# Integracija trigonometrijskih f-ja

I $\int R(\sin x, \cos x)\,dx \qquad tg\frac{x}{2} = t \,, \quad \sin x = \frac{2t}{1+t^2}$

$$dx = \frac{2\,dt}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

1. $$\int \frac{dx}{(2+\cos x)\sin x} = \begin{bmatrix} tg\frac{x}{2} = t & \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2\,dt}{1+t^2} & \cos x = \frac{1-t^2}{1+t^2} \end{bmatrix} = \int \frac{\frac{\cancel{2}dt}{\cancel{1+t^2}}}{\left(2+\frac{1-t^2}{1+t^2}\right)\cdot \frac{\cancel{2}t}{\cancel{1+t^2}}}$$

$$= \int \frac{1+t^2}{t(3+t^2)}\,dt = \frac{1}{3}\int \frac{dt}{t} + \frac{2}{3}\int \frac{t}{3+t^2}\,dt = \begin{bmatrix} 3+t^2 = z \\ t\,dt = \frac{1}{2}dz \end{bmatrix} =$$

$$\frac{1+t^2}{t(3+t^2)} = \frac{A}{t} + \frac{Bt+C}{3+t^2} \quad \dots \quad A = \frac{1}{3}, \; B = \frac{2}{3}, \; C = 0$$

$$= \frac{1}{3}\ln|t| + \frac{2}{3}\cdot\frac{1}{2}\int \frac{dz}{z} = \frac{1}{3}\ln|t| + \frac{1}{3}\ln|z| + C =$$

$$= \frac{1}{3}\ln\left|tg\frac{x}{2}\right| + \frac{1}{3}\ln\left|3+tg^2\frac{x}{2}\right| + C$$

$\int R(\sin x, \cos x)\,dx$

R neparna po $\sin x$, $\cos x = t$

R neparna po $\cos x$, $\sin x = t$

2. $\int \frac{\cos^3 x}{2+\sin x}\,dx$

$R(\sin x, \cos x) = \frac{\cos^3 x}{2+\sin x}$

$R(\sin x, -\cos x) = \frac{(-\cos x)^3}{2+\sin x} = -\frac{\cos^3 x}{2+\sin x} = -R(\sin x, \cos x) \Rightarrow$

$\Rightarrow$ R je neparna po $\cos x$

$$\int \frac{\cos^3 x}{2+\sin x}\,dx = \int \frac{\cos^2 x \cos x}{2+\sin x}\,dx = \begin{bmatrix} \sin x = t \\ \cos x\,dx = dt \end{bmatrix} = \int \frac{1-t^2}{2+t}\,dt =$$

$$= -\int \frac{t^2-1}{t+2}\,dt = -\int \frac{t^2-4+4-1}{t+2}\,dt = -\int (t-2)\,dt - 3\int \frac{dt}{t+2} =$$

$$= -\frac{t^2}{2} + 2t - 3\ln|t+2| + C = -\frac{1}{2}\sin^2 x + 2\sin x - 3\ln|\sin x + 2| + C$$

II $\int R(\sin x, \cos x)\,dx$ R parna po $\sin x$ i po $\cos x$

$\operatorname{tg} x = t$ $\sin x = \frac{t}{\sqrt{1+t^2}}$

$dx = \frac{dt}{1+t^2}$ $\cos x = \frac{1}{\sqrt{1+t^2}}$

3. $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx =$

$R(\sin x, \cos x) =$

$$R(-\sin x, -\cos x) = \frac{-\sin x - (-\cos x)}{-\sin x + (-\cos x)} = \frac{(\sin x - \cos x)}{-(\sin x + \cos x)} = R(\sin x, \cos x)$$

R je parna i po $\sin x$ i po $\cos x$

$$\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = \left\lceil \begin{array}{ll} tg\,x = t & \sin x = \frac{t}{\sqrt{1+t^2}} \\ dx = \frac{dt}{1+t^2} & \cos x = \frac{1}{\sqrt{1+t^2}} \end{array} \right\rfloor =$$

$$= \int \frac{\frac{t}{\sqrt{1+t^2}} - \frac{1}{\sqrt{1+t^2}}}{\frac{t}{\sqrt{1+t^2}} + \frac{1}{\sqrt{1+t^2}}} \frac{dt}{1+t^2} = \int \frac{t-1}{t+1} \cdot \frac{dt}{1+t^2} = \int \frac{t-1}{(t+1)(t^2+1)}\,dt =$$

$$\left\lceil \frac{t-1}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1} \right.$$

$$t-1 = (A+B)t^2 + (B+C)t + A + C$$

$A+B=0 \rightarrow A=-B$

$B+C=1 \rightarrow C=1-B$

$A+C=-1$

$-B+1-B=-1$

$B=1$, $A=-1$, $C=0$

$$= -\int \frac{dt}{t+1} + \int \frac{t}{t^2+1}\,dt = -\ln|t+1| + \frac{1}{2}\ln|t^2+1| + C =$$

$$= -\ln|tg\,x+1| + \frac{1}{2}\ln|tg^2x+1| + C$$

$\underline{IV}$ $\int \sin^m x \cos^n x \, dx$

$m$ - neparno, $n$ - parno $\rightarrow$ smjena $\cos x = t$

$m$ - parno, $n$ - neparno $\rightarrow$ smjena $\sin x = t$

1. $$\int \sin^3 x \cos^4 x\,dx = \int \sin^2 x \sin x \cos^4 x\,dx = \left\lceil \begin{array}{l} \cos x = t \\ \sin x\,dx = -dt \end{array} \right\rfloor =$$

$$= -\int (1-t^2)t^4\,dt = -\frac{t^5}{5} + \frac{t^7}{7} + C = -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

