Интеграција рационалних функција

I $\displaystyle\int \frac{Ax+B}{\underbrace{ax^2+bx+c}}\,dx \quad , \quad \int \frac{dx}{ax^2+bx+c}$

нема нула
несводљив

1. $\displaystyle\int \frac{3x-2}{5x^2-3x+2}\,dx = \int \frac{\frac{3}{10}(10x-3)-2+\frac{9}{10}}{5x^2-3x+2}\,dx =$

$\boxed{D<0}$

$\displaystyle = \frac{3}{10}\int \frac{10x-3}{5x^2-3x+2}\,dx - \frac{11}{10}\int \frac{dx}{5x^2-3x+2} =$

$= \left\lceil \begin{array}{l} 5x^2-3x+2 = t \\ 10x-3\,dx = dt \end{array} \right\rfloor$

$\displaystyle \left\lceil\, 5x^2-3x+2 = 5\left(\underbrace{x^2-\frac{3}{5}x}+\frac{2}{5}\right) = \right.$

$\displaystyle = 5\left(x^2-\frac{3}{5}x+\frac{9}{100}+\frac{2}{5}-\frac{9}{100}\right) =$

$\displaystyle = 5\cdot\left[\left(x-\frac{3}{10}\right)^2+\frac{31}{100}\right]$

$$\frac{3}{10}\int\frac{dt}{t} - \frac{11}{10}\int\frac{1\,dx}{5\cdot\left[\left(x-\frac{3}{10}\right)^2+\frac{31}{100}\right]} = \qquad \left[\left(x-\frac{3}{10}\right)^2 = \frac{31}{100}z^2\right]$$

$$= \frac{3}{10}\ln|t| - \frac{11}{50}\cdot\int\frac{dx}{\left(x-\frac{3}{10}\right)^2+\frac{31}{100}} =$$

$$\frac{3}{10}\ln|t| - \frac{11}{50}\int\frac{dx}{\left(x-\frac{3}{10}\right)^2+\frac{31}{100}} = \qquad \left[x-\frac{3}{10} = \frac{\sqrt{31}}{10}\cdot z \qquad dx = \frac{\sqrt{31}}{10}\cdot dz \qquad z = \frac{10x-3}{\sqrt{31}}\right]$$

$$\frac{3}{10}\ln|t| - \frac{11}{50}\cdot\frac{\sqrt{31}}{10}\int\frac{dz}{\frac{31}{100}\cdot z^2+\frac{31}{100}} =$$

$$\frac{3}{10}\ln|t| - \frac{11}{5\cancel{00}}\cdot\frac{\sqrt{31}}{}\cdot\frac{1\cancel{00}}{31}\cdot\int\frac{dz}{z^2+1} =$$

$$\frac{3}{10}\ln|t| - \frac{11\sqrt{31}}{5\cdot 31}\operatorname{arctg} z + C$$

$$\frac{3}{10}\ln|5x^2-3x+2| - \frac{11\cdot\sqrt{31}}{155}\cdot\operatorname{arctg}\left(\frac{10x-3}{\sqrt{31}}\right)+C$$

(2.) $$\int\frac{3-4x}{x^2-4x+5}\,dx$$

(3.) $$\int\frac{3x-2}{x^2+2x+1}\,dx = \int\frac{3x-2}{(x+1)^2}\,dx = \qquad \left[x+1=t \Rightarrow x=t-1 \qquad dx=dt\right]$$

$\boxed{D=0}$

$$= \int\frac{3(t-1)-2}{t^2}\,dt = \int\frac{3t-5}{t^2}\,dt =$$

$$= 3\int\frac{1}{t}\,dt - 5\int\frac{1}{t^2}\,dt =$$

$$t^{-2+1} \qquad \frac{t^{-1}}{-1}$$

$$= 3\ln|t| - 5\cdot\left(-\frac{1}{t}\right)+C$$

$$= 3\ln|x+1| + \frac{5}{x+1}+C$$

$\int \frac{\quad}{Q(x)}\,dx$,

МЕТОД НЕОДРЕЂЕНИХ КОЕФИЦЈЕНАТА

$Q(x)$ – ИЗВРШИТИ ФАКТОРИЗАЦИЈУ

(1) $\int \frac{2x+1}{x^2-x-2}\,dx$

$x^2-x-2=0$

$x_1/x_2 = \frac{1 \pm \sqrt{1+8}}{2}$

$x_1 = 2 \qquad x_2 = -1$

$x^2-x-2 = 1(x-2)(x+1)$

$\frac{2x+1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$\frac{2x+1}{(x-2)(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$

$2x+1 = A(x+1) + B(x-2)$

$2x+1 = (A+B)x + A - 2B$

$A+B=2$

$A-2B=1 \quad / \cdot (-1)$

$A+B=2 \qquad A=\frac{5}{3}$

$-3B=-1 \Rightarrow B=\frac{1}{3}$

$\int \frac{2x+1}{(x-2)(x-1)}\,dx = \frac{5}{3}\int \frac{1}{(x-2)}\,dx + \frac{1}{3}\int \frac{1}{(x+1)}\,dx$

