1.1. The initial idea lies on the following simple approach. Let $\eta \in S$ and $f(\eta) = q \in \partial\Omega$. We can suppose that $q = 0$ and the tangent plane of q at $\partial\Omega$ is $x_n = 0$. This can be obtained in the following way: Using a isometry L we can postcompose f such that we get a function $\tilde{f}$ from B to Ω', $\tilde{f}(\eta) = 0$ and the tangent plane of this point on $\partial\Omega'$ is $x_n = 0$. Observe that $\tilde{f}$ is also harmonic and quasiconformal, because it is composed by a isometry. The Lipschitz continuity for function $\tilde{f}$ would yield the proof of this property for the function f also, because the isometry preserves the distances.

The proof is given in Section 3. It uses an iteration procedure. Before that, in next section, we give some basic preparations through Theorems 2.1-2.4.

2 Auxiliary Results

The next theorem is of general interest; on the other side it plays an important role in proving Theorem 1.1. Some versions of it for $n = 2$ can be found in [7] and [22].

Theorem 2.1 *Let $u : \overline{B} \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$, be a real harmonic function, $\eta \in S$. Assume that $|u(\xi)-u(\eta)| \leq M|\xi-\eta|^{\mu}$, $\forall \xi \in S$, for some $\mu \in (0,1)$. Then we have $C = C(M, \mu, n)$ such that*

$$|\nabla u(x)|(1-|x|)^{1-\mu} \leq C,$$

where $x = r\eta$, $r \in [0, 1)$.

Proof Throught the proof, the constant C can change its value. Using the Poisson integral formula we have

$$u(x) = \int_S \frac{1-|x|^2}{(1+|x|^2-2\langle\xi,x\rangle)^{\frac{n}{2}}} u(\xi)d\sigma(\xi).$$

Observe that

$$\nabla u(x) = \int_S Q(x,\xi)u(\xi)d\sigma(\xi), \tag{2.1}$$

where

$$\begin{aligned} Q(x,\xi) &= \frac{(-2x)\left(1+|x|^2-2\langle\xi,x\rangle\right)^{\frac{n}{2}} - n(1-|x|^2)\left(1+|x|^2-2\langle\xi,x\rangle\right)^{\frac{n}{2}-1}(x-\xi)}{\left(1+|x|^2-2\langle\xi,x\rangle\right)^{n}} \\ &= \frac{(-2x)\left(1+|x|^2-2\langle\xi,x\rangle\right) - n(1-|x|^2)(x-\xi)}{\left(1+|x|^2-2\langle\xi,x\rangle\right)^{\frac{n}{2}+1}} \\ &= \frac{(-2x)\left(1+|x|^2-2\langle\xi,x\rangle\right) - n(1-|x|^2)(x-\xi)}{\left(1+|x|^2-2\langle\xi,x\rangle\right)} \cdot \frac{1}{\left(1+|x|^2-2\langle\xi,x\rangle\right)^{\frac{n}{2}}}. \end{aligned} \tag{2.2}$$

Let $h \in R^n$ be an arbitrary vector. Then

$$\langle \nabla u(x), h \rangle = \int_S \langle Q(x,\xi), h \rangle u(\xi) d\sigma(\xi). \tag{2.3}$$

Since (2.3) is true for every harmonic function $u : \overline{B} \to \mathbb{R}$, taking the constant function $u(\eta)$, we get

$$0 = \int_S \langle Q(x,\xi), h \rangle u(\eta) d\sigma(\xi), \tag{2.4}$$

which, together with (2.3), gives us

$$\langle \nabla u(x), h \rangle = \int_S \langle Q(x,\xi), h \rangle [u(\xi) - u(\eta)] d\sigma(\xi). \tag{2.5}$$

On the other side

$$\begin{aligned} &\left| \frac{-2\langle x, h\rangle \left(1 + |x|^2 - 2\langle \xi, x\rangle\right) - n(1 - |x|^2)\langle x - \xi, h\rangle}{\left(1 + |x|^2 - 2\langle \xi, x\rangle\right)} \right| \\ &\le 2|x||h| + n \frac{\left(1 - |x|^2\right)|x - \xi||h|}{|x - \xi|^2} \le \\ &= 2|x||h| + 2n|h| \frac{1 - |x|}{|x - \xi|} \le (2 + 2n)|h|. \end{aligned} \tag{2.6}$$

In the last inequality it is used the fact that $1 - |x| \le |x - \xi|$, which is obviously true from the geometrical point of view, but it is also equivalent to $\langle \xi, x\rangle \le |x|$ (Cauchy-Schwarz inequality).

