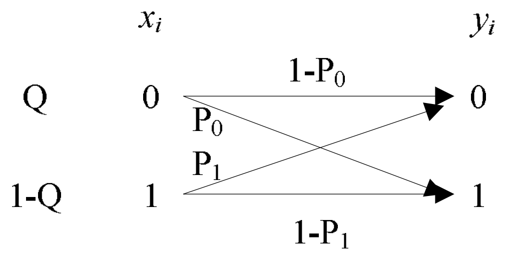
Vježbe 6

1. Odrediti kapacitet binarnog nesimetričnog kanala.

Rješenje:



$C = \max I(X;Y)$,

$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$,

$H(Y) = -\sum_{i=1}^{N} p(y_i)\log p(y_i) = -P(Y=0)\log P(Y=0) - P(Y=1)\log P(Y=1)$.

Razmotrimo $p(y)$:

$P(Y=0) = Q(1-P_0) + (1-Q)P_1$,

$P(Y=1) = (1-Q)(1-P_1) + QP_0$,

pa uvrštavanjem u formulu za entropiju dobijamo:

$$H(Y) = -[(Q(1-P_0)+(1-Q)P_1)\log(Q(1-P_0)+(1-Q)P_1) + ((1-Q)(1-P_1)+QP_0)\log((1-Q)(1-P_1)+QP_0)]$$

Posmatrajmo uslovnu entropiju:

$$\begin{aligned}
H(Y|X) &= -\sum_x\sum_y p(x,y)\log p(y|x) = -\sum_x\sum_y p(x)p(y|x)\log p(y|x) = \\
&= -\sum_y (p(0)p(y|0)\log p(y|0) + p(1)p(y|1)\log p(y|1)) = \\
&= -[p(0)p(0|0)\log p(0|0) + p(0)p(1|0)\log p(1|0) + p(1)p(0|1)\log p(0|1) + p(1)p(1|1)\log p(1|1)] = \\
&= -[Q(1-P_0)\log(1-P_0) + QP_0\log P_0 + (1-Q)P_1\log P_1 + (1-Q)(1-P_1)\log(1-P_1)] = \\
&= -Q[(1-P_0)\log(1-P_0) + P_0\log P_0] - (1-Q)[P_1\log P_1 + (1-P_1)\log(1-P_1)].
\end{aligned}$$

$$\begin{aligned}
I(X;Y) &= H(Y) - H(Y|X) = \\
&= -[(Q(1-P_0)+(1-Q)P_1)\log(Q(1-P_0)+(1-Q)P_1) + ((1-Q)(1-P_1)+QP_0)\log((1-Q)(1-P_1)+QP_0)] + \\
&\quad + Q[(1-P_0)\log(1-P_0) + P_0\log P_0] + (1-Q)[P_1\log P_1 + (1-P_1)\log(1-P_1)].
\end{aligned}$$

Imajući u vidu definiciju kapaciteta kanala, tražimo prvi izvod združene informacije:

*** $(\log y)' = \dfrac{y'}{y}$

$$\frac{dI}{dQ}=-[(1-P_0-P_1)\log(Q(1-P_0)+(1-Q)P_1)+(Q(1-P_0)+(1-Q)P_1)\frac{(1-P_0-P_1)}{(Q(1-P_0)+(1-Q)P_1)}]-$$

$$-[(-1+P_1+P_0)\log((1-Q)(1-P_1)+QP_0)+((1-Q)(1-P_1)+QP_0)\frac{(-1+P_1+P_0)}{((1-Q)(1-P_1)+QP_0)}]+$$

$$+[(1-P_0)\log(1-P_0)+P_0\log P_0]-[P_1\log P_1+(1-P_1)\log(1-P_1)]=$$

$$=-(1-P_0-P_1)\log(Q(1-P_0)+(1-Q)P_1)-(1-P_0-P_1)+$$

$$+(-1+P_1+P_0)\log((1-Q)(1-P_1)+QP_0)+(1-P_0-P_1)-H(P_0)+H(P_1)=$$

$$=(1-P_0-P_1)\log\frac{(1-Q)(1-P_1)+QP_0}{Q(1-P_0)+(1-Q)P}-H(P_0)+H(P_1)=0$$

Odnosno:

$$\log\frac{(1-Q)(1-P_1)+QP_0}{Q(1-P_0)+(1-Q)P}=\frac{H(P_0)-H(P_1)}{(1-P_0-P_1)}$$