$\frac{2x^3-4x+1}{(x-7)^2 \cdot (x+2)^3} =$

$= \frac{A}{x-7} + \frac{B}{(x-7)^2} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2}$

$\frac{5x^2-4x+2}{(x-1)^2 \cdot (x^2-3x+5)} =$ $\longrightarrow$ несводљив

$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2-3x+5}$

$$= \frac{5}{3} \cdot \ln|x-2| + \frac{1}{3} \ln|x+1| + c$$

⑤ $$\int \frac{x}{x^3+x^2-x-1}\,dx$$

$$(x^3+x^2-x-1) : (x-1) = x^2+2x+1$$
$$\underline{-x^3 \mp x^2}$$
$$2x^2-x-1$$
$$\underline{-2x^2 \mp 2x}$$
$$x-1$$
$$\underline{-x \mp 1}$$
$$0$$

$$x^3+x^2-x-1 = (x-1)(x^2+2x+1) =$$
$$= (x-1)(x+1)^2$$

$$\frac{x}{x^3+x^2-x-1} = \frac{x}{(x-1)(x+1)^2}$$

$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\frac{x}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$x = (A+B)\cdot x^2 + (2A+C)\cdot x + A - B - C$$

$$A+B=0 \qquad A=\frac{1}{4} \quad B=-\frac{1}{4} \quad C=\frac{1}{2}$$
$$2A+C=1$$
$$A-B-C=0$$

$$\int \frac{x}{x^3+x^2-x-1}\,dx = \frac{1}{4}\int\frac{dx}{x-1} - \frac{1}{4}\int\frac{dx}{x+1} + \frac{1}{2}\int\frac{dx}{(x+1)^2} =$$
$$= \left| \begin{array}{l} x+1=t \\ dx=dt \end{array} \right|$$

$\frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\int\frac{dt}{t^2} =$

$\frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\cdot\frac{1}{x+1} + c$

(6.) $\int \frac{x^4-3x}{x^3+1}\,dx$

$$\begin{array}{l} (\cancel{x^4}-3x) : (x^3+1) = x \\ -\cancel{x^4} \pm x \\ \hline \quad -4x \end{array}$$

$x^4-3x = x(x^3+1)-4x$

$\frac{x^4-3x}{x^3+1} = x - \frac{4x}{x^3+1}$

$\int\frac{x^4-3x}{x^3+1}\,dx = \int\left(x - \frac{4x}{x^3+1}\right)dx =$

$= \int x\,dx - 4\cdot\int\frac{x}{x^3+1}\,dx \overset{(*)}{=}$

(*)

(*) $\int\frac{-4x}{x^3+1}\,dx = \int\frac{4x}{(x+1)(x^2-x+1)}\,dx$

$\frac{-4x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$\frac{-4x}{(x+1)(x^2-x+1)} = \frac{A(x^2-x+1)+(Bx+C)\cdot(x+1)}{(x+1)(x^2-x+1)}$

$-4x = (A+B)x^2 + (-A+B+C)\cdot x + A + C$

$A+B=0$

$-A+B+C=-4$

$A+C=0$

$A=\frac{4}{3}$ $B=-\frac{4}{3}$ $C=-\frac{4}{3}$

$$\int \frac{-4x}{x^3+1}\,dx = \frac{4}{3}\int\frac{dx}{x+1} - \frac{4}{3}\int\frac{x+1}{x^2-x+1}\,dx \qquad x^2-x+1$$

$$= \frac{4}{3}\ln|x+1| - \frac{4}{3}\int \frac{\frac{1}{2}(2x-1)+1+\frac{1}{2}}{x^2-x+1}\,dx =$$

$$= \frac{4}{3}\ln|x+1| - \frac{2}{3}\int\frac{2x-1}{x^2-x+1}\,dx - 2\int\frac{dx}{x^2-x+1} =$$

$$= \frac{4}{3}\ln|x+1| - \frac{2}{3}\int\frac{2x-1}{x^2-x+1}\,dx - 2\int\frac{dx}{x^2-x+1} =$$

$x^2-x+1=t$

$(2x-1)\,dx = dt$

$x^2-x+1 = x^2-x+\frac{1}{4}+1-\frac{1}{4} = \left(x-\frac{1}{2}\right)^2+\frac{3}{4}$

$$= \frac{4}{3}\ln|x+1| - \frac{2}{3}\int\frac{dt}{t} - 2\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} =$$