From (2.2), (2.5), (2.6) we get

$$|\langle \nabla u(x), h\rangle| \le (2n+2)|h| \int_S \frac{|u(\xi) - u(\eta)|}{\left(1 + |x|^2 - 2\langle \xi, x\rangle\right)^{\frac{n}{2}}} d\sigma(\xi) \tag{2.7}$$

As h was taken arbitrary, then

$$|\nabla u(x)| \le (2n+2) \int_S \frac{|u(\xi) - u(\eta)|}{\left(1 + |x|^2 - 2\langle \xi, x\rangle\right)^{\frac{n}{2}}} d\sigma(\xi), \tag{2.8}$$

which is equivalent to

$$\begin{aligned} |\nabla u(r\eta)| &\le (2n+2) \int_S \frac{|u(\xi) - u(\eta)|}{\left(1 + r^2 - 2r\langle \xi, \eta\rangle\right)^{\frac{n}{2}}} d\sigma(\xi) \\ &= (2n+2) \int_S \frac{|u(\xi) - u(\eta)|}{\left((1-r)^2 + r|\xi - \eta|^2\right)^{\frac{n}{2}}} d\sigma(\xi), \end{aligned} \tag{2.9}$$

where $x = r\eta$, $r = |x| \in [0, 1)$.

Using the condition of the theorem we get

$$|\nabla u(r\eta)| \le M(2n+2) \int_S \frac{|\xi - \eta|^\mu}{\left((1-r)^2 + r|\xi - \eta|^2\right)^{\frac{n}{2}}} d\sigma(\xi). \tag{2.10}$$

Because of the symmetry, it is enough to show the required inequality for $\eta = (1, 0, \dots, 0)$.

1st Case $r = |x| \geq \frac{1}{2}$.

As the integrand function in (2.10) depends only on the first coordinate of ξ, we use the following representation ([3], Appendix A5):

$$|\nabla u(r\eta)| \leq M(2n+2)C_1 \int_{-1}^{1} \int_{S_{n-2}} \frac{(2-2x)^{\frac{\mu}{2}}}{((1-r)^2 + r(2-2x))^{\frac{n}{2}}} \left(1-x^2\right)^{\frac{n-3}{2}} d\sigma_{n-2}(\zeta) dx,$$

where σ_{n-2} denotes the respective normalized surface-area measure on the unit sphere S_{n-2} in $\mathbb{R}^{n-1}$. The constant C_1 depends on n and the volumes of the unit balls in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$. From this, it follows

$$|\nabla u(r\eta)| \leq C \int_{S_{n-2}} d\sigma_{n-2}(\zeta) \int_{-1}^{1} \frac{(2-2x)^{\frac{\mu}{2}}}{((1-r)^2 + r(2-2x))^{\frac{n}{2}}} \left(1-x^2\right)^{\frac{n-3}{2}} dx \tag{2.11}$$

$$\leq C \int_{-1}^{1} \frac{(2-2x)^{\frac{\mu}{2}}}{((1-r)^2 + r(2-2x))} \frac{2^{\frac{n-3}{2}}(1-x)^{\frac{n-3}{2}}}{((1-r)^2 + r(2-2x))^{\frac{n-2}{2}}} dx$$

$$= C 2^{\frac{\mu}{2}} 2^{\frac{n-3}{2}} \int_{-1}^{1} \frac{(1-x)^{\frac{\mu-1}{2}}}{((1-r)^2 + r(2-2x))} \left(\frac{1-x}{(1-r)^2 + r(2-2x)} \right)^{\frac{n-2}{2}} dx.$$

Since $r \geq \frac{1}{2}$, we easily get

$$\frac{1-x}{(1-r)^2 + r(2-2x)} \leq \frac{1-x}{(1-r)^2 + (1-x)} \leq 1,$$

so

$$|\nabla u(r\eta)| \leq C \int_{-1}^{1} \frac{(1-x)^{\frac{\mu-1}{2}}}{(1-r)^2 + (1-x)} dx. \tag{2.12}$$

First, using the substitution $1 - x = t^2$, then $s = \frac{t}{1-r}$, we have

$$|\nabla u(r\eta)| \leq C \int_0^{\sqrt{2}} \frac{2t^\mu}{(1-r)^2 + t^2} dt = C \frac{(1-r)^\mu}{(1-r)^2} \int_0^{\frac{\sqrt{2}}{1-r}} \frac{2s^\mu}{1+s^2} (1-r) ds,$$

so

$$|\nabla u(r\eta)| \leq C(1-r)^{\mu-1} \int_0^{\infty} \frac{s^\mu}{1+s^2} ds.$$

As the last integral converges we finally have

$$|\nabla u(r\eta)|(1-r)^{1-\mu} \leq C, \tag{2.13}$$

for $r \in [\frac{1}{2}, 1)$, where C depends on M, μ and n only.

