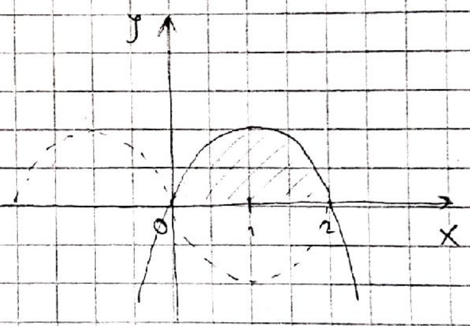
11. Naći V tijela koje nastaje rotacijom figure ograničene krivom $y = 2x - x^2$ i pravom $y = 0$ oko : a) Ox - ose b) Oy - ose

$y = -x^2 + 2x$

$y = 0 \iff x(2-x) = 0$

$x = 0 \vee x = 2$

$A(0,0) \quad B(2,0)$



$$V = \pi \int_a^b f^2(x)\,dx$$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$T\left(-\frac{2}{-2}, -\frac{4}{-4}\right)$

$T(1, 1)$

a) $V = \pi \int_0^2 (2x - x^2)^2\,dx = \frac{4\pi}{3} x^3 \Big|_0^2 - \pi x^4 \Big|_0^2 + \frac{\pi}{5} x^5$

$= \frac{16}{15}\pi$

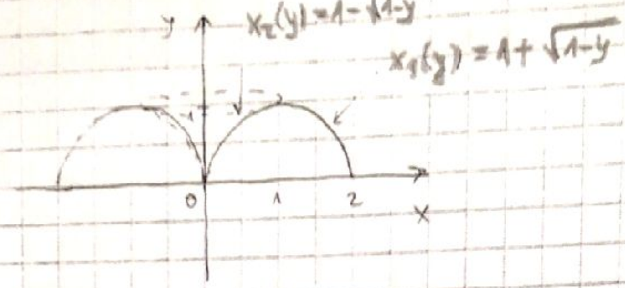
b) $y = -x^2 + 2x$

$x^2 - 2x + y = 0$

$x_{1,2} = \frac{2 \pm \sqrt{4-4y}}{2}$

$x_1(y) = 1 + \sqrt{1-y}$

$x_2(y) = 1 - \sqrt{1-y}$



$$V = \pi \int_0^1 \left(x_1^2(y) - x_2^2(y)\right) dy = \pi \int_0^1 \left(\left(1+\sqrt{1-y}\right)^2 - \left(1-\sqrt{1-y}\right)^2\right) dy =$$

$$= \pi \int_0^1 4\sqrt{1-y}\, dy = 4\pi \int_0^1 \sqrt{1-y}\, dy = \left[\begin{array}{c} 1-y = t \\ dy = -dt \end{array} \quad \begin{array}{c|c|c} y & 0 & 1 \\ \hline t & 1 & 0 \end{array}\right] =$$

$$= -4\pi \int_1^0 \sqrt{t}\, dt = 4\pi \int_0^1 t^{\frac{1}{2}} dt = 4\pi \cdot \frac{2}{3} \sqrt{t^3} \Big|_0^1 = \frac{8\pi}{3}$$

Napomena: $V_y = 2\pi \int_a^b x f(x)\, dx$

$$V_y = 2\pi \int_0^2 x\left(-x^2 + 2x\right) dx = -\frac{\pi}{2} x^4 \Big|_0^2 + \frac{4\pi}{3} x^3 \Big|_0^2 = -8\pi + \frac{32\pi}{3} = \frac{8\pi}{3}$$

12. U tački $P(3, y_0)$ parabole $y^2 = 2(x-1)$ povučena je tangenta na parabolu. Izračunati V tijela koje nastaje rotacijom oko Ox-ose figure ograničene parabolom i Ox-osom.

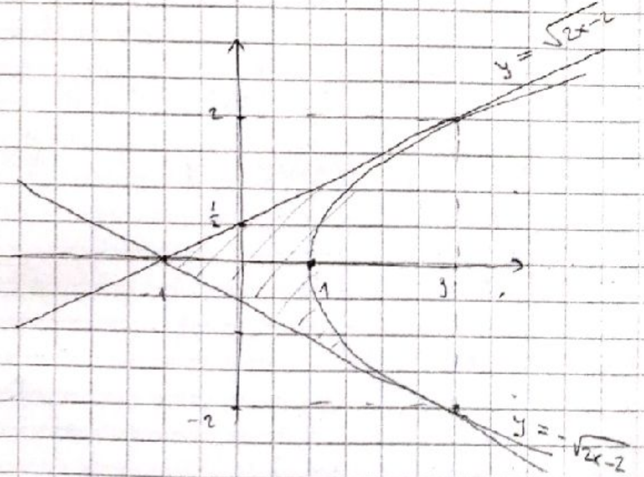
$y^2 = 2x - 2$

$x = \frac{1}{2} y^2 + 1$

Nema presjeka sa Oy osom

Presjek sa Ox-osom $(y=0)$

$x = 1 \qquad T(1,0)$



$P(3, y_0) \in$ paraboli

$y_0^2 = 2 \cdot 3 - 2 = 4$

$y_0 = \pm 2$

$P_1(3,2), \; P_2(3,-2)$

Tangenta u tački $P_1(3,2)$

$t_1: \; y - 2 = y'(3)(x-3) \qquad y = \pm\sqrt{2x-2}$

$y^2 = 2x - 2$

$2yy' = 2 \Rightarrow y' = \frac{1}{y} \qquad y'(3) = \frac{1}{2}$

$y - 2 = \frac{1}{2}(x-3)$

$y = \frac{1}{2}x + \frac{1}{2}$

Rotacijom date figure i njene polovine nastaje isto tijelo

$V = V_1 - V_2$ $\qquad\qquad \frac{4\pi}{3}$

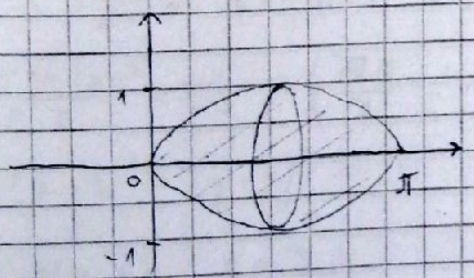
$V_1 = \pi \int_{-1}^{3} \left(\frac{1}{2}x + \frac{1}{2}\right)^2 dx$

$V_2 = \pi \int_{1}^{3} (2x-2)\, dx$ ...