Uvedimo da je:

$$A=\frac{H(P_0)-H(P_1)}{(1-P_0-P_1)}\ \Rightarrow 2^A=\frac{(1-Q)(1-P_1)+QP_0}{Q(1-P_0)+(1-Q)P}\ \Rightarrow Q=\frac{1-P_1-2^A P_1}{(1-P_0-P_1)(1+2^A)}$$

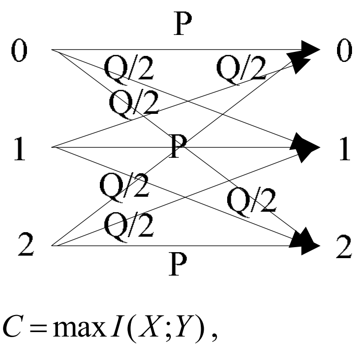
$$\Rightarrow Q=\frac{1-P_1-2^{\frac{H(P_0)-H(P_1)}{(1-P_0-P_1)}}P_1}{(1-P_0-P_1)(1+2^{\frac{H(P_0)-H(P_1)}{(1-P_0-P_1)}})}$$

Uvrštavanjem ove vrijednosti Q u formulu za kapacitet kanala i sređivanjem izraza,dobija se njegova konačna vrijednost:

$$C=\log\left[1+2^{\frac{H(P_0)-H(P_1)}{1-P_0-P_1}}\right]-\frac{H(P_0)(1-P_1)-H(P_1)P_0}{1-P_0-P_1}.$$

2. Odrediti kapacitet kanala koji prenosi ternarne poruke i napisati njegovu tranzicionu matricu. Vjerovatnoća prenosa ispravnog simbola je P dok je vjerovatnoća prenosa pogrešnog Q (po Q/2 da će biti prenijeta ostala dva simbola).

Rješenje:



$C=\max I(X;Y)$,

$I(X;Y)=H(Y)-H(Y|X)$.

Ranije je dokazano da je:

$$H(Y\mid X)=\sum_{x\in A}p(x)H(Y\mid X=x)=$$

$$=P(X=0)\mathrm{H}(Y\mid X=0)+P(X=1)H(Y\mid X=1)+P(X=2)H(Y\mid X=2).$$

Posmatrajmo pojedinačno uslovne vjerovatnoće:

$$\mathrm{H}(Y|X=0)=-P(Y=0|X=0)\log P(Y=0|X=0)-P(Y=1|X=0)\log P(Y=1|X=0)-$$

$$-P(Y=2|X=0)\log P(Y=2|X=0)=-P\log P-\frac{Q}{2}\log\frac{Q}{2}-\frac{Q}{2}\log\frac{Q}{2}=$$

$$=-P\log P-Q\log\frac{Q}{2}.$$

Lako ćemo uočiti da su:

$$\mathrm{H}(Y|X=0)=\mathrm{H}(Y|X=1)=\mathrm{H}(Y|X=2)=-P\log P-Q\log\frac{Q}{2}=-P\log P-Q\log Q+Q\log 2=$$

$$=H(P)+Q\log 2=H(P)+Q.$$

Imajući u vidu da je: $P(X=0)+P(X=1)+P(X=2)=1$, jer je riječ o potpunom sistemu događaja, dobijamo:

$$H(Y\mid X)=\mathrm{H}(Y\mid X=0)(P(X=0)+P(X=1)+P(X=2))=\mathrm{H}(Y\mid X=0).$$

Dalje je:

$$I(X;Y)=H(Y)-H(Y|X)=H(Y)-H(P)-Q,$$

odnosno:

$$C=\max I(X;Y)=\log 3-H(P)-Q.$$

Tranziciona matrica je:

$$\begin{bmatrix} P & Q/2 & Q/2 \\ Q/2 & P & Q/2 \\ Q/2 & Q/2 & P \end{bmatrix}$$