2nd Case $r = |x| < \frac{1}{2}$

Here the proof is quite straightforward. Since

$$\frac{|\xi - \eta|^\mu (1-r)^{1-\mu}}{((1-r)^2 + r|\xi - \eta|^2)^{\frac{n}{2}}} < \frac{2^\mu \cdot 1}{\left(\frac{1}{2}\right)^n} = 2^{n+\mu}, \tag{2.14}$$

using (2.10) we get

$$|\nabla u(r\eta)|(1-r)^{1-\mu} \leq M(2n+2)2^{n+\mu}. \tag{2.15}$$

We conclude that the inequality is true for all $r \in [0, 1)$, with the final C being the larger of the obtained constants on the RHS of (2.13) and (2.15). □

The idea of the proof in Section 3 will be based on obtaining locally the C^μ condition of f on the unit sphere for $\mu < 1$, by increasing μ. In relation to a fixed point $\eta \in S$ this will, in one moment, give us a similar inequality as the one from Theorem 2.1, but for $\mu > 1$. So, on this step, we need a different version of the previuos statement which is given in the following theorem. However, the proof of it is very similar to the proof of the previous one.

Theorem 2.2 *Let $u : \overline{B} \subset \mathbb{R}^n \to \mathbb{R}$, be a harmonic function, $\eta \in S$. Assume that $|u(\xi) - u(\eta)| \leq M|\xi - \eta|^\mu$, $\forall \xi \in S$, for some $\mu > 1$. Then we have $C = C(M, \mu, n)$ such that*

$$|\nabla u(r\eta)| \leq C,$$

for every $r \in [0, 1)$.

Proof The proof of the theorem for $r \in [\frac{1}{2}, 1)$ is identical to the previous theorem until (2.12).

$$\int_{-1}^{1} \frac{(1-x)^{\frac{\mu-1}{2}}}{(1-r)^2 + (1-x)} dx \leq \int_{-1}^{1} (1-x)^{\frac{\mu-3}{2}} dx = \frac{2^{\frac{\mu+1}{2}}}{\mu - 1}$$

shows that the inequality is true.

For $r \in [0, \frac{1}{2})$, similar to (2.14) we see that

$$\frac{|\xi - \eta|^\mu}{((1-r)^2 + r|\xi - \eta|^2)^{\frac{n}{2}}}$$

is bounded, so therefore again from (2.10) we have our inequality. □

The next celebrated theorem will also be used. The proof can be found in [5].

Theorem 2.3 (Mori's theorem) *Let g be a K-quasiconformal mapping of B onto B, $n \geq 2$, with $g(0) = 0$. Then*

$$|g(x) - g(y)| \leq M(n, K)|x - y|^\beta,$$

for all $x, y \in B$, where $\beta = K^{\frac{1}{1-n}}$.

We collect now the following useful result. The proof can be found in [25]. We will formulate it in the form which corresponds to our notation and use.

Theorem 2.4 *Let u be a real harmonic function on $\overline{B}$ and $\mu \in (0, 1)$ such that*

$$||u(r\eta)| - |u(\eta)|| \leq C(1-r)^\mu, \forall r \in [0, 1), \eta \in S, \tag{2.16}$$

where C is independent of r and η, then u is μ-Hölder continuous in $\overline{B}$, i.e.:

$$|u(x) - u(y)| \leq M|x - y|^\mu,$$

for all $x, y \in \overline{B}$.

Using the previous theorem we can easily prove the following lemma.

Lemma 2.5 *Let u be a real harmonic function on $\overline{B}$ and $\mu \in (0, 1)$ such that*

$$|\nabla u(r\eta)| \leq C(1-r)^{\mu-1}, \forall r \in (0, 1), \eta \in S,$$

where C does not depend on r and η, then u is μ-Hölder continuous in $\overline{B}$.

Proof To prove this lemma, from Theorem 2.4, it is sufficient to show the relation (2.16).

We have

$$u(\eta) - u(r\eta) = \int_{\gamma_r} D_1 u dx_1 + \ldots + D_n u dx_n, \tag{2.17}$$

where γ_r is the radial segment with endpoints $r\eta$ and η.