18. mart

1. Naći V tijela koje nastaje rotacijom krive $y = \sin x$ i prave $y=0$ na segmentu $[0, \pi]$ oko a) Ox-ose b) Oy-ose

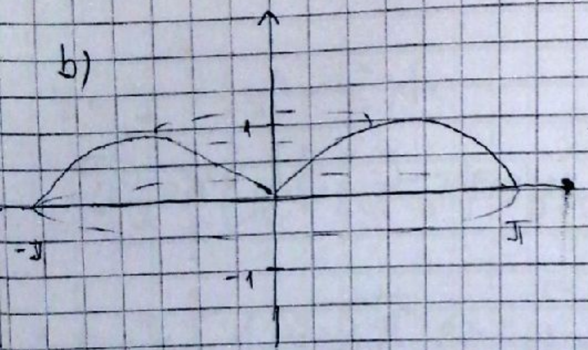
a)



$V = \pi \int_0^{\pi} \sin^2 x\, dx = \pi \int_0^{\pi} \frac{1-\cos 2x}{2}\, dx =$

$= \frac{\pi}{2} x \Big|_0^{\pi} - \frac{\pi}{2} \cdot \frac{1}{2} \sin 2x \Big|_0^{\pi} = \frac{\pi^2}{2}$

b)



$V = 2\pi \int_0^{\pi} x\, f(x)\, dx$

$V = 2\pi \int_0^{\pi} x \sin x\, dx = \left| \cancel{u = \sin x \Rightarrow du = \cos x\, dx} \quad \cancel{v = \int x\, dx = \frac{x^2}{2}} \right|$

$= \left| \begin{matrix} u = x \Rightarrow du = dx \\ v = \int \sin x\, dx = -\cos x \end{matrix} \right| = 2\pi \left(-x\cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x\, dx \right.$

$= 2\pi \left(-\pi \cdot (-1) + \sin x \Big|_0^{\pi} \right) = 2\pi \cdot \pi = 2\pi^2$

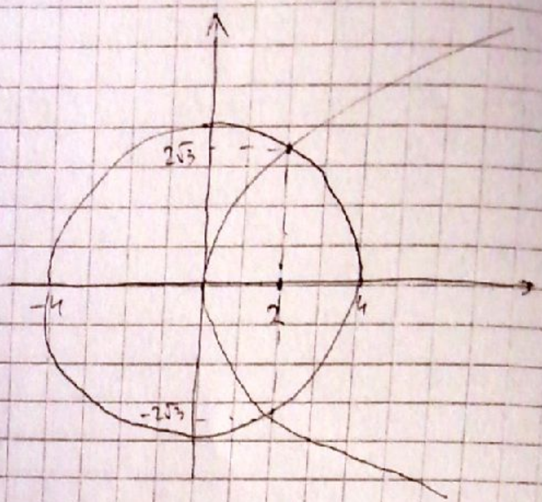
2. Izračunati V nastalu rotacijom figure ograničene krivom $x^2+y^2=16$ krivom $y^2=6x$: a) oko Ox-ose b) oko Oy-ose

Presjek kruga i parabole

$$\begin{cases} x^2+y^2=16 \\ y^2=6x \end{cases} \qquad \begin{aligned} & x^2+6x-16=0 \\ & x_{1,2}=-3\pm5 \\ & x_1=2,\ \cancel{x_2=-8} \quad x>0 \\ & \downarrow \\ & y=\pm 2\sqrt{3} \end{aligned}$$

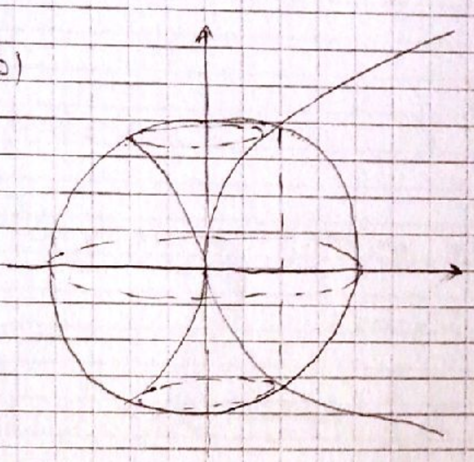
$A_1(2,2\sqrt{3}),\ A_2(2,-2\sqrt{3})$

a)



$$V=\pi\int_0^2 6x\,dx+\pi\int_2^4(16-x^2)\,dx=\frac{76}{3}\pi$$

b)



$$x^2=16-y^2\ ,\ y^2=6x \Rightarrow x=\frac{1}{6}y^2 \rightarrow x^2=\frac{1}{36}y^4$$

$$V=2\pi\int_0^{2\sqrt{3}}\left(16-y^2-\frac{1}{36}y^4\right)dy$$

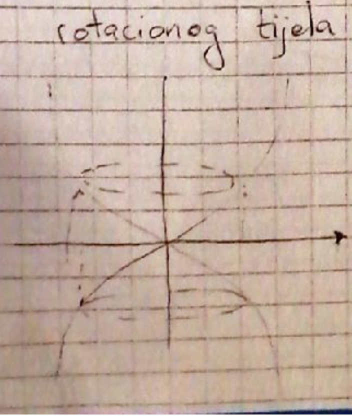
$$V=2\pi\left(16y\Big|_0^{2\sqrt{3}}-\frac{y^3}{3}\Big|_0^{2\sqrt{3}}-\frac{1}{36}\frac{y^5}{5}\Big|_0^{2\sqrt{3}}\right)=$$

$$=2\pi\left(32\sqrt{3}-8\sqrt{3}-\frac{8\sqrt{3}}{5}\right)=2\pi\cdot\frac{112\sqrt{3}}{5}$$

$$V=\frac{224\sqrt{3}}{5}\pi$$

Površina rotacionog tijela

3. Luk krive $y=x^3$ rotira oko Ox-ose na segmentu $\left[-\frac{2}{3},\frac{2}{3}\right]$. Naći P rotacionog tijela



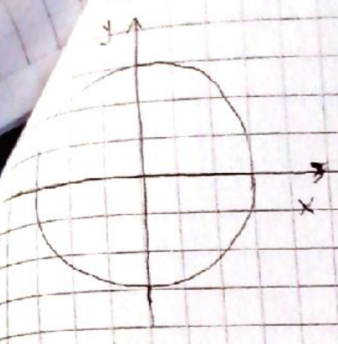
$$P=2\cdot2\pi\int_0^{2/3}y(x)\sqrt{1+y'^2(x)}\,dx \qquad y(x)=x^3\ ;\ y'(x)=3x^2$$

$$P=4\pi\int_0^{2/3}x^3\sqrt{1+9x^4}\,dx=\left[\begin{array}{l} 1+9x^4=t \\ x^3dx=\frac{1}{36}dt \end{array}\ \begin{array}{c|c|c} x & 0 & \frac{2}{3} \\ \hline t & 1 & \frac{25}{9} \end{array}\right]$$

$$=4\pi\cdot\frac{1}{36}\int_1^{25/9}\sqrt{t}\,dt=\frac{2\pi}{27}\sqrt{t^3}\Big|_1^{25/9}=\frac{196}{729}\pi$$