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$\underline{V}$ $\int \sin^m x \cos^n x\, dx$

$n, m$ - parno $\quad \sin^2 x = \frac{1-\cos 2x}{2}$, $\quad \cos^2 x = \frac{1+\cos 2x}{2}$

1. $\int \sin^2 x \cos^4 x\, dx = \int \frac{1-\cos 2x}{2}\left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{8}\int (1-\cos^2 2x)(1+\cos$

$= \frac{1}{8}\int dx + \frac{1}{8}\int \cos 2x\, dx - \frac{1}{8}\int \cos^2 2x\, dx - \frac{1}{8}\int \cos^3 2x\, dx =$

$= \left[ \begin{array}{l} \sin 2x = t \\ \cos 2x dx = \frac{1}{2} dt \end{array} \right] = \frac{1}{8}x + \frac{1}{8}\cdot\frac{1}{2}\sin 2x - \frac{1}{8}\int \frac{1+\cos 4x}{2} dx - \frac{1}{8}\cdot\frac{1}{2}\int (1-t^2) dt$

$= \frac{1}{8}x + \frac{1}{16}\sin 2x - \frac{1}{16}x - \frac{1}{16}\cdot\frac{1}{4}\sin 4x - \frac{1}{16}t + \frac{1}{16}\frac{t^3}{3} + C =$

$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$

$\underline{VI}$ $\int \frac{\sin^m x}{\cos^n x} dx$

$m, n$ - parno $\quad$ smjena $tgx = t$

2. $\int \frac{\sin^2 x}{\cos^6 x} dx = \left[ \begin{array}{ll} tgx = t & \sin x = \frac{t}{\sqrt{1+t^2}} \\ dx = \frac{dt}{1+t^2} & \cos x = \frac{1}{\sqrt{1+t^2}} \end{array} \right] = \int \frac{\frac{t^2}{1+t^2}}{\frac{1}{(1+t^2)^3}} \cdot \frac{dt}{1+t^2} =$

$= \int t^2(1+t^2) dt = \frac{t^3}{3} + \frac{t^5}{5} + C = \frac{tg^3 x}{3} + \frac{tg^5 x}{5} + C$

$\underline{VII}$ $\int \sin \alpha x \cos \beta x\, dx$, adicione formule

3. $\int \sin 5x \cos x\, dx = \frac{1}{2}\int (\sin 6x + \sin 4x) dx = \frac{1}{2}\cdot\left(-\frac{1}{6}\right)\cos 6x + \frac{1}{2}\left(-\frac{1}{4}\right)c$

$+$

1. $I_n = \int \frac{dx}{(x^2+a^2)^n} = \left[ \begin{array}{l} u = \frac{1}{(x^2+a^2)^n} \Rightarrow du = \frac{-n}{(x^2+a^2)^{n+1}} 2x\,dx \\ v = \int dx = x \end{array} \right] =$

$$= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2}{(x^2+a^2)^{n+1}} dx = \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}} dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n \underbrace{\int \frac{dx}{(x^2+a^2)^n}}_{I_n} - 2na^2 \underbrace{\int \frac{dx}{(x^2+a^2)^{n+1}}}_{I_{n+1}}$$

$$I_n = \frac{x}{(x^2+a^2)^n} + 2n I_n - 2na^2 \cdot I_{n+1}$$

$$I_{n+1} = \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2} I_n, \quad n \geq 1$$

$$I_1 = \int \frac{dx}{x^2+a^2} = \left[ \begin{array}{l} x = at \\ dx = a\,dt \end{array} \right] = \frac{a}{a^2} \int \frac{dt}{t^2+1} = \frac{1}{a} \operatorname{arctg} t = \frac{1}{a} \operatorname{arctg}\left(\frac{x}{a}\right)$$

2. $I_n = \int \frac{dx}{\sin^n x} = \int \frac{\sin x}{\sin^{n+1} x} dx = \left[ \begin{array}{l} u = \frac{1}{\sin^{n+1} x} \Rightarrow du = -\frac{n+1}{\sin^{n+2} x} \cos x\, dx \\ v = \int \sin x\, dx = -\cos x \end{array} \right] =$

$$= -\frac{\cos x}{\sin^{n+1} x} - (n+1) \int \frac{\cos^2 x}{\sin^{n+2} x} dx = -\frac{\cos x}{\sin^{n+1} x} - (n+1) \int \frac{1-\sin^2 x}{\sin^{n+2} x} dx =$$

$$= -\frac{\cos x}{\sin^{n+1} x} - (n+1) \underbrace{\int \frac{dx}{\sin^{n+2} x}}_{I_{n+2}} + (n+1) \underbrace{\int \frac{dx}{\sin^n x}}_{I_n}$$

$$I_n = -\frac{\cos x}{\sin^{n+1} x} - (n+1) I_{n+2} + (n+1) I_n$$

$$I_{n+2} = -\frac{\cos x}{(n+1)\sin^{n+1} x} + I_n \cdot \frac{n}{n+1}, \quad n \geq 0$$

$$I_0 = \int dx = x + C$$

$$I_1 = \int \frac{dx}{\sin x} = \int \frac{\sin x}{\sin^2 x} dx = \left[ \begin{array}{l} \cos x = t \\ \sin x\, dx = -dt \end{array} \right] = -\int \frac{dt}{1-t^2} = -\frac{1}{2} \ln \left| \frac{1-t}{1+t} \right| + C = -\frac{1}{2} \ln \left| \frac{1-\cos x}{1+\cos x} \right|$$

$$\int \frac{dx}{x^4+1} = \int \frac{dx}{x^4+2x^2+1-2x^2} = \int \frac{dx}{(x^2+1)^2-2x^2} = \int \frac{dx}{(x^2+1-\sqrt{2}x)(x^2+1+\sqrt{2}x)} =$$

$$\frac{1}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1} \quad \dots \text{Sami}$$