Therefore, we have

$$\begin{aligned} ||u(r\eta)| - |u(\eta)|| &\leq |u(r\eta) - u(\eta)| \leq \int_r^1 |\langle \nabla u(t\eta), \eta \rangle| dt \\ &\leq C \int_r^1 (1-t)^{\mu-1} dt \leq C \frac{(1-r)^\mu}{\mu}. \end{aligned} \tag{2.18}$$

□

3 Proof of the Main Result - Theorem 1.1

Proof First, let we prove the Hölder continuity of $f = (f_1, \ldots, f_n)$. Indeed, let G be a quasiconformal diffeomorphism (recall that Ω has $C^{1,\alpha}$ boundary) from B^n to Ω which is Lipschitz continuous mapping up to the boundary, such that $G(0) = f(0)$. Then the mapping $g = G^{-1} \circ f$ is a K' quasiconformal mapping (as a composition of two quasiconformal mappings) of B onto B, where $g(0) = 0$. According to Mori's Theorem 2.3, there exists a constant $M_1(n, K')$ such that

$$|g(x) - g(y)| \leq M_1(n, K') |x - y|^{K'^{\frac{1}{1-n}}},$$

for all $x, y \in B^n$.

As $f = G \circ g$, then f satisfies a similar inequality, being a composition of Lipschitz and Hölder continuous functions:

$$|f(x) - f(y)| \leq C_1 |x - y|^\beta, \tag{3.1}$$

for all $x, y \in \overline{B^n}$, where $\beta \in (0, 1)$, and the constant C_1 depends on M_1 and the Lipschitz constant of G.

In view of the remark after the formulation of Theorem 1.1, there exists a neighbourhood $\mathcal{O}$ of the origin in R^{n-1} which is the projection of $\partial\Omega \cap B(0, \rho)$ in R^{n-1} and a $C^{1,\alpha}$ function $\Phi : \mathcal{O} \to \mathbb{R}$ such that $\partial\Omega \cap B(0, \rho)$ can be expressed as the graph of Φ, i.e. points of $\partial\Omega \cap B(0, \rho)$ are of the form:

$$(\zeta_1, \ldots, \zeta_{n-1}, \Phi(\zeta_1, \ldots, \zeta_{n-1})), \tag{3.2}$$

where $(\zeta_1, \ldots, \zeta_{n-1}) \in \mathcal{O}$.

The function Φ has the properties $\Phi(0, \ldots, 0) = 0$ and $D_j \Phi(0, \ldots, 0) = 0$, for all $j \in \{1, 2, \ldots, n-1\}$, and

$$|\nabla\Phi(\zeta) - \nabla\Phi(\omega)| \leq C_2 |\zeta - \omega|^\alpha. \tag{3.3}$$

The constant C_2 is the same for all points $q \in \partial\Omega$, because of the $C^{1,\alpha}$ condition of $\partial\Omega$. Also,

$$|\Phi(\zeta) - \Phi(\omega)| = |\langle \nabla\Phi(c), \zeta - \omega \rangle| \leq |\nabla\Phi(c)||\zeta - \omega|, \tag{3.4}$$

where c belongs to the segment $[\zeta, \omega]$.

Using (3.3) we get

$$\begin{aligned} |\nabla\Phi(c)| &\le |\nabla\Phi(\zeta)| + |\nabla\Phi(c) - \nabla\Phi(\zeta)| \\ &\le C_2(|\zeta|^\alpha + |c-\zeta|^\alpha) \le C_2\left(|\zeta|^\alpha + |\zeta-\omega|^\alpha\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} |\nabla\Phi(c)| &\le |\nabla\Phi(\omega)| + |\nabla\Phi(c) - \nabla\Phi(\omega)| \\ &\le C_2(|\omega|^\alpha + |c-\omega|^\alpha) \le C_2\left(|\omega|^\alpha + |\zeta-\omega|^\alpha\right), \end{aligned} \tag{3.6}$$

which yields

$$|\nabla\Phi(c)| \le C_2 \min\left\{\{|\zeta|^\alpha, |\omega|^\alpha\} + |\zeta-\omega|^\alpha\right\}.$$

Therefore, from (3.4) we have:

$$|\Phi(\zeta) - \Phi(\omega)| \le C_2|\zeta-\omega|\left(\min\left\{|\zeta|^\alpha, |\omega|^\alpha\right\} + |\zeta-\omega|^\alpha\right), \tag{3.7}$$

for all ζ, ω in $\mathcal{O}$.