Površina lopte poluprečnika r

$$x^2+y^2=r^2 \to y^2=r^2-x^2 \to y=\pm\sqrt{r^2-x^2}$$



lopta nastaje rotacijom ovog kruga oko Ox-ose

svejedno je da li rotira cio ili pola kruga

neka rotira $\frac{1}{4}$ pa množimo sa 2

$$P = 2\cdot 2\pi\int_0^r y(x)\sqrt{1+y'^2(x)}\,dx$$

$y(x)=\sqrt{r^2-x^2}$ jer uzimamo + dio

$y'(x) = -\frac{x}{\sqrt{r^2-x^2}}$

$$P=4\pi\int_0^r \sqrt{r^2-x^2}\cdot\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = \dots$$

$$=4\pi\int_0^r\sqrt{r^2-x^2}\cdot\frac{r}{\sqrt{r^2-x^2}}\,dx = 4\pi r\int_0^r dx = 4\pi r^2$$

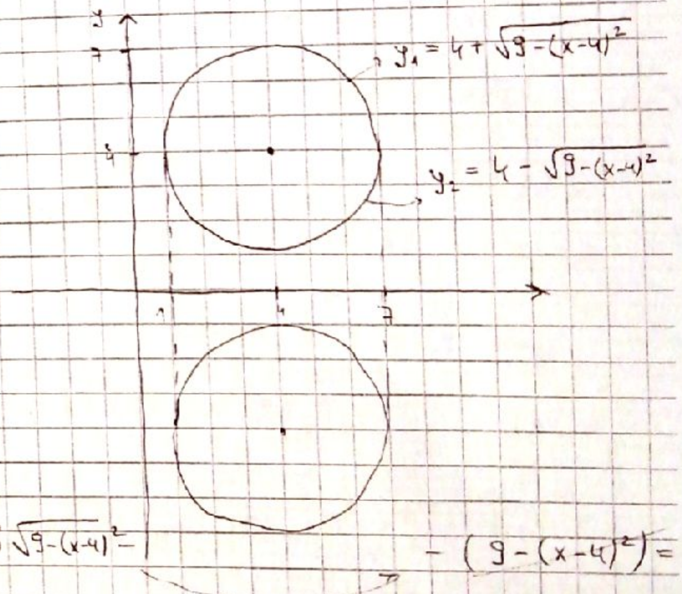
Kombinovano → kolokvijumi

5. Izračunati površinu i zapreminu tijela koje se dobija rotacijom krive $x^2+y^2-8x-8y+23=0$ oko Ox-ose

$(x-4)^2-16+(y-4)^2-16+23=0$

$(x-4)^2+(y-4)^2=9$

$A(4,4)$, $r=3$



$$V=\pi\int_1^7 (y_1^2-y_2^2)\,dx =$$

$$=\pi\int_1^7\left(16+8\sqrt{9-(x-4)^2}+9-(x-4)^2-16+8\sqrt{9-(x-4)^2}-(9-(x-4)^2)\right)=$$

$$=\pi\int_1^7 16\sqrt{9-(x-4)^2}\,dx = \left[\begin{array}{l} x-4=t \\ dx=dt \end{array}\quad \begin{array}{c|c|c} x & 1 & 7 \\ t & -3 & 3 \end{array}\right]$$

$$=16\pi\int_{-3}^{3}\sqrt{9-t^2}\,dt = \left[\begin{array}{l} t=3\sin z \\ dt=3\cos z\,dz \\ z=\arcsin\frac{t}{3} \end{array}\quad \begin{array}{c|c|c} t & -3 & 3 \\ z & -\frac{\pi}{2} & \frac{\pi}{2} \end{array}\right] = 16\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\cdot 3\sqrt{1-\sin^2 z}\cos z\,dz =$$

$$=144\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 z\,dz = 72\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2z)\,dz = \cancel{72\pi \ \frac{\pi}{2}}$$

$$\left[\begin{array}{l} 2z=s \\ dz=\frac{ds}{2} \end{array}\quad \begin{array}{c|c|c} z & -\frac{\pi}{2} & \frac{\pi}{2} \\ s & -\pi & \pi \end{array}\right]$$

$$= 72\pi\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{1}{2}\sin s\Big|_{-\pi}^{\pi}\right) = 72\pi\cdot\pi = 72\pi^2$$

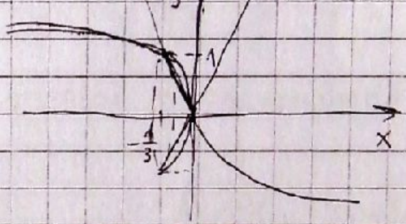
<u>Površina</u> → gornji dio rotira + donji dio rotira

$$P = 2\pi\int_1^7 y_1(x)\sqrt{1+y_1'^2(x)}\,dx + 2\pi\int_1^7 y_2(x)\sqrt{1+y'^2(x)}\,dx$$

$$y_1'(x) = \frac{-(x-4)}{\sqrt{9-(x-4)^2}} \quad ; \quad y_2'(x) = \frac{x-4}{\sqrt{9-(x-4)^2}}$$

6. Figura F ograničena krivama $y=-\sqrt[3]{3x}$, $y=3|x|$ rotira oko Oy-ose. Skicirati figuru F i izračunati P tjela koje nastaje rotacijom.