$x-\frac{1}{2} = \frac{\sqrt{3}}{2}\cdot z$

$dx = \frac{\sqrt{3}}{2}\cdot dz$

$z = \frac{2x-1}{\sqrt{3}}$

$$= \frac{4}{3}\ln|x+1| - \frac{2}{3}\ln|t| - \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}}\int\frac{dz}{\frac{3}{4}z^2+\frac{3}{4}} =$$

$$= \frac{4}{3}\ln|x+1| - \frac{2}{3}\ln|t| - \frac{4\sqrt{3}}{3}\cdot\int\frac{dz}{z^2+1} =$$

$$= \frac{4}{3}\ln|x+1| - \frac{2}{3}\ln|x^2-x+1| - \frac{4\sqrt{3}}{3}\cdot \operatorname{arctg} z + C$$

$$= \frac{4}{3}\ln|x+1| - \frac{2}{3}\ln|x^2-x+1| - \frac{4\sqrt{3}}{3}\cdot\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

$$(*)\quad \int x\cdot dx - 4\int\frac{x}{x^3+1} = \frac{x^2}{2} + \frac{4}{3}\ln|x+1| - \frac{2}{3}\ln|x^2-x+1| - \frac{4\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

Интеграција ирационалних функција

$\frac{1}{273}, 16$

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}, \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

цијели квадрат

(1.) $$\int \frac{x+3}{\sqrt{4x^2+4x+3}}\,dx = \int \frac{\frac{1}{8}(8x+4)+3-\frac{1}{2}}{\sqrt{4x^2+4x+3}}\,dx =$$

$$= \frac{1}{8}\int \frac{8x+4}{\sqrt{4x^2+4x+3}}\,dx + \frac{5}{2}\int \frac{dx}{\sqrt{4x^2+4x+3}} =$$

$4x^2+4x+3 = t$

$(8x+4)dx = dt$

$4x^2+4x+3 = 4\left(x^2+x+\frac{3}{4}\right) =$

$= 4\left(x^2+\frac{1}{4}x+\frac{1}{4}+\frac{3}{4}-\frac{1}{4}\right) =$

$= 4\cdot\left[\left(x+\frac{1}{2}\right)^2+\frac{1}{2}\right]$

$$= \frac{1}{8}\int \frac{dt}{\sqrt{t}} + \frac{5}{2}\frac{1}{\sqrt{4}}\cdot\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{1}{2}}} =$$

$= x+\frac{1}{2} = \frac{1}{\sqrt{2}}z \Rightarrow z = \frac{2x+1}{\sqrt{2}}$

$dx = \frac{1}{\sqrt{2}}dz$

$$\frac{1}{8}\int t^{-\frac{1}{2}}dt + \frac{5}{\cancel{8}}\cdot\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{\frac{1}{2}z^2+\frac{1}{2}}}$$

$$\frac{1}{8}\cdot 2\cdot\sqrt{t}+\frac{5}{4}\cdot\frac{1}{\cancel{\sqrt{2}}}\cdot\cancel{\sqrt{2}}\int\frac{dz}{\sqrt{z^2+1}}=$$

$$\frac{1}{4}\sqrt{t}+\frac{5}{4}\ln\left|z+\sqrt{z^2+1}\right|+C$$

$$\frac{1}{4}\sqrt{4x^2+4x+3}+\frac{5}{4}\ln\left|\frac{2x+1}{\sqrt{2}}+\sqrt{\left(\frac{2x+1}{\sqrt{2}}\right)^2+1}\right|+C$$

(2.) $$\int\frac{dx}{\sqrt{1-x-3x^2}}=$$

$$\ulcorner -3x^2-x+1=-3\left(x^2+\frac{1}{3}x-\frac{1}{3}\right)=$$

$$=-3\left(x^2+\frac{1}{3}x+\frac{1}{36}-\frac{1}{3}-\frac{1}{36}\right)$$

$$=-3\cdot\left[\left(x+\frac{1}{6}\right)^2-\frac{13}{36}\right]=$$

$$=3\cdot\left[\frac{13}{36}-\left(x+\frac{1}{6}\right)^2\right]\lrcorner$$

$$=\int\frac{dx}{\sqrt{3\cdot\left[\frac{13}{36}-\left(x+\frac{1}{6}\right)^2\right]}}=\frac{1}{\sqrt{3}}\cdot\int\frac{dx}{\sqrt{\frac{13}{36}-\left(x+\frac{1}{6}\right)^2}}=$$

$$\ulcorner x+\frac{1}{6}=\frac{\sqrt{13}}{6}\cdot t$$

$$dx=\frac{\sqrt{13}}{6}\cdot dt$$

$$t=\frac{6x+1}{\sqrt{13}}\lrcorner$$

$$=\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{13}}{6}\cdot\int\frac{dt}{\sqrt{\frac{13}{36}-\frac{13}{36}\cdot t^2}}=$$

$$=\frac{1}{\sqrt{3}}\cdot\frac{\cancel{6}}{\cancel{\sqrt{13}}}\cdot\frac{\cancel{\sqrt{13}}}{\cancel{6}}\cdot\int\frac{dt}{\sqrt{1-t^2}}=$$

$$=\frac{\sqrt{3}}{3}\cdot\arcsin t+C=$$

$$=\frac{\sqrt{3}}{3}\cdot\arcsin\left(\frac{6x+1}{\sqrt{13}}\right)+C$$

II група

a) $\int R\left(x, x^{\frac{m_1}{n_1}}, x^{\frac{m_2}{n_2}}, \ldots, x^{\frac{m_s}{n_s}}\right) \cdot dx$