4. $$\int \frac{dx}{(x^2-3x+2)\sqrt{x^2-4x+5}}$$

$\ulcorner \; x^2-3x+2=0$

$x_{1,2} = \frac{3 \pm 1}{2} \qquad x_1 = 2, \; x_2 = 1$

$$\frac{1}{x^2-3x+2} = \frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \quad \dots A=2, B=-1$$

$$\int \frac{dx}{-\,''\,-} = \int \left(\frac{2}{x-2} - \frac{1}{x-1}\right) \frac{dx}{\sqrt{x^2-4x+5}} = 2\int \frac{dx}{(x-2)\sqrt{x^2-4x+5}} - \int \frac{dx}{(x-1)\sqrt{x^2-4x}}$$

$x-2 = \frac{1}{t} \qquad x-1 = \frac{1}{z}$

...

5. $$\int \frac{dx}{\sqrt[4]{(x-1)^3(x+1)^5}} = \int \frac{dx}{\sqrt[4]{(x-1)^4(x+1)^4 \frac{x+1}{x-1}}} = \int \frac{dx}{(x-1)(x+1)\sqrt[4]{\frac{x+1}{x-1}}} =$$

$= \ulcorner \; x, \left(\frac{x+1}{x-1}\right)^{\frac{1}{4}}$

$\frac{x+1}{x-1} = t^4 \lrcorner \qquad \dots$

6. $$\int \frac{1+x\cos x}{\sin^3 x}\,dx = \int \frac{dx}{\sin^3 x} + \int \frac{x\cos x}{\sin^3 x}\,dx = \ulcorner \; u = x \Rightarrow du = dx$$

$v = \int \frac{\cos x}{\sin^3 x}\,dx = \ulcorner \sin x = \; \cos x\, dx =$

$$= \int \frac{\sin x}{\sin^4 x}\,dx + \frac{-x}{2}\cdot\frac{1}{\sin^2 x} + \frac{1}{2}\int \frac{dx}{\sin^2 x} = \ulcorner \cos x = t \quad \sin x\, dx = -dt \lrcorner$$

$= \int \frac{dt}{t^3} = -\frac{1}{2}\cdot\frac{1}{t^2} = -\frac{}{2s}$

$$= \underbrace{-\int \frac{dt}{(1-t^2)^2}}_{I} - \frac{x}{2\sin^2 x} - \frac{1}{2}\,\mathrm{ctg}\,x$$

$$I = \int \frac{dt}{(1-t)^2(1+t)^2} \qquad \frac{1}{(1-t)^2(1+t)^2} = \frac{A}{1-t} + \frac{B}{(1-t)^2} + \frac{C}{1+t} + \frac{D}{(1+t)^2} \quad \dots$$

$$7. \quad \int \ln\left(\frac{x^{\sqrt{3x}}}{\sqrt{x+2}+\sqrt{x}}\right)dx = \int \left(\ln x^{\sqrt{3x}} - \ln\left(\sqrt{x+2}+\sqrt{x}\right)\right)dx =$$

$$= \underbrace{\int \sqrt{3x}\,\ln x\,dx}_{I_1} - \underbrace{\int \ln\left(\sqrt{x+2}+\sqrt{x}\right)dx}_{I_2}$$

$$I_1 = \sqrt{3}\int \sqrt{x}\,\ln x\,dx = \begin{bmatrix} u = \ln x \Rightarrow du = \frac{dx}{x} \\ v = \int \sqrt{x}\,dx = \frac{2}{3}\sqrt{x^3} \end{bmatrix} = \sqrt{3}\left(\frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\int \sqrt{x}\,dx\right)$$

$$=$$

$$I_2 = \int \ln\left(\sqrt{x+2}+\sqrt{x}\right)dx = \begin{bmatrix} u = \ln\left(\sqrt{x+2}+\sqrt{x}\right) \Rightarrow du = \frac{1}{\sqrt{x+2}+\sqrt{x}}\left(\frac{1}{2\sqrt{x+2}}+\frac{1}{2\sqrt{x}}\right)dx \\ v = \int dx = x \qquad du = \frac{1}{2\sqrt{x+2}\,\sqrt{x}}\,dx = \frac{dx}{2\sqrt{x^2+2x}} \end{bmatrix}$$

$$= x\ln\left(\sqrt{x+2}+\sqrt{x}\right) - \frac{1}{2}\int \frac{x\,dx}{\sqrt{x^2+2x}} \qquad \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$$

$$8. \quad \int \frac{\sqrt{ctg x}}{1+\sin 2x - 4\cos^2 x}\,dx = \begin{bmatrix} tg\,x = t & \sin x = \frac{t}{\sqrt{1+t^2}} \\ dx = \frac{dt}{1+t^2} & \cos x = \frac{1}{\sqrt{1+t^2}} \end{bmatrix} =$$

$$= \int \frac{\sqrt{\frac{1}{t}}}{1+2\frac{t}{1+t^2} - 4\cdot\frac{1}{1+t^2}}\cdot\frac{dt}{1+t^2} = \int \frac{\frac{1}{\sqrt{t}}\cdot\cancel{\frac{1}{1+t^2}}}{\frac{1+t^2+2t-4}{\cancel{1+t^2}}}\,dt = \int \frac{dt}{\sqrt{t}\,(t^2+2t-3)}$$

$$= \begin{bmatrix} t = z^2 \\ dt = 2z\,dz \end{bmatrix} = 2\int \frac{\cancel{z}\,dz}{\cancel{z}\,(z^4+2z^2-3)} = 2\int \frac{dz}{z^4+2z^2-3} = \begin{bmatrix} z^2 = t \\ t^2+2t-3=0 \\ t_{1,2} = -1 \pm 2 \\ t_1 = 1 \quad t_2 = -3 \end{bmatrix}$$

$$= 2\int \frac{dz}{(z^2-1)(z^2+3)} = 2\int \frac{dz}{(z-1)(z+1)(z^2+3)} = \ldots$$

$$t^2+2t-3 = (t-1)(t+3)$$

$$z^4+2z^2-3 = (z^2-1)(z^2+3)$$

$$\frac{1}{(z-1)(z+1)(z^2+3)} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{Cz+D}{z^2+3}$$