$$y = 3|x| = \begin{cases} -3x, & x<0 \\ 3x, & x\geq 0\end{cases}$$



presjek krive i prave

$y=-\sqrt[3]{3x}$, $y=3x$

$x\geq 0$, $-\sqrt[3]{3x} = 3x \,/^3 \rightarrow -3x = 27x^3$

$3x(9x^2+1)=0$

$x=0 \quad \vee \quad 9x^2+1=0$

$x\notin\mathbb{R}$

$-\sqrt[3]{3x} = -3x \,/^3$, $x<0$

$3x = 27x^3$

$3x(9x^2-1)=0$

$x=0 \quad \vee \quad x^2=\frac{1}{9}$

$x = \pm\frac{1}{3} \Rightarrow x = -\frac{1}{3}$

$y = \sqrt[3]{3x}$

$y^3 = 3x$

$x = \frac{1}{3}y^3$

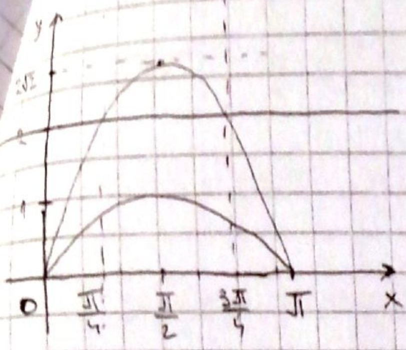
$x' = y^2$

$$P_1 = 2\pi\int_0^1 x(y)\cdot\sqrt{1+x'^2(y)}\,dy$$

$$P_1 = 2\pi\int_0^1 \frac{1}{3}y^3\sqrt{1+y^4}\,dy = \left[\begin{array}{l} 1+y^4=t \\ 4y^3dy=dt \\ y^3dy=\frac{1}{4}dt\end{array}\quad \begin{array}{c|c|c} y & 0 & 1 \\ \hline t & 1 & 2\end{array}\right] =$$

$$= \frac{2\pi}{12}\int_1^2 \sqrt{t}\,dt = \frac{\pi}{6}\cdot\frac{2}{3}\sqrt{t^3}\Big|_1^2 = \frac{\pi}{9}\left(\sqrt{8}-1\right)$$

...gura ograničena lukovima krivih $y = \sin x$ i $y = 2\sqrt{2}\sin x$ na $[0, \pi]$ ... pravam $y = 2$ rotira oko Ox - ose. Izračunati V tako dobijenog rot. tijela



$$\begin{cases} y = 2 \\ y = 2\sqrt{2}\sin x \end{cases} \to \quad 2\sqrt{2}\sin x = 2$$

$$\sin x = \frac{\sqrt{2}}{2}$$

$$x = \frac{\pi}{4} \lor x = \frac{3\pi}{4}$$

$$V = V_1 + V_2 + V_3, \quad V_1 = V_3$$

$$V_1 = \pi \int_0^{\frac{\pi}{4}} (8\sin^2 x - \sin^2 x)\,dx = 7\pi \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{2}\,dx = \frac{7\pi}{2} \cdot x \Big|_0^{\frac{\pi}{4}} - \frac{7\pi}{2} \cdot \frac{1}{2} \sin 2x \Big|$$

$$= \frac{7\pi^2}{8} - \frac{7\pi}{4} = \frac{7\pi}{8}(\pi - 2)$$

$$V_2 = \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4 - \sin^2 x)\,dx = 4\pi x \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} - \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1 - \cos 2x}{2}\,dx =$$

$$= 4\pi \cdot \frac{\pi}{2} - \frac{\pi}{2}\left(x \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} - \frac{1}{2}\sin 2x \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \right) = 2\pi^2 - \frac{\pi}{2}\left(\frac{\pi}{2} + 1\right) = \frac{7\pi^2}{4} - \frac{\pi}{2} =$$

$$= \frac{\pi}{4}(7\pi - 2)$$

2. Neka je F figura određena pozitivnim djelovima koordinatnih osa i krivama $x-y-x^2=0$ i $2x+3y-3=0$. Skicirati figuru F, a zatim izračunati zapreminu tijela koje nastaje njenom rotacijom oko Oy ose.

2. $x-y-x^2=0 \quad 2x+3y-3=0$

$y=x-x^2$, $y=-x^2+x$ $\qquad$ $y=0 \Leftrightarrow x(1-x)=0$

$a=-1,\ b=1\ c=0$ $\qquad$ $\Leftrightarrow x=0 \vee x=1$

$D=1$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$T\left(-\frac{1}{-2}, -\frac{1}{-4}\right)$

$T\left(\frac{1}{2}, \frac{1}{4}\right)$

$\begin{cases} y=-x^2+x \\ y=\frac{1}{3}(3-2x) \end{cases}$

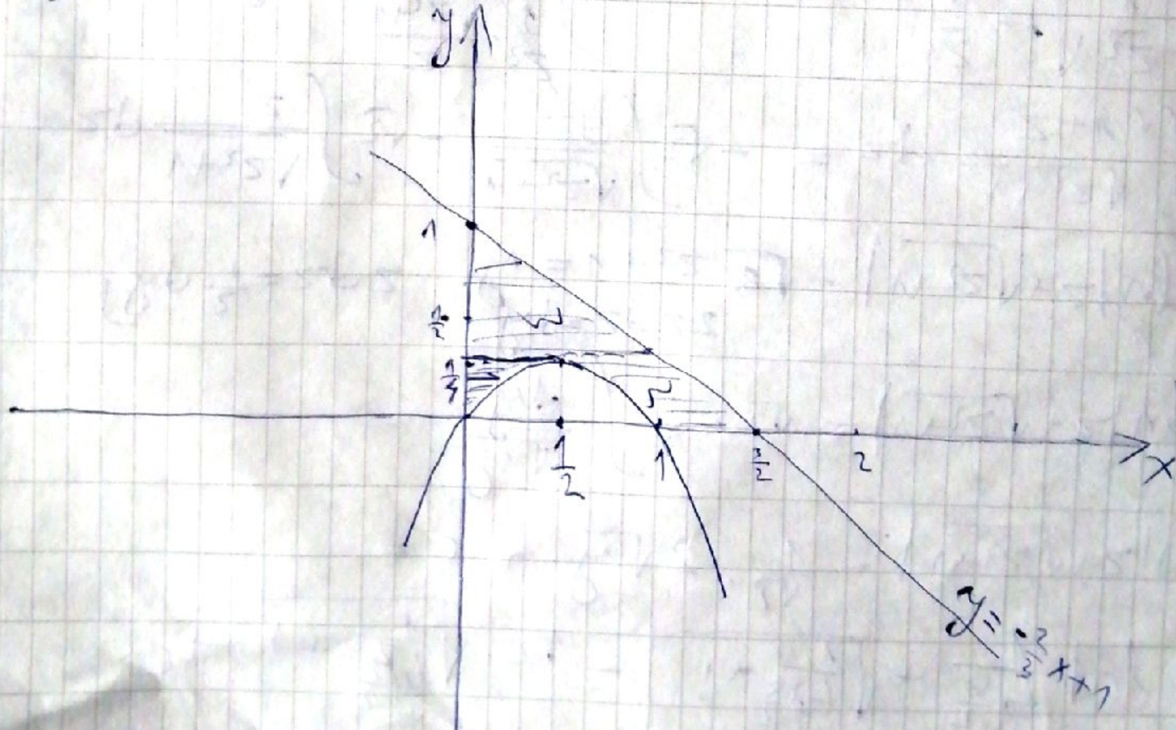
$-x^2+x=\frac{1}{3}(3-2x)$

$-3x^2+3x=3-2x$

$3x^2-5x+3=0$

$D=25-36<0$ nema presj.