Let $F = (F_1, \dots, F_n) = f|_S$ or $P[F] = f$. Notice that F is also C^β in S. We will use the notation $\tilde{F}(\xi) = (F_1(\xi), \dots, F_{n-1}(\xi))$. $\tilde{F}$, as F, also satisfies (3.1). In view of (3.2) we have that in a small neighbourhood of η in S, F_n is of the form

$$F_n(\xi) = \Phi(F_1(\xi), \dots, F_{n-1}(\xi)).$$

We may also assume that this neighbourhood of η is of the form $V(\eta) = B(\eta, \delta) \cap S$, where δ is small enough positive constant for all $q \in \partial\Omega$. Indeed, let $\tilde{U}(q) = B(q, r_q) \cap \partial\Omega$ be the neighbourhood of q in $\partial\Omega$ such that after the isometry L_q (the one that sends q to 0 and which makes the plane $x_n = 0$ the tangent plane of $\partial\Omega$ at point 0), $L_q(\tilde{U}(q))$ is the neighbourhood of 0 which is the graphic of a function as in (3.2). Furthermore, we can choose r_q small enough, such that for every point $p \in \tilde{U}(q)$, the image of $\tilde{U}(q)$ under the respective isometry L_p is a graphic of a function.

Observe now $U(q) = B\left(q, \frac{r_q}{2}\right) \cap \partial\Omega$. The collection $\{U(q)\}_{q\in\partial\Omega}$ is a cover of $\partial\Omega$. As $\partial\Omega$ is compact, there exists a finite subcollection $\{U(q_k)\}_{k=1}^m$ which covers $\partial\Omega$. Let $\rho = \min\left\{\frac{r_{q_1}}{2}, \dots, \frac{r_{q_m}}{2}\right\}$. Since F is continuous on a compact, there is a $\delta > 0$ such that if $|\xi_1 - \xi_2| < \delta$, $\xi_1, \xi_2 \in S$, then $|F(\xi_1) - F(\xi_2)| < \frac{\rho}{2}$. This ensures that the image of every $V(\eta) = B(\eta, \delta) \cap S$ under F is contained in a $B(q_j, r_{q_j}) \cap \partial\Omega = \tilde{U}(q_j)$, and further, after the mentioned isometry is done, this image is the graphic of a function as in (3.2).

We get back to our fixed η, such that $f(\eta) = 0$. Now

$$\begin{aligned} |F_n(\xi) - F_n(\eta)| &= |\Phi\left(\tilde{F}(\xi)\right) - \Phi(0)| \\ &\le C_2|\tilde{F}(\xi)|\left(\min\left\{|\tilde{F}(\xi)|^\alpha, 0\right\} + |\tilde{F}(\xi) - 0|^\alpha\right) \\ &= C_2|\tilde{F}(\xi)|^{1+\alpha} \le C_1^{1+\alpha}C_2|\xi-\eta|^{(1+\alpha)\beta}, \end{aligned} \tag{3.8}$$

for all $\xi \in V(\eta)$. The function F_n is bounded, because $F = f|_S$ is bounded ($|F(\xi)| \le \widetilde{M}$, for all $\xi \in S$), so if $\xi \in S\setminus V(\eta)$ then

$$|F_n(\xi) - F_n(\eta)| \le 2\widetilde{M} \le \frac{2\widetilde{M}}{\delta^{(1+\alpha)\beta}}|\xi-\eta|^{(1+\alpha)\beta}. \tag{3.9}$$

Taking $M = \max\left\{C_1^{1+\alpha}C_2, \frac{2\widetilde{M}}{\delta^{(1+\alpha)\beta}}\right\}$ we get

$$|F_n(\xi) - F_n(\eta)| \le M|\xi-\eta|^{(1+\alpha)\beta}, \tag{3.10}$$

for all $\xi \in S$.

Now, from Theorem 2.1, we have

$$|\nabla f_n(r\eta)| \le C(1-r)^{(1+\alpha)\beta-1}, \forall r \in [0,1).$$

As f is quasiconformal mapping then

$$\frac{\max_{|h_1|=1} |f'(x)h_1|}{\min_{|h_2|=1} |f'(x)h_2|} \le K < \infty, \forall x \in B.$$

Taking, $h_1 = e_j$ and $h_2 = e_n$, for $x = r\eta$ we have

$$|\nabla f_j(r\eta)| \le K|\nabla f_n(r\eta)| \le K \cdot C(1-r)^{(1+\alpha)\beta-1},$$

for all $j \in \{1, \dots, n-1\}$.