9. $\int \frac{\sin 2x \sqrt[3]{\cos x+2}}{\sqrt[3]{\cos x+2}+\cos x}dx = \left[\begin{array}{l} \cos x = t \\ \sin x\, dx = -dt \end{array}\right] = -2\int \frac{t\sqrt[3]{t+2}}{\sqrt[3]{t+2}+t}dt = \left[\begin{array}{l} t,\ (t+2)^{\frac{1}{3}} \\ t+2 = z^3 \end{array}\right] \ldots$

10. $\int \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2(1+e^{3\sqrt[3]{x}})}}dx = \left[\begin{array}{l} \sqrt[3]{x} = t \\ \frac{1}{3}x^{-\frac{2}{3}}dx = dt \\ \frac{dx}{\sqrt[3]{x^2}} = 3dt \end{array}\right] = 3\int \frac{e^t}{\sqrt[3]{1+e^{3t}}}dt = \left[\begin{array}{l} e^t = z \\ e^t dt = dz \end{array}\right]$

$= 3\int \frac{dz}{\sqrt[3]{1+z^3}} = 3\int (1+z^3)^{-\frac{1}{3}}dz = \left[\begin{array}{l} m=0,\ n=3,\ p=-\frac{1}{3} \\ \frac{m+1}{n} = \frac{1}{3} \notin Z \quad \frac{m+1}{n}+p = 0 \in Z \\ \frac{1}{z^3}+1 = t^3 \end{array}\right.$ ...

11. $\int \frac{x-2}{(x^2-x+1)^2}dx$

I $\frac{x-2}{(x^2-x+1)^2} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{(x^2-x+1)^2}$ ...

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II $\int \frac{x-2}{(x^2-x+1)}dx = \int \frac{\frac{1}{2}(2x-1)-2+\frac{1}{2}}{(x^2-x+1)^2}dx = \frac{1}{2}\int \frac{2x-1}{(x^2-x+1)^2}dx - \frac{3}{2}\underbrace{\int \frac{dx}{(x^2-x+1)^2}}_{I}$

$I = \int \frac{dx}{(x^2-x+\frac{1}{4}+\frac{3}{4})^2} = \int \frac{dx}{((x-\frac{1}{2})^2+\frac{3}{4})^2} = \left[\begin{array}{l} x-\frac{1}{2} = \frac{\sqrt{3}}{2}t \\ dx = \frac{\sqrt{3}}{2}dt \\ t = \frac{2x-1}{\sqrt{3}} \end{array}\right] =$

$= \frac{\sqrt{3}}{2}\cdot\frac{16}{9}\int \frac{dt}{(t^2+1)^2}$ parcijalna int.

12. $\int \sqrt{2e^{2x}+e^{3x}}\ \mathrm{arctg}\sqrt{1+e^x}\,dx = \int \sqrt{e^{2x}(2+e^x)}\ \mathrm{arctg}\sqrt{1+e^x}\,dx =$

$= \int e^x\sqrt{2+e^x}\ \mathrm{arctg}\sqrt{1+e^x}\,dx = \left[\begin{array}{l} e^x = t \\ e^x dx = dt \end{array}\right] = \int \sqrt{2+t}\cdot \mathrm{arctg}\sqrt{1+t}\,dt =$

$= \left[\begin{array}{l} u = \mathrm{arctg}\sqrt{1+t} \Rightarrow \\ v = \int \sqrt{2+t}\,dt \end{array}\right.$

# Određeni integrali

1. $\int_0^8 (1+\sqrt{2x}+\sqrt[3]{x})\,dx = \int_0^8 dx + \sqrt{2}\int_0^8 \sqrt{x}\,dx + \int_0^8 \sqrt[3]{x}\,dx = x\Big|_0^8 + \sqrt{2}\cdot\frac{2}{3}\sqrt{x^3}\Big|_0^8 + \frac{3}{4}\sqrt[3]{x^4}\Big|_0^8$

$= 8-0+\frac{2\sqrt{2}}{3}(\sqrt{8^3}-\sqrt{0^3})+\frac{3}{4}(\sqrt[3]{8^4}-\sqrt[3]{0^4}) = \frac{124}{3}$

$\int_0^5 \sqrt{x+4}\,dx = \begin{bmatrix} x+4=t \\ dx=dt \end{bmatrix} \begin{array}{c|c|c} x & 0 & 5 \\ \hline t & 4 & 9 \end{array} = \int_4^9 \sqrt{t}\,dt = \frac{2}{3}\sqrt{t^3}\Big|_4^9 = \frac{2}{3}(\sqrt{9^3}-\sqrt{4^3}) =$

$= \frac{2}{3}(27-8) = \frac{38}{3}$

2. $\int_0^2 f(x)\,dx,\quad f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \end{cases}$

$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx = \frac{x^3}{3}\Big|_0^1 + 2x\Big|_1^2 - \frac{x^2}{2}\Big|_1^2$

3. $\int_{1/e}^{e} |\ln x|\,dx = \int_{1/e}^{1} |\ln x|\,dx + \int_1^e |\ln x|\,dx = -\int_{1/e}^{1} \ln x\,dx + \int_1^e \ln x\,dx =$