$y=-\frac{2}{3}x+1$



$2x+3y-3=0$

$2x=3-3y$

$x=\frac{3}{2}-\frac{3}{2}y$

$V_1 = \frac{1}{3} B \cdot H, \quad B = r^2\pi \quad H = 1$

$r = \frac{3}{2}$

$$V_1 = \pi \int_0^1 \left(\frac{3}{2} - \frac{3}{2}y\right)^2 dy = \pi\left(\frac{9}{4}y\Big|_0^1 - \frac{9}{2}\cdot\frac{y^2}{2}\Big|_0^1 + \frac{9}{4}\frac{y^3}{3}\Big|_0^1\right) =$$

$$= \pi\left(\frac{9}{4} - \frac{9}{4} + \frac{9}{12}\right) = \frac{3}{4}\pi$$

$$V_2 = 2\pi \cdot \int_0^1 x \cdot f(x)\,dx = 2\pi \int_0^1 x \cdot (-x^2 + x)\,dx =$$

$$= 2\pi \int_0^1 (-x^3 + x^2)\,dx = -2\pi \cdot \frac{x^4}{4}\Big|_0^1 + 2\pi \cdot \frac{x^3}{3}\Big|_0^1 =$$

$$= -\frac{\pi}{2} + \frac{2\pi}{3} = \frac{-3\pi + 4\pi}{6} = \frac{\pi}{6}$$

$$V = V_1 - V_2 = \frac{3}{4}\pi - \frac{\pi}{6} = \frac{9\pi - 2\pi}{12} = \frac{7\pi}{12}$$

II $\quad y = x - x^2$ ← Drugi način za V_2

$x^2 - x + y = 0$

$x_{1,2} = \frac{1 \pm \sqrt{1-4y}}{2} = \frac{1}{2} \pm \frac{\sqrt{1-4y}}{2}$

$x_1(y) = \frac{1 + \sqrt{1-4y}}{2}$

$x_2(y) = \frac{1 - \sqrt{1-4y}}{2}$

$$V_2 = \pi \int_0^{1/4} (x_1^2(y) - x_2^2(y))\,dy = \pi \int_0^{1/4} \frac{1}{4}\left(\cancel{1} + 2\sqrt{1-4y} + \cancel{1-4y} - \cancel{1} + 2\sqrt{1-4y} - \cancel{(1-4y)}\right)dy$$

$$= \frac{\pi}{4} \cdot 4 \cdot \int_0^{1/4} \sqrt{1-4y}\,dy = \pi \cdot \int_0^{1/4} \sqrt{1-4y}\,dy = \quad \begin{matrix} 1-4y = t \\ dy = -\frac{1}{4}dt \end{matrix} \quad \begin{array}{c|c|c} y & 0 & \frac{1}{4} \\ \hline t & 1 & 0 \end{array}$$

$$= -\frac{\pi}{4} \int_1^0 \sqrt{t}\,dt = \frac{\pi}{4} \int_0^1 \sqrt{t}\,dt = \frac{\pi}{4} \cdot \frac{2}{3}\sqrt{t^3}\Big|_0^1 = \frac{\pi}{6}$$

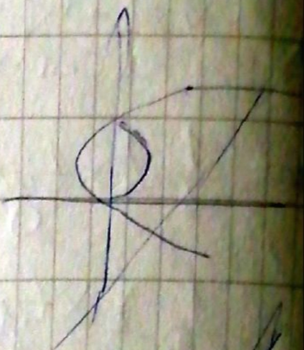
III $$V = \underbrace{\pi \int_0^{1/4} x_2(y)\,dy}_{1} + \underbrace{\pi \int_0^{1/4} (x^2(y) - x_1^2(y))\,dy}_{2} + \underbrace{\pi \int_{1/4}^{1} x^2(y)\,dy}_{3}$$

$x(y) = \frac{1}{2}(3 - 3y)$

2. Neka je F figura ograničena krivama $x = |y^2 - y|$ i $y = x - 3$ koja leži u prvom kvadrantu. Skicirati figuru F, a zatim izračunati površinu tijela koje nastaje rotacijom figure F oko Ox ose.

2. $x = |y^2 - y|$

$x = y^2 - y = y^2 - \cancel{2}y + \frac{1}{4} - \frac{1}{4} = (y - \frac{1}{2})^2 - \frac{1}{4}$