смјена $x = t^k$, $k =$ најмањи заједнички садржалац $(n_1, \ldots, n_s)$

(1.) $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \begin{bmatrix} x^{\frac{1}{2}}, x^{\frac{1}{3}} \\ x = t^6 \\ dx = 6t^5 dt \\ x^{\frac{1}{2}} = t^3 \\ x^{\frac{1}{3}} = t^2 \end{bmatrix}$

подијели

$\int \frac{6t^5 dt}{t^3 + t^2} = 6 \cdot \int \frac{t^5 dt}{t^2(t+1)} = 6 \int \frac{t^3}{t+1} dt =$

$= 6 \int \frac{t^3 + 1 - 1}{t+1} dt = 6 \int \frac{t^3 + 1}{t+1} dt - 6 \int \frac{dt}{t+1} =$

$6 \int \frac{\cancel{(t+1)}(t^2 - t + 1)}{\cancel{(t+1)}} dt - 6 \ln|t+1| =$

$= 6 \int t^2 \cdot dt - 6 \int t dt + 6 \int dt - 6 \ln|t+1| =$

$= 6 \cdot \frac{t^3}{3} - 6 \cdot \frac{t^2}{2} + 6t - 6 \ln|t+1| + C =$

$= 2\sqrt{x} - 3\sqrt[3]{x} + 6 \cdot \sqrt[6]{x} - 6 \ln|\sqrt[6]{x} + 1| + C$

① $\int R\left(x, (ax+b)^{\frac{m_1}{n_1}}, (ax+b)^{\frac{m_2}{n_2}}, \dots (ax+b)^{\frac{m_s}{n_s}}\right) dx$

смјена $ax+b=t^k$, $k = \text{НЗС}(n_1, \dots n_s)$.

② $\int \frac{x^2+\sqrt{1+x}}{\sqrt[3]{1+x}}\,dx = \ulcorner x, (1+x)^{\frac{1}{2}}, (1+x)^{\frac{1}{3}}$

$1+x = t^6$

$t = \sqrt[6]{1+x}$ $\qquad x = t^6 - 1$

$dx = 6t^5 dt$

$(1+x)^{\frac{1}{2}} = t^3$

$(1+x)^{\frac{1}{3}} = t^2 \lrcorner$

$\int \frac{(t^6-1)^2 + t^3}{t^2}\, 6 \cdot t^5 dt =$ $\qquad (1+x)^{\frac{1}{6}} = t \;/^{16}$

$(1+x)^{\frac{16}{6}} = t^{16}$

$6\int t^3 (t^{12} - 2\cdot t^6 + 1 + t^3)\,dt =$ $\qquad (1+x)^{\frac{8}{3}} = \sqrt[3]{(1+x)^8}$

$6\int t^{15}dt - 12\int t^9 \cdot dt \quad 6\cdot\int dt + 6\int t^6 dt$

$6\cdot\frac{1}{16}\cdot t^{16} - 12\cdot\frac{t^{10}}{10} + 6\cdot t + 6\cdot\frac{t^7}{7} + C =$

$= \frac{3}{8}\cdot\sqrt[3]{(1+x)^8} - \frac{6}{5}\cdot\sqrt[3]{(1+x)^5} + 6\cdot\sqrt[6]{1+x} + \frac{6}{7}\cdot\sqrt[6]{(1+x)^7} +$

⑥ $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \dots\dots, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_s}{n_s}}\right) dx$

смена
$\frac{ax+b}{cx+d} = t^k$
$k = НЗS(n_1 \dots n_s)$

⑤ $\int \sqrt{\frac{x+1}{x-1}}\, dx =$

$\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}$

$\frac{x+1}{x-1} = t^2$

$x+1 = xt^2 - t^2$

$(t^2-1)\cdot x = 1+t^2$

$x = \frac{1+t^2}{t^2-1}$

$dx = \frac{2t(t^2-1) - (1+t^2)\cdot 2t}{(t^2-1)^2} \cdot dt$

$dx = \frac{-4t}{(t^2-1)^2} \cdot dt$

$\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} = t$

$^{*}\frac{t^2}{(t-1)^2(t+1)^2} =$
$= \frac{A_1}{t-1} + \frac{A_2}{(t-1)^2} + \frac{B_1}{t+1} + \frac{B_2}{(t+1)}$