$= \begin{bmatrix} u=\ln x \Rightarrow du=\frac{dx}{x} \\ v=\int dx = x \end{bmatrix} = -\left(x\ln x\Big|_{1/e}^{1} - \int_{1/e}^{1} dx\right) + x\ln x\Big|_1^e - \int_1^e dx =$

$= -\left(1\cdot\ln 1 - \frac{1}{e}\ln\frac{1}{e} - x\Big|_{1/e}^{1}\right) + e\ln e - 1\cdot\ln 1 - x\Big|_1^e = -\left(\frac{1}{e} - 1 + \frac{1}{e}\right) + e - e + 1$

$= 2 - \frac{2}{e}$

4. $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \begin{bmatrix} x = 2\sin t \\ dx = 2\cos t\,dt \\ \sin t = \frac{x}{2} \\ t = \arcsin\frac{x}{2} \end{bmatrix} \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 0 & \frac{\pi}{6} \end{array} = \int_0^{\pi/6} \frac{4\sin^2 t}{\sqrt{4-4\sin^2 t}}\cdot 2\cos t\,dt =$

$= 4\int_0^{\pi/6} \frac{\sin^2 t\,\cos t}{|\cos t|}\,dt = 4\int_0^{\pi/6} \sin^2 t\,dt = 2\int_0^{\pi/6}(1-\cos 2t)\,dt = 2t\Big|_0^{\pi/6} - 2\cdot\frac{1}{2}\sin 2t\Big|_0^{\pi/6} =$

$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right) \rightarrow T\left(-\frac{D}{4a}, -\frac{b}{2a}\right)$

$D = b^2 - 4ac$

$y = ax^2 + bx + c$ $\qquad x = ay^2 + by + c$

Transformacije grafika

1° $f(cx)$

2° $f(x - c)$

3° $c f(x)$

4° $f(-x)$

5° $-f(x)$

6° $f(x) + c$

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1. Izračunati površinu figure ograničene krivom $y = x^2 + 2x + 2$ i pravama $y = 0$, $x = 2$, $x = -3$.

$a = 1 \quad b = 2 \quad c = 2$

$D = 4 - 4 \cdot 1 \cdot 2 = -4 < 0 \Rightarrow$ nema presjeka sa $Ox$-osom

Presjek sa $Oy$-osom $\begin{cases} y = x^2 + x + 2 \\ x = 0 \end{cases}$ $\quad y = 2 \quad A(0, 2)$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$T\left(-\frac{2}{2}, -\frac{-4}{4}\right)$

$T(-1, 1)$

$\begin{cases} y = x^2 + 2x + 2 \\ x = 2 \end{cases}$ $\quad y = 4 + 4 + 2 = 10 \quad B(2, 10)$

$\begin{cases} y = x^2 + 2x + 2 \\ x = -3 \end{cases}$ $\quad y = 9 - 6 + 2 = 5 \quad C(-3, 5)$


$$P = \int_{-3}^{2} (x^2 + 2x + 2)\,dx = \frac{x^3}{3}\Big|_{-3}^{2} + 2\frac{x^2}{2}\Big|_{-3}^{2} + 2x\Big|_{-3}^{2}$$

$$= \frac{1}{3}\left(8 - (-3)^3\right) + \left(4 - (-3)^2\right) + 2\left(2 - (-3)\right) = \frac{50}{3}$$

2. Izračunati površinu ograničenu krivom $y = x^2 - 1$ osom $O_x$ i pr

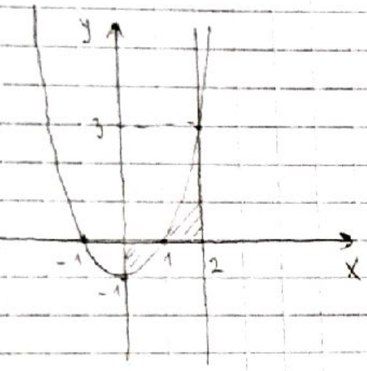
$x = 0$ i $x = 2$.

$y = x^2 - 1$ $\qquad y = 0 \iff x^2 - 1 = 0$ $\qquad$ Presjek sa $O_y$-osom (kako je $b = 0$ biće

$x = \pm 1$

$y = 0^2 - 1 = -1 \quad T(0, -1)$

$\begin{cases} y = x^2 - 1 \\ x = 2 \end{cases} \quad y = 4 - 1 = 3 \quad A(2, 3)$




$P = \int_0^2 |f(x)|\,dx, \quad f(x) = x^2 - 1$

$P = \int_0^1 |x^2 - 1|\,dx + \int_1^2 |x^2 - 1|\,dx = -\int_0^1 (x^2 - 1)\,dx + \int_1^2 (x^2 - 1)\,dx$

$= -\frac{x^3}{3}\Big|_0^1 + x\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 - x\Big|_1^2 = 2$

3. Izračunati P figure ograničene parabolom $y = x^2 + 4x$ i pravom $y = x + 4$.

$y = x^2 + 4x$ $\qquad T\left(-\frac{4}{2}, -\frac{16}{4}\right)$

$y = 0 \iff x(x+4) = 0$ $\qquad T(-2, -4)$

$x = 0 \lor x = -4$

(biće i presjek sa $O_y$-osom)