$\begin{cases} x = y^2 - y \\ x = y + 3 \end{cases}$

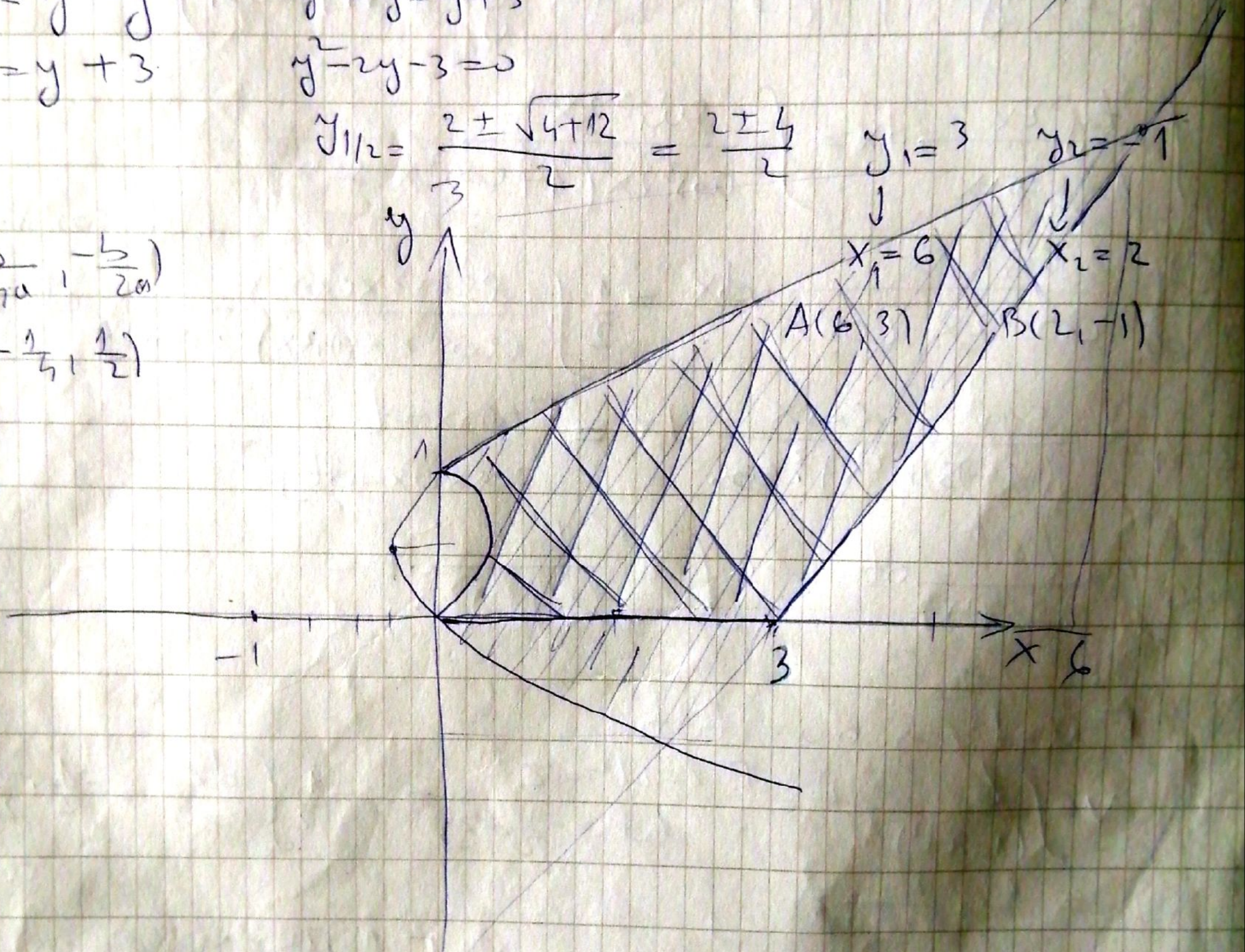
$y^2 - y = y + 3$

$y^2 - 2y - 3 = 0$

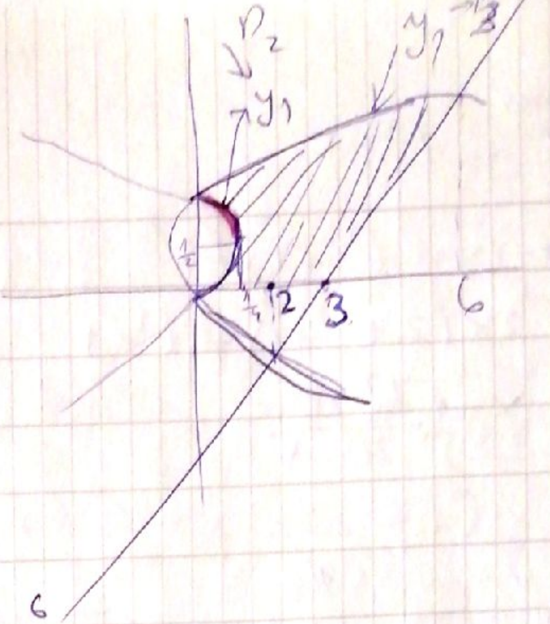
$y_{1/2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2}$ $y_1 = 3$ $y_2 = -1$

$T(-\frac{D}{4a}, -\frac{b}{2a})$

$T(-\frac{1}{4}, \frac{1}{2})$



$$2\pi\int f(x)\sqrt{1+f'^2(x)}\,dx$$



$P = P_1 + P_2 + P_3$

$P_1 \quad y = x+3 \quad [3,6]$

$P_2 \quad \cancel{y}\, x = -(y^2-y) \quad [0, \frac{1}{4}]$

$P_3 \quad x = y^2-y \quad [0,6]$

$$P_1 = 2\pi\int_3^6 (x-3)\cdot\sqrt{1+1}\,dx = 2\pi\sqrt{2}\int_3^6 (x-3)\,dx =$$

$$= 2\pi\sqrt{2}\cdot\left(\frac{x^2}{2}-3x\right)\Big|_3^6 = 2\pi\sqrt{2}\left(\frac{36}{2}-18-\frac{9}{2}+9\right) =$$

P_2: $\quad x = -(y^2-y)$

$y^2 - y + x = 0$

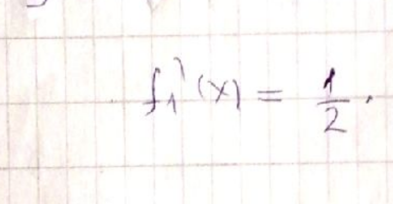
$y_{1/2} = \frac{1 \pm \sqrt{1-4x}}{2}$

$y_1 = \frac{1}{2} + \frac{\sqrt{1-4x}}{2}$

$y_2 = \frac{1}{2} - \frac{\sqrt{1-4x}}{2}$

$$f_1'(x) = \frac{1}{2}\cdot\frac{1}{2\sqrt{1-4x}}\cdot(-4) = \frac{-1}{\sqrt{1-4x}}$$

$$f_2'(x) = -\frac{1}{2}\cdot\frac{1}{2\sqrt{1-4x}}\cdot(-4) = \frac{1}{\sqrt{1-4x}}$$