$\int t \cdot \frac{-4t}{(t^2-1)^2} dt = -4\int t \cdot \frac{t}{(t^2-1)^2}{}^{*} dt =$

$u = t \Rightarrow du = dt$
$V = \int \frac{t}{(t^2-1)^2} dt$
смена
$t^2 - 1 = z$
$2t\,dt = dz$
$t\,dt = \frac{dz}{2}$

$\frac{1}{2}\int \frac{dz}{z^2} = -\frac{1}{2} \cdot \frac{1}{z} = -\frac{1}{2} \cdot \frac{1}{t^2-}$

$= -4\left(\frac{-t}{2\cdot(t^2-1)} + \frac{1}{2}\int \frac{dt}{t^2-1}\right) = \frac{2t}{t^2-1} - 2 \cdot \frac{1}{2} \ln\left|\frac{t-1}{t+1}\right| +$

⑥

$$2\cdot\sqrt{\frac{x+1}{x-1}}\cdot\frac{1}{\frac{x+1}{x-1}-1}-\ln\left|\frac{\sqrt{\frac{x+1}{x-1}}-1}{\sqrt{\frac{x+1}{x-1}}+1}\right|+C=$$

$$\cancel{2}\cdot\sqrt{\frac{x+1}{x-1}}\cdot\frac{x-1}{\cancel{2}}-\ln\left|\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right|+C$$

$$\sqrt{\frac{(x+1)\cdot(x-1)^2}{x-1}}-\ln\left|\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right|+C$$

$$\sqrt{x^2-1}-\ln\left|\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right|+C$$

III

$$\int\frac{Mx+N}{(x-\alpha)^k\cdot\sqrt{ax^2+bx+c}}\cdot dx$$

смена ; $x-\alpha=\frac{1}{t}$

6. $$\int\frac{x}{(x-1)^2\cdot\sqrt{-x^2+2x+1}}\,dx=$$

$$x-1=\frac{1}{t} \qquad x=\frac{1}{t}+1=\frac{t+1}{t}$$

$$dx=-\frac{1}{t^2}\,dt$$

$$t=\frac{1}{x-1}$$

$$=\int\frac{\frac{t+1}{t}}{\cancel{\frac{1}{t^2}}\cdot\sqrt{-\left(\frac{t+1}{t}\right)^2+2\cdot\left(\frac{t+1}{t}\right)+1}}\cdot\left(-\cancel{\frac{1}{t^2}}\right)\cdot dt=$$

$$-\int\frac{\frac{t+1}{t}}{\sqrt{\frac{\cancel{-t^2}-2t-1+2t^2+\cancel{2t}+\cancel{t^2}}{t^2}}}\,dt=$$

$\sqrt{2}\cdot t=y$

$$-\int\frac{t+1}{\sqrt{2t^2-1}}\,dt=$$

$$-\int\frac{t}{\sqrt{2t^2-1}}\,dt+\int\frac{dt}{\sqrt{2t^2-1}}=$$

смяна

$2t^2-1=z$

$4t\,dt=dz$

$t\,dt=\frac{1}{4}dz$

$\sqrt{2}\cdot t=y$

$dt=\frac{1}{\sqrt{2}}\cdot dy$

$$=-\frac{1}{4}\int\frac{dz}{\sqrt{z}}-\frac{1}{\sqrt{2}}\cdot\int\frac{dy}{\sqrt{y^2-1}}=$$

$$=\frac{1}{4}\cdot 2\sqrt{z}-\frac{1}{\sqrt{2}}\cdot\ln\left|y+\sqrt{y^2-1}\right|+C$$

$$=\frac{1}{\cancel{4}}\cdot\cancel{2}\sqrt{2\left(\frac{1}{x-1}\right)^2-1}-\frac{1}{\sqrt{2}}\cdot\ln\left|\sqrt{2}\cdot\frac{1}{x-1}+\sqrt{2\cdot\left(\frac{1}{x-1}\right)^2-1}\right|+C$$

$$=\frac{1}{2}\cdot\ \ldots$$

МЕТОД ОСТРОГРАДСКОГ

IV $$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c} + k\cdot\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

1. $$\int \frac{x^2}{\sqrt{x^2+x+1}}\,dx = (Ax+B)\cdot\sqrt{x^2+x+1} + C\cdot\int\frac{dx}{\sqrt{x^2+x+1}}$$

$$\frac{x^2}{\sqrt{x^2+x+1}} = A\cdot\sqrt{x^2+x+1} + (Ax+B)\cdot\frac{1}{2\sqrt{x^2+x+1}}\cdot(2x+1)$$

$$+ C\,\frac{1}{\sqrt{x^2+x+1}} =$$

$$\frac{x^2}{\sqrt{x^2+x+1}} = \frac{A(x^2+x+1) + \frac{(Ax+B)}{2} \cdot (2x+1) + C}{\sqrt{x^2+x+1}}$$