$\begin{cases} y = x^2 + 4x \\ y = x + 4 \end{cases}$ $\qquad x^2 + 4x = x + 4$

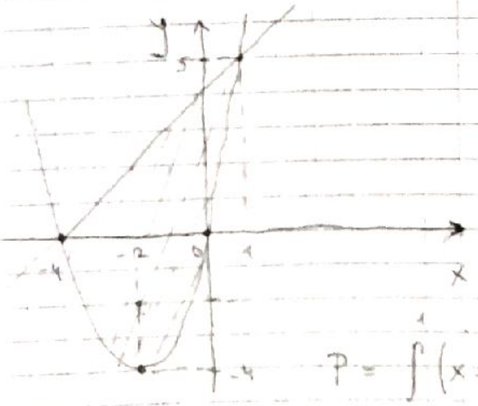
$x^2 + 3x - 4 = 0$

$x_{1,2} = \frac{-3 \pm 5}{2}$

$x_1 = 1 \qquad x_2 = -4$

$\Downarrow \qquad \Downarrow$

$y_1 = 5 \qquad y_2 = 0$




$P = \int_{-4}^{1} (x + 4 - (x^2 + 4x))\,dx = \int_{-4}^{1} (-x^2 - 3x + 4)\,dx =$

$= -\frac{x^3}{3}\Big|_{-4}^{1} - 3\frac{x^2}{2}\Big|_{-4}^{1} + 4x\Big|_{-4}^{1} = \frac{125}{6}$

Izračunati površinu figure ograničene krivom $y^2 = 2x+1$ i pravom $y = x$

$y^2 = 2x+1$

$x = \frac{1}{2}y^2 - \frac{1}{2}$

$x = 0 \Rightarrow y^2 = 1$

$y = \pm 1$

$A_1(0,1),\ A_2(0,-1)$

Presjek sa $Ox$ osom $(y=0)$

($b=0$ pa je tačka tjeme

$\begin{cases} x = \frac{1}{2}y^2 - \frac{1}{2} \\ y = 0 \end{cases}$ $x = -\frac{1}{2}$

$T(-\frac{1}{2}, 0)$

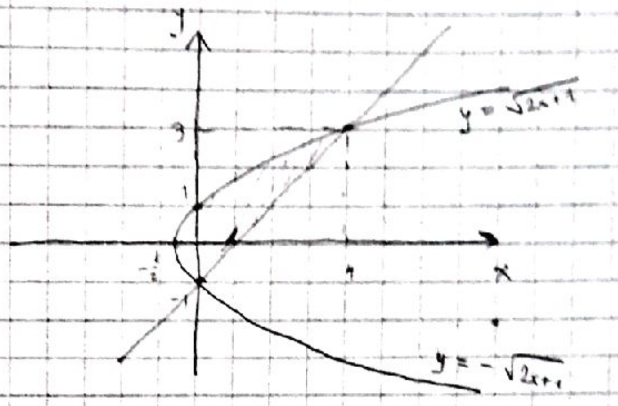
$\begin{cases} y^2 = 2x+1 \\ y = x-1 \end{cases}$ $(x-1)^2 = 2x+1$

$x^2 - 4x = 0$

$x_1 = 0 \lor x = 4$

$y = -1 \quad y = 3$

$A_2(0,-1) \quad B(4,3)$




I način $P = \int_{-1}^{3} \left(y+1 - \left(\frac{1}{2}y^2 - \frac{1}{2}\right)\right)dy = \int_{-1}^{3}\left(-\frac{1}{2}y^2 + y + \frac{3}{2}\right)dy = \ldots = \frac{16}{3}$

II način $P = 2P_1 + P_2$

$P_1 = \int_{-\frac{1}{2}}^{0} \sqrt{2x+1}\,dx = \begin{bmatrix} 2x+1 = t \\ dx = \frac{1}{2}dt \end{bmatrix} \begin{array}{c|c|c} x & -\frac{1}{2} & 0 \\ \hline t & 0 & 1 \end{array} =$

$y^2 = 2x+1$

$y = \pm\sqrt{2x+1}$

$= \frac{1}{2}\int_{0}^{1} \sqrt{t}\,dt = \frac{1}{2} \cdot \frac{2}{3} t^{\frac{3}{2}}\Big|_0^1 = \frac{1}{3}$

$P_2 = \int_0^4 \left(\sqrt{2x+1} - (x-1)\right)dx = \int_0^4 \sqrt{2x+1}\,dx - \int_0^4 x\,dx + \int_0^4 dx = \ldots$

5. U presječnim tačkama prave $x - y + 1 = 0$ i parabole $y = x^2 - 4x + 5$ povučene su tangente na parabolu. Izračunati površinu figure ograničene parabolom i tangentama