$P_2 = P_2' + P_2''$

$$P_2' = 2\pi\int_0^{1/4} f_1(x)\sqrt{1+f_1'^2(x)}\,dx = 2\pi\int_0^{1/4}\left(\frac{1}{2}+\frac{\sqrt{1-4x}}{2}\right)\cdot\sqrt{1+\frac{1}{1-4x}}\,dx =$$

$$= 2\pi\int_0^{1/4}\left(\frac{1}{2}+\frac{\sqrt{1-4x}}{2}\right)\cdot\frac{\sqrt{2-4x}}{\sqrt{1-4x}}\,dx =$$

$$= \pi\int_0^{1/4}\sqrt{\frac{2-4x}{1-4x}}\,dx + \pi\int_0^{1/4}\sqrt{2-4x}\,dx =$$

$$= \sqrt{2}\pi\int_0^{1/4}\sqrt{\frac{1-2x}{1-4x}}\,dx + \pi\sqrt{2}\int_0^{1/4}\sqrt{1-2x}\,dx =$$

$$\sqrt{\frac{1-2x}{1-4x}} = t^2$$

$$1-2x = t^2 - 4xt^2$$

$$2x(2t^2-1) = t^2-1$$

$$x = \frac{1}{2}\cdot\frac{t^2-1}{2t^2-1}$$

$$dx = \frac{1}{2}\cdot\frac{2t\cdot(2t^2-1) - (t^2-1)4t}{(2t^2-1)^2}$$

$$dx = \frac{1}{2}\,\frac{2t}{(2t^2-1)^2}\,dt$$

$$dx = \frac{t}{(2t^2-1)^2}\,dt$$

...

$$P_2'' = 2\pi\int_0^{1/4} f_2(x)\sqrt{1+f_2'(x)}\,dx =$$

$$= 2\pi\int_0^{1/4}\left(\frac{1}{2} - \frac{\sqrt{1-4x}}{2}\right)\sqrt{1+\frac{1}{1-4x}}\,dx = \dots$$

P_3':

$$x = y^2 - y$$

$$y^2 - y - x = 0$$

$$y_{1/2} = \frac{1 \pm \sqrt{1+4x}}{2}$$

$$y_1 = \frac{1}{2} + \frac{\sqrt{1+4x}}{2}$$

$$y_2 = \frac{1}{2} - \frac{\sqrt{1+4x}}{2}$$

$$f_1'(x) = \frac{1}{2}\cdot\frac{1}{2\sqrt{1+4x}}\cdot(4)$$

$$f_1'(x) = \frac{1}{\sqrt{1+4x}}$$

$$P_3 = 2\pi\int_0^6\left(\frac{1}{2} + \frac{\sqrt{1+4x}}{2}\right)\cdot\sqrt{1+\frac{1}{1+4x}}\,dx = \dots$$

Nesvojstveni integrali

1. a) $\int_0^{+\infty} x e^{-2x}\,dx = \lim_{B\to+\infty} \int_0^B x e^{-2x}\,dx = \left\lceil \begin{array}{l} x = u \Rightarrow du = dx \\ v = \int e^{-2x}\,dx = -\frac{1}{2} e^{-2x} \end{array} \right\rfloor =$

$= -\frac{1}{2} \lim_{B\to+\infty} \left(x e^{-2x} \Big|_0^B - \int_0^B e^{-2x}\,dx \right) = -\frac{1}{2} \lim_{B\to+\infty} \left(\underbrace{\frac{x}{e^{2x}}}_{0} \Big|_0^B + \frac{1}{2}\Big(\underbrace{\frac{1}{e^{2B}}}_{0} - 1\Big)\right) = \frac{1}{4}$

b) $\int_1^{+\infty} \frac{dx}{x\sqrt{1+x^2}} = \lim_{B\to+\infty} \int_1^B \frac{dx}{x\sqrt{1+x^2}} = \left\lceil \begin{array}{l} \frac{1}{x} = t \to x = \frac{1}{t} \\ dx = -\frac{dt}{t^2} \end{array} \quad \begin{array}{c|c|c} x & 1 & B \\ \hline t & 1 & \frac{1}{B} \end{array} \right\rfloor = -\lim_{B\to+\infty} \int_1^{\frac{1}{B}} \frac{\frac{dt}{t^2}}{\frac{1}{t}\sqrt{1+\frac{1}{t^2}}}$

$= -\lim_{B\to+\infty} \int_1^{\frac{1}{B}} \frac{dt}{\sqrt{t^2+1}} = -\lim_{B\to+\infty} \left(\ln\left|t + \sqrt{t^2+1}\right| \right)\Big|_1^{\frac{1}{B}} = -\lim_{B\to+\infty} \Big(\underbrace{\ln\Big|\frac{1}{B} + \sqrt{\frac{1}{B^2}+1}\Big|}_{\ln 1 = 0} - \ln\left|1+\sqrt{2}\right| \Big) =$

$= \ln(1+\sqrt{2})$

2. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{B\to 1^-} \int_0^B \frac{dx}{\sqrt{1-x^2}} = \lim_{B\to 1^-} \arcsin x \Big|_0^B = \lim_{B\to 1^-} (\arcsin B - \underbrace{\arcsin 0}_{0}) =$

3. $\int_2^3 \frac{dx}{\sqrt{6x-x^2-8}} = \left[\begin{array}{l} D = 36-32=4 \\ x_{1,2} = \frac{-3\pm 1}{-1} \begin{array}{l} \nearrow 4 \\ \searrow 2 \end{array} \\ -(x-2)(x-4) \end{array} \right] = \int_2^3 \frac{dx}{\sqrt{-(x-2)(x-4)}} = \lim_{A\to 2^+} \int_A^3 \frac{dx}{\sqrt{-(x-2)(x-4)}} = \left[1-(x-3)^2 \right]$

$= \lim_{A\to 2^+} \int_A^3 \frac{dx}{\sqrt{1-(x-3)^2}} = \lim_{A\to 2^+} \left(\arcsin(x-3) \Big|_A^3 \right) = \lim_{A\to 2^+} (\underbrace{\arcsin 0}_{0} - \arcsin(a-3)) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

problem u tački 0

4. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx = \int_{-\frac{\pi}{2}}^{0} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx = \lim_{B\to 0^-} \int_{-\frac{\pi}{2}}^{B} \frac{\cos x\, dx}{\sqrt[3]{\sin^2 x}} + \lim_{A\to 0^+} \int_A^{\frac{\pi}{2}} \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx$

$= \left[\begin{array}{l} \sin x = t \\ \cos x\, dx = dt \end{array} \right] = \lim_{B\to 0^-} 3\sqrt[3]{\sin x} \Big|_{-\frac{\pi}{2}}^{B} + \lim_{A\to 0^+} 3\sqrt[3]{\sin x} \Big|_A^{\frac{\pi}{2}} = 3 + 3 = 6$