$$x^2 = 2Ax^2 + \left(\frac{3A}{2} + B\right)x + A + \frac{B}{2} + C$$

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$\frac{3}{2}A + B = 0 \Rightarrow B = -\frac{3}{4}$$

$$A + \frac{B}{2} + C = 0 \Rightarrow C = -\frac{1}{8}$$

$$\int \frac{x^2}{\sqrt{x^2+x+1}}\,dx = \left(\frac{1}{2}x - \frac{3}{4}\right) \cdot \sqrt{x^2+x+1} - \frac{1}{8}\int \frac{dx}{\sqrt{x^2+x+1}}^{\,*}$$

$$* \int \frac{dx}{\sqrt{x^2+x+1}} = \quad \left[\ x^2+x+1 = x^2+x+\frac{1}{4}+1-\frac{1}{4} = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\ \right]$$

$$= \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}} = \quad \left[\ x + \frac{1}{2} = \frac{\sqrt{3}}{2} \cdot z \quad dx = \frac{\sqrt{3}}{2} \cdot dz \quad z = \frac{2x+1}{\sqrt{3}}\ \right]$$

$$= \frac{\sqrt{3}}{2}\int \frac{dz}{\sqrt{\frac{3}{4}z^2 + \frac{3}{4}}} = \cancel{\frac{\sqrt{3}}{2}} \cdot \cancel{\frac{2}{\sqrt{3}}} \cdot \int \frac{dz}{\sqrt{z^2+1}} =$$

$$= \ln\left|z + \sqrt{z^2+1}\right| + C = \ln\left|\frac{2x+1}{\sqrt{3}} + \sqrt{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1}\right| + C$$

V $\int x^m \cdot (a+bx^n)^p dx$, a, b – реални бројеви
m, n, p – рационални бројеви

ИНТЕГРАЦИЈА ДИФЕРЕНЦИЈАЛНОГ БИНОМА

1° p – цио број

а) $p > 0$

$(a+bx^n)^p$ – растављамо по биномној формули

б) $p < 0$

смјена $x = t^k$ гдје је k – НЗС имениоца бројева m и n

2° $\frac{m+1}{n} \in \mathbb{Z}$

ако јесте ⇒ смјена $a + bx^n = t^k$, k – имениоц броја p

3° $\frac{m+1}{n} + p \in \mathbb{Z}$

смјена $\frac{a}{x^n} + b = t^k$ k – имениоц броја p

1. $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} \cdot (1+x^{\frac{1}{4}})^{\frac{1}{3}} dx =$

$m = -\frac{1}{2}$

$n = \frac{1}{4}$

$p = \frac{1}{3}$

$a = 1$

$b = 1$

$\frac{m+1}{n} = \frac{-\frac{1}{2}+1}{\frac{1}{4}} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2 \in \mathbb{Z}$

$1 + x^{\frac{1}{4}} = t^3$

$x^{\frac{1}{4}} = t^3 - 1 \quad /^4$

$x = (t^3-1)^4$

$dx = 4(t^3-1)^3 \cdot 3t^2 dt$

$dx = 12t^2(t^3-1)^3 dt$

$x^{-\frac{1}{2}} = (t^3-1)^{-2}$

$(1+x^{\frac{1}{4}})^{\frac{1}{3}} = (t^3)^{\frac{1}{3}} = t$

$$= \int (t^3-1)^{-2} \cdot t \cdot 12t^2 \cdot (t^3-1)^3 dt =$$

$$12\int t^3(t^3-1)\cdot dt = 12\int t^6 dt - 12\int t^3 dt =$$

$$= 12\cdot \frac{t^7}{7} - 12\,\frac{t^4}{4} + C$$

$$= 12\cdot \sqrt[3]{(1+\sqrt[4]{x})^7} - 3\sqrt[3]{(1+\sqrt[4]{x})^4} + C$$

(2.)

$$\int \frac{dx}{x^2\cdot\sqrt[3]{(2+x^3)^5}} = \int x^{-2}(2+x^3)^{-\frac{5}{3}}\cdot dx =$$

$m=-2;\ n=3,\ p=-\frac{5}{3},\ a=2,\ b=1$

$\frac{m+1}{n} = \frac{-2+1}{3} = -\frac{1}{3} \notin Z$

$\frac{m+1}{n}+p = -\frac{1}{3}-\frac{5}{3} = -\frac{6}{3} = -2 \in Z$

$\frac{2}{x^3}+1=t^3$

$\frac{2}{x^3}=t^3-1$

$x^3=\frac{2}{t^3-1}$

$x^3 = 2\cdot(t^3-1)^{-1} \;/\; ^{\frac{1}{3}}$

$x = 2^{\frac{1}{3}}(t^3-1)^{-\frac{1}{3}}$

$dx = 2^{\frac{1}{3}}(-\frac{1}{3}\,(t^3-1)^{-\frac{4}{3}}\cdot 3t^2 dt$

$dx = -2^{\frac{1}{3}}\cdot t^2\cdot(t^3-1)^{-\frac{4}{3}}dt$

$x^{-2} = 2^{-\frac{2}{3}}\cdot(t^3-1)^{\frac{2}{3}}$

$(2+x^3)^{-\frac{5}{3}} = (2+\frac{2}{t^3-1})^{-\frac{5}{3}} = \left(\frac{2t^3}{t^3-1}\right)^{-\frac{3}{3}} =$