$y = x^2 - 4x + 5$

$T\left(-\frac{-4}{2}, -\frac{-4}{4}\right)$

$T(2, 1)$

$\begin{cases} y = x^2 - 4x + 5 \\ y = x+1 \end{cases}$

$x^2 - 4x + 5 = x + 1$

$x^2 - 5x + 4 = 0$

$x_{1,2} = \frac{5 \pm 3}{2}$

$D = 16 - 20 = -4 < 0$

Nema presjeka sa Ox-osom

Presjek sa Oy-osom

$\begin{cases} y = x^2 - 4x + 5 \\ x = 0 \end{cases}$ $y = 5 \quad A(0,5)$

$x_1 = 1 \quad x_2 = 4$

$y_1 = 2 \quad y_2 = 5$

$C(1,2) \quad B(4,5)$

Tangenta u tački $B(4,5)$

$y - 5 = y'(4)(x-4)$

$y'(x) = 2x - 4$

$y'(4) = 4$

$t_1: \ y - 5 = 4(x-4)$

$t_1: \ y = 4x - 11$

Tangenta u tački $C(1,2)$

$y - 2 = y'(1)(x-1)$

$y - 2 = -2(x-1)$

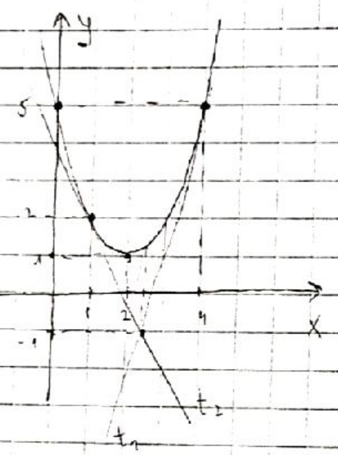
$t_2: \ y = -2x + 4$

$$\begin{cases} t_1: y = 4x - 11 \\ t_2: y = -2x + 4 \end{cases}$$

$4x - 11 = -2x + 4$

$6x = 15$

$x = \frac{5}{2} \Rightarrow y = -1$ $D(\frac{5}{2}, -1)$

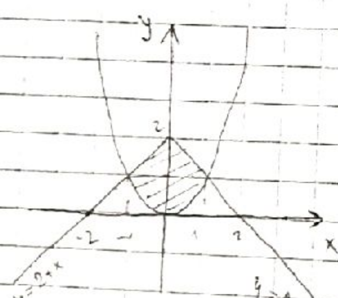



$P = P_1 + P_2$

$$P = \int_1^{5/2} \left(x^2 - 4x + 5 - (-2x+4)\right) dx + \int_{5/2}^{4} \left(x^2 - 4x + 5 - (4x - 11)\right)$$

$= \qquad \frac{9}{4}$

6. Izračunati $P$ figure ograničene krivama $y = 2 - |x|$ i $y = x^2$




$y = 2 - |x|$

$y = 0 \Leftrightarrow |x| = 2$

$x = 2 \lor x = -2$

$$y = \begin{cases} 2 - x, & x \geq 0 \\ 2 + x, & x < 0 \end{cases}$$

$$\begin{cases} y = 2 - x, & x \geq 0 \\ y = x^2 \end{cases}$$

$x^2 = 2 - x, \quad x \geq 0$

$x^2 + x - 2 = 0$

$x_{1,2} = \frac{-1 \pm 3}{2}$

$x_1 = 1, \ x_2 = -2$

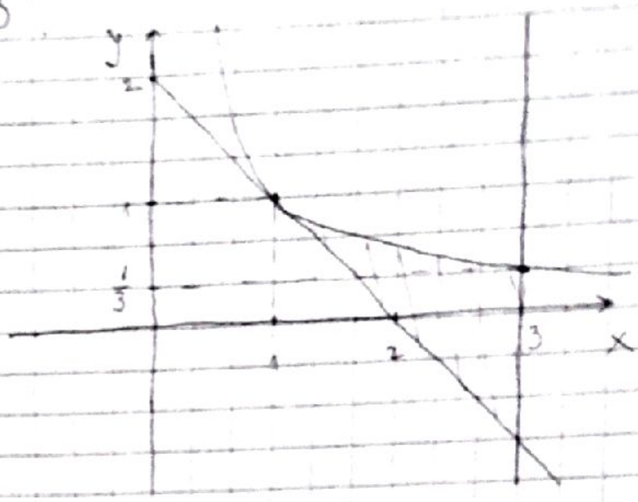
$A(1,1)$

$$\begin{cases} y = 2 + x, & x < 0 \\ y = x^2 \end{cases}$$

$B(-1,1)$

$$P = 2\int_0^1 (2 - x - x^2)\, dx$$

Izračunati P figure ograničene krivom $y = \frac{1}{x}$, njenom tangentom u tački $M(1,1)$ i pravom $x = 3$.



Tangenta u $M(1,1)$

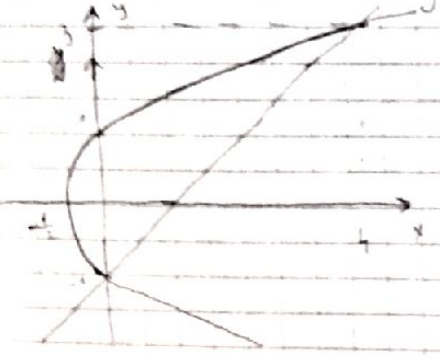
$y - 1 = y'(1)(x-1)$

$y'(x) = -\frac{1}{x^2} \Rightarrow y'(1) = -1$

$t: \; y = -x + 2$

$$P = \int_1^3 \left(\frac{1}{x} - (-x+2)\right) dx = \ln|x| \Big|_1^3 + \frac{x^2}{2}\Big|_1^3 + 2x\Big|_1^3 = 8 + \ln 3$$

8. Naći dužinu luka koji na paraboli $y^2 = 2x+1$ odsijeca prava $y = x - 1$



$x = \frac{1}{2}y^2 - \frac{1}{2}$

$$l = \int_{-1}^{3} \sqrt{1 + x'^2(y)}\, dy$$

$x'(y) = y$

$$l = \int_{-1}^{3} \sqrt{1+y^2}\, dy = \quad u = \sqrt{1+y^2} \quad du = \frac{y}{\sqrt{1+y^2}}\, dy$$

$$v = \int dy = y$$

$$= y\sqrt{1+y^2}\Big|_{-1}^{3} - \int_{-1}^{3} \frac{y^2}{\sqrt{1+y^2}}\, dy = 3\sqrt{10} + \sqrt{2} - \int_{-1}^{3} \frac{y^2+1-1}{\sqrt{y^2+1}}$$

$$= 3\sqrt{10} + \sqrt{2} - \int_{-1}^{3} \sqrt{y^2+1}\, dy + \int_{-1}^{3} \frac{dy}{\sqrt{y^2+1}}$$