This implies

$$|\nabla f_j(r\eta)| \le C(1-r)^{(1+\alpha)\beta-1}, \tag{3.11}$$

where C is a new global constant for all $j \in \{1, \dots, n\}$, and all $r \in [0,1)$.

We want to prove (3.11) in B. Let $\eta_1 \ne \eta$ be an arbitrary point on S and $f(\eta_1) = q_1$. Let L_{q_1} be the isometry that sends q_1 to 0, with $x_n = 0$ being the tangent plane of $L_{q_1}(\partial\Omega)$ at $L_{q_1}(q_1) = 0$.

Let $L_{q_1} \circ f = \tilde{f} = \left(\tilde{f}_1, \dots, \tilde{f}_n\right)$. Then $\tilde{f}$ has all the properties of the function f with η_1 in place of η: at $\tilde{f}(\eta_1) = 0$ the tangent plane of the surface $L_{q_1}(\partial\Omega)$ is $x_n = 0$ and $\tilde{f}(\eta_1)$ has a neighbourhood in $L_{q_1}(\partial\Omega)$ which can be expressed as a part of a graphic of the form (3.2). Using the same procedure, we conclude that

$$|\nabla \tilde{f}_j(r\eta_1)| \le C(1-r)^{(1+\alpha)\beta-1},$$

for all $j \in \{1, \dots, n\}$, and all $r \in [0,1)$. Constant C is universal and it does not depend on η_1, because δ and M are independent of the choice of $\eta \in S$. As $f = L_{q_1}^{-1}\tilde{f}$, $\left(L_{q_1}^{-1} \text{ is also an isometry}\right)$ we get

$$f_j(\xi) = b_j + \sum_{k=1}^{n} a_{j,k}\tilde{f}_k(\xi),$$

$j \in \{1, \dots, n\}$, so

$$\nabla f_j(\xi) = \sum_{k=1}^{n} a_{j,k}\nabla \tilde{f}_k(\xi) \tag{3.12}$$

where $\left\{a_{i,j}\right\}_{1 \le i,j \le n}$ is an orthogonal matrix. From (3.12) we have:

$$\begin{aligned} |\nabla f_j(\xi)| &\le \sum_{k=1}^{n} |a_{j,k}||\nabla \tilde{f}_k(\xi)| \\ &\le \left(\sum_{k=1}^{n} |\nabla \tilde{f}_k(\xi)|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.13}$$

In the last inequality it is used the Cauchy-Schwarz inequality and the orthogonality of matrix $\{a_{j,k}\}_{1 \le j,k \le n}$. Taking $\xi = r\eta_1$ we get

$$|\nabla f_j(r\eta_1)| \le \sqrt{n}C(1-r)^{(1+\alpha)\beta-1}.$$

As the point η_1 was arbitrary we conclude

$$|\nabla f_j(x)| \le C(1-r)^{(1+\alpha)\beta-1}, r = |x|,$$

for all $x \in B$.

From Lemma 2.5 it follows that $f_j \in C^{(1+\alpha)\beta}(\overline{B})$, for all $j \in \{1, \dots, n\}$ and so $f \in C^{(1+\alpha)\beta}(\overline{B})$.

We could have chosen $\beta < \frac{1}{2}$ (by decreasing it, if necessary) so the numbers $(1+\alpha)^k\beta \neq 1$, for every k. As $1+\alpha > 1$ there exists a unique integer k_0 such that $(1+\alpha)^{k_0}\beta < 1$ and $(1+\alpha)^{k_0+1}\beta > 1$. Repeating the procedure, we get that $f \in C^{(1+\alpha)^2\beta}(\overline{B}), \dots, C^{(1+\alpha)^{k_0}\beta}(\overline{B})$. Note that such procedure for two-dimensional setting and different purpose has been used in [22] and in [14]. Similar to (3.8) it follows that $|F_n(\xi) - F_n(\eta)| \le M|\xi - \eta|^{(1+\alpha)^{k_0+1}\beta}$, $\forall \xi \in S$. This time, using Theorem 2.2 we obtain

$$|\nabla f_n(r\eta)| \le C, \forall r \in [0, 1).$$

Using the same order of implications, first we get the same inequality for every f_k on points $r\eta$. Then, using the isometries, we get the inequality on every point of B for a global constant C. This implies trivially, by mean value inequality, the Lipschitz continuity of function f in $\overline{B}$. □

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