$= 2^{-\frac{5}{3}}\cdot t^{-5}\cdot(t^3-1)^{\frac{5}{3}}$

$$\int 2^{-\frac{2}{3}}(t^3-1)^{\frac{2}{3}} \cdot 2^{-\frac{5}{3}} \cdot t^{-5} \cdot (t^3-1)^{\frac{5}{3}} \cdot (-2^{\frac{1}{3}}) \cdot t^2 \cdot (t^3-1)^{-\frac{4}{3}}\, dt =$$

$$= -2^{-2}\int t^{-3} \cdot (t^3-1)\, dt = -\frac{1}{4}\int dt + \frac{1}{4}\int t^{-3}\, dt$$

$$= -\frac{1}{4}t + \frac{1}{4}\left(-\frac{1}{2}\right)\cdot\frac{1}{t^2} + C$$

$$= -\frac{1}{4}\sqrt[3]{\frac{2}{x^3}+1} + \frac{1}{8}\cdot\frac{1}{\sqrt[3]{\left(\frac{2}{x^3}+1\right)^2}} + C$$

ОЈЛЕРОВЕ СМЈЕНЕ

VI $\int R(x, \sqrt{ax^2+bx+c})\cdot dx$

1° $a>0$

$\sqrt{ax^2+bx+c} = \pm\sqrt{a}\cdot x \pm t$ $\quad (\sqrt{\ldots} = \pm\sqrt{a}\, x \pm t)$

$\qquad\qquad -\sqrt{a}\, x + t$

2° $c>0$

$\sqrt{ax^2+bx+c} = \sqrt{c} + xt$

3° $a<0 \wedge c<0 \wedge ax^2+bx+c$ – има различите реалне корјене

$ax^2+bx+c = a(x-\alpha)(x-\beta)$

смјена $\sqrt{ax^2+bx+c} = t(x-\alpha)$

$$① \int \frac{dx}{x+\sqrt{1+x+x^2}} =$$

$$\sqrt{x^2+x+1} + x = t$$

$$\sqrt{x^2+x+1} = t - x \quad /^2$$

$$\cancel{x^2}+x+1 = t^2 - 2tx + \cancel{x^2}$$

$$(1+2t)x = t^2 - 1$$

$$x = \frac{t^2-1}{1+2t}$$

$$dx = \frac{2t(1+2t) - (t^2-1)\cdot 2}{(1+2t)^2}\,dt$$

$$dx = \frac{2t^2+2t+2}{(1+2t)^2}\,dt$$

$$dx = \frac{2(t^2+t+1)}{(1+2t)^2}\,dt$$

$$= 2\int \frac{\frac{t^2+t+1}{(1+2t)^2}}{t}\,dt =$$

$$= 2\int \frac{t^2+t+1}{t(1+2t)^2}\,dt$$

$1+2t$

$$\frac{t^2+t+1}{t(1+2t)^2} = \frac{A}{t} + \frac{B}{1+2t} + \frac{C}{(1+2t)^2}$$

$$A = 1, \quad B = -\frac{3}{2}, \quad C = -\frac{3}{2}$$

$$2\cdot\int \frac{1}{t}\cdot dt - 3\int \frac{dt}{1+2t} - 3\int \frac{dt}{(1+2t)^2}$$

$$1+2t = z$$

$$dt = \frac{1}{2}\,dz \quad = 2\cdot \ln|t| - 3\cdot\frac{1}{2}\ln|1+2t| - 3\cdot\frac{1}{2}\int\frac{dz}{z^2} =$$

$$= 2\ln|t| - \frac{3}{2}\ln|1+2t| + \frac{3}{2}\,\frac{1}{z} + C$$

$$2\ln\left|\sqrt{x^2+x+1}+x\right| - \frac{3}{2}\ln\left|1+2\sqrt{x^2+x+1}+2x\right| + \frac{3}{2}\cdot\frac{1}{1+2\sqrt{x^2+x+1}\ldots}$$

(2) $$\int \frac{x}{(\sqrt{7x-10-x^2})^3}dx = \int \frac{x}{(\sqrt{-x^2+7x-10})^3}dx =$$