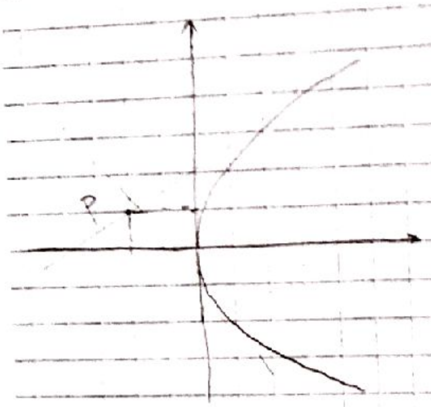
$$2l = 3\sqrt{10} + \sqrt{2} + \ln\left|y + \sqrt{1+y^2}\right| \Big|_{-1}^{3}$$

$$l = \frac{3\sqrt{10} + \sqrt{2}}{2} + \frac{1}{2}\left(\ln(3+\sqrt{10}) - \ln(1+\sqrt{2})\right)$$

9. Iz tačke $P(-2,1)$ van parabole $y^2=4x$ povučene su tangente na ...

a) Naći j-ne tangenti

b) Izračunati dužinu luka od njenog tjemena do one dodirne tačke sa tangentama koja ima veću apcisu.



1) $t: y=kx+n$ - tangenta na parabolu iz tačke $P(-2,1)$

$P \in t \Rightarrow 1=-2k+n$

$\begin{cases} t: y=kx+n \\ y^2=4x \end{cases}$ $(kx+n)^2=4x$

$\underbrace{k^2}_{a}x^2+(\underbrace{2kn-4}_{b})x+\underbrace{n^2}_{c}=0$

Ova kv. j-na ima jedno rješenje jer t ima 1 presječnu tačku

$D=0$

$(2kn-4)^2-4k^2n^2=0$

$-4kn+4=0$

$kn=1$

2) $\begin{cases} -2k+n=1 \quad /\cdot(-k) \\ kn=1 \end{cases}$

$2k^2=-k+1$

$2k^2+k-1=0$

$k_{1,2}=\frac{-1\pm 3}{4}$

$k_2=-1, \; k_1=1/2$

$n_2=-1 \quad n_1=2$

$t_1: y=\frac{1}{2}x+2$

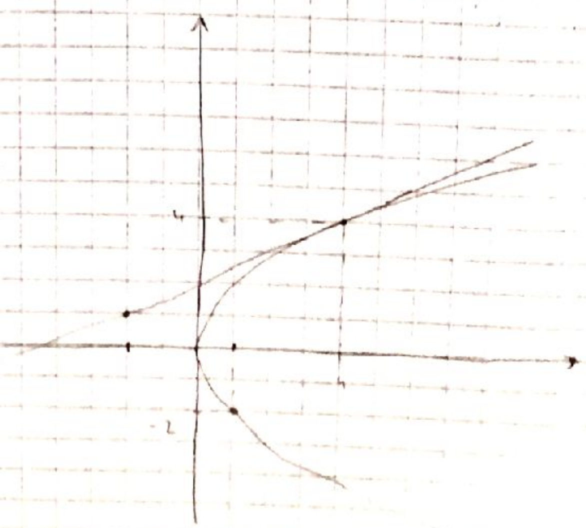
$t_2: y=-x+1$

3) I $\begin{cases} y^2=4x \\ y=-x+1 \end{cases}$ $A(1,-2)$ $k_1=y'(x)$ $y'(x)=-1$

$\begin{cases} y^2=4x \\ y=\frac{1}{2}x+2 \end{cases}$ $B(4,4)$ $k_2=y'(x)$ $y'(x)=\frac{1}{2}$

$k_t=y'(x)$

~~$y'(x)=-1$~~



$y^2 = 4x \Rightarrow x = \frac{1}{4}y^2$

$$l = \int_0^4 \sqrt{1+x'^2(y)}\,dy = \int_0^4 \sqrt{1+\frac{1}{4}y^2}\,dy = \left[\begin{array}{l} y = 2z \\ dy = 2dz \end{array} \quad \begin{array}{c|c|c} y & 0 & 4 \\ \hline z & 0 & 2 \end{array}\right] =$$

$$= 2\int_0^2 \sqrt{1+z^2}\,dz = \left| \begin{array}{l} u = \sqrt{1+z^2} \Rightarrow du = \frac{z\,dz}{\sqrt{1+z^2}} \\ v = \int dz = z \end{array} \right| = 2z\sqrt{1+z^2}\Big|_0^2 - 2\int_0^2 \frac{z^2\,dz}{\sqrt{1+z^2}}$$

parc. integracija

10. Naći dužinu luka krive $\begin{cases} x = \frac{t^6}{6} \\ y = 2 - \frac{t^4}{4} \end{cases}$ između presječnih tačaka sa koo osom

$$l = \int_{t_1}^{t_2} \sqrt{x'^2(t) + y'^2(t)}\,dt \qquad \begin{array}{l} x'(t) = t^5 \\ y'(t) = -t^3 \end{array}$$

Presjek sa $Oy$-osom $\quad x = 0$

$t^6 = 0$

$t = 0$

Presjek sa $Ox$-osom $\quad y = 0$

$2 - \frac{t^4}{4} = 0$

$t^4 = 8$

$t = \sqrt[4]{8}$

$$l = \int_0^{\sqrt[4]{8}} \sqrt{t^{10} + t^6}\,dt = \int_0^{\sqrt[4]{8}} |t^3| \sqrt{t^4+1}\,dt = \left[\begin{array}{l} t^4+1 = z \\ t^3\,dt = \frac{1}{4}dz \end{array} \quad \begin{array}{c|c|c} t & 0 & \sqrt[4]{8} \\ \hline z & 1 & 9 \end{array}\right] =$$

$t^3$ (pozitivan interval)

$$= \frac{1}{4}\int_1^9 \sqrt{z}\,dz = \frac{1}{4}\cdot\frac{2}{3}\cdot\sqrt{z^3}\Big|_1^9 = \frac{1}{6}(27-1) = \frac{13}{3}$$