$a=-1,\ b=7,\ c=-10$

$-x^2+7x-10=0$

$$x_1/x_2 = \frac{-7\pm\sqrt{49-40}}{-2}$$

$$x_1 = \frac{-7+3}{-2} = 2 \quad x_2 = \frac{-7-3}{-2} = 5$$

$-x^2+7x-10 = (-1)(x-2)(x-5) = (x-2)(5-x)$

$\sqrt{-x^2+7x-10} = t\cdot(x-2)$

$\sqrt{(x-2)(5-x)} = t(x-2) \,/^2$

$(x-2)(5-x) = t^2(x-2)^2$

$$(5-x) = \frac{t^2(x-2)^2}{(x-2)}$$

$(5-x) = t^2x-2t^2$

$(1+t^2)x = 5+2t^2$

$$x = \frac{5+2t^2}{1+t^2}$$

$$dx = \frac{4t(1+t^2)-(5+2t^2)\cdot 2t}{(1+t^2)^2}dt$$

$$dx = \frac{-6t}{(1+t^2)^2}dt$$

$$\sqrt{-x^2+7x-10} = t(x-2) = t\cdot\left(\frac{5+2t^2}{1+t^2}-2\right) = \frac{3t}{1+t^2}$$

$$-6\int \frac{\frac{5+2t^2}{1+t^2}}{\frac{27\cdot t^3}{(1+t^2)^3}}\cdot\frac{t}{(1+t^2)^2}\,dt = -\frac{6}{27}\int\frac{5+2t^2}{t^2}\,dt =$$

$$-\frac{10}{9}\int t^{-2}\,dt - \frac{4}{9}\int dt = -\frac{10}{9}\cdot\left(-\frac{1}{t}\right) - \frac{4}{9}t + C =$$

$$= \frac{10}{9}\cdot\sqrt{\frac{x-2}{5-x}} - \frac{4}{9}\sqrt{\frac{5-x}{x-2}} + C$$

VII $\int R(x, \sqrt{a^2-x^2})\,dx$, $x = a\sin t$

$\int R(x, \sqrt{a^2+x^2}\,dx$, $x = a\,tg\,t$

b) ③ $(x^2+2x-1) =$

⑤ $\int x\sqrt{4-x^2}\,dx = \begin{bmatrix} x = 2\sin t \\ dx = 2\cdot\cos t\,dt \\ \sin t = \frac{x}{2} \\ t = \arcsin\frac{x}{2} \end{bmatrix}$

$= \int 2\cdot\sin t\cdot\sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt =$

$8\int \sin t\cdot\sqrt{1-\sin^2 t}\cdot\cos t\,dt =$

$= 8\cdot\int \sin t\sqrt{1-\sin^2 t}\cdot\cos t\cdot dt =$

$= 8\cdot\int \sin t\cdot\cos^2 t\cdot dt = \Big)$ $\begin{matrix} \cos t = z \\ -\sin t\,dt = dz \\ \sin t\,dt = dz \end{matrix}$

$= 8\int z^2(-dz) = -8\cdot\frac{z^3}{3}+C =$

$$= -\frac{8}{3} \cdot \cos^3 t + c = -\frac{8}{3}\left(\sqrt{1-\left(\frac{x}{2}\right)^2}\right)^3 + C$$

3. $\int \sqrt{1-2x-x^2}\,dx =$

$$\Gamma \; -x^2-2x+1 = -(x^2+2x-1) =$$
$$= -(x^2+2x+1-1-1) =$$
$$-\left[(x+1)^2-2\right] =$$
$$2-(x+1)^2$$

$$= \int \sqrt{2-(x+1)^2}\,dx$$

$$\Gamma \; x+1=t$$
$$dx = dt$$

$$= \int \sqrt{2-t^2}\,dt = \Gamma \; t=\sqrt{2}\cdot\sin z$$
$$dt = \sqrt{2}\cdot\cos z\,dz$$
$$\sin z = \frac{t}{\sqrt{2}}$$
$$z = \arcsin\frac{t}{\sqrt{2}}$$

$$= \int \sqrt{2-2\sin^2 z}\cdot\sqrt{2}\cdot\cos z\cdot dz =$$

$$= 2\int \sqrt{1-\sin^2 z}\cdot\cos z\cdot dz$$

$$2\cdot\int \cos^2 z\,dz = 2\cdot\int \frac{1+\cos 2z}{2}\,dz = \int dz + \int \cos 2z\,dz$$

$$z + \frac{1}{2}\sin 2z + C$$

$$\cancel{z + \frac{1}{2}\sin z \cdot \cos z + C}$$

$$\arcsin\frac{(x+1)}{\sqrt{2}} + \frac{t}{\sqrt{2}}\cdot\sqrt{1-\frac{t^2}{2}} + C$$

$$\arcsin\frac{(x+1)}{\sqrt{2}} + \frac{(x+1)}{\sqrt{2}}\sqrt{1-\frac{(x+1)^2}{2}} + C$$

5.
$$\int x\sqrt{4-x^2}\,dx = \left[\begin{array}{l} x = 2\sin t \\ dx = 2\cdot\cos t\,dt \end{array}\right.$$