1. Ispitati konvergenciju integrala $\int_0^1 \frac{dx}{x^\alpha}$

$\alpha \neq 1 \qquad \int_0^1 \frac{dx}{x^\alpha} = \lim_{a\to 0^+} \int_a^1 x^{-\alpha}\, dx = \lim_{a\to 0^+} \frac{x^{-\alpha+1}}{1-\alpha} \Big|_a^1 = \lim_{a\to 0^+} \left(\frac{1}{1-\alpha} - \frac{a^{1-\alpha}}{1-\alpha} \right) =$

$= \begin{cases} +\infty, & \alpha > 1 \\ \frac{1}{1-\alpha}, & \alpha < 1 \end{cases}$

$\alpha = 1 \qquad \int_0^1 \frac{dx}{x} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x} = \lim_{a\to 0^+} \ln|x| \Big|_a^1 = \lim_{a\to 0^+} (\underbrace{\ln 1}_{0} - \underbrace{\ln a}_{-\infty}) = +\infty$

→ Za $\alpha < 1$ → integral konvergira i njegova vrijednost je $\frac{1}{1-\alpha}$, a za $\alpha \geq 1$ integral divergira

2. Ispitati konvergenciju $\int_1^{+\infty} \frac{dx}{x^\alpha}$

$\alpha \neq 1 : \int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{B\to +\infty} \frac{x^{1-\alpha}}{1-\alpha} \Big|_1^B = \lim_{B\to +\infty} \left(\frac{B^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} \right) = \begin{cases} -\frac{1}{1-\alpha}, & \alpha > 1 \\ +\infty, & \alpha < 1 \end{cases}$

$\alpha = 1 \quad \int_1^{+\infty} \frac{dx}{x} = \lim_{B\to +\infty} \ln x \Big|_1^B = \lim_{B\to +\infty} (\ln B - \ln 1) = +\infty$

pitati konvergenciju integrala $\int_0^{+\infty} \frac{\sin x \cdot \operatorname{arctg} x}{1+x^2}\,dx$

Neka je $f(x) = \frac{\sin x \cdot \operatorname{arctg} x}{1+x^2}$

$$|f(x)| = \left| \frac{\sin x \cdot \operatorname{arctg} x}{1+x^2} \right| \leq \frac{1 \cdot \frac{\pi}{2}}{1+x^2}, \quad \forall x \geq 0$$

$$|f(x)| \leq g(x), \quad g(x) = \frac{\pi}{2} \cdot \frac{1}{1+x^2}$$

$$\int_0^{+\infty} g(x)\,dx = \int_0^{+\infty} \frac{\pi}{2} \cdot \frac{1}{1+x^2}\,dx = \frac{\pi}{2} \lim_{B \to +\infty} \int_0^{B} \frac{dx}{1+x^2} = \frac{\pi}{2} \lim_{B \to +\infty} (\operatorname{arctg} B - \operatorname{arctg} 0) = \frac{\pi}{4}$$

$\Rightarrow \int_0^{+\infty} g(x)\,dx$ konvergira

$$\left.\begin{array}{l} |f(x)| \leq g(x), \ \forall x \geq 0 \\ \int_0^{+\infty} g(x)\,dx \text{ konvergira} \end{array}\right\} \Rightarrow \int_0^{+\infty} f(x)\,dx \text{ takođe konvergira}$$

25. mart

$$\int_2^{+\infty} \frac{dx}{e^x + 2\sqrt{x} + 3}$$

нека је $f(x) = \dfrac{1}{e^x + 2\sqrt{x} + 3}$

$$|f(x)| = \left| \frac{1}{e^x + 2\sqrt{x} + 3} \right| \leq \frac{1}{e^x}, \quad \forall x \in [2, +\infty)$$

нека је $g(x) = \dfrac{1}{e^x}$

$$\int_2^{+\infty} g(x) \cdot dx = \lim_{b \to +\infty} \int_2^b \frac{1}{e^x} dx = \lim_{b \to +\infty} -e^{-x} \Big|_2^b =$$

$$= \lim_{b \to +\infty} \left(-\frac{1}{e^b} + \frac{1}{e^2} \right) = \frac{1}{e^2} \Rightarrow \int_2^{+\infty} g(x)\, dx$$

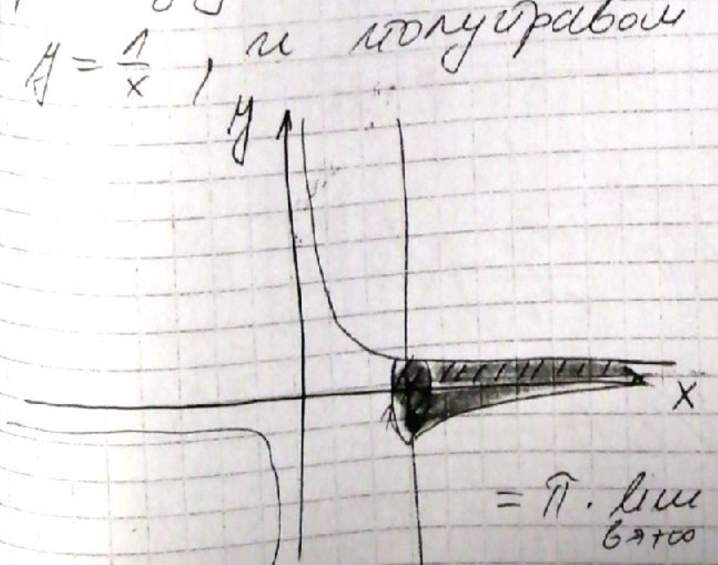
конвергира

1° $|f(x)| \leq g(x), \quad \forall x \in [-2, +\infty)$

2° $\int_2^{+\infty} g(x) \cdot dx$ конв.

$\Rightarrow \int_2^{+\infty} f(x) \cdot dx$ конвергира

7. Израчунати запремину тела које се добију ротацијом ~~фиг~~ фигуре ограничене кривом $y = \frac{1}{x}$, и полуправом $(y = 0,\ x > 1)$ око 0-x осе



$$V = \pi \int_1^{+\infty} \left(\frac{1}{x} \right)^2 dx =$$

$$= \pi \cdot \lim_{b \to +\infty} \int_1^b \frac{1}{x^2} dx$$

$$= \pi \cdot \lim_{b \to +\infty} -\frac{1}{x} \Big|_1^b = \pi \cdot \lim_{b \to +\infty} \left(-\frac{1}{b} + 1 \right) = \pi$$

⑤ Израчунати површину дијела равни који лежи између криве $y = \frac{e^x-1}{e^x+1}$ и њене хоризонталне асимптоте у првом квадранту.

$$f(x) = \frac{e^x-1}{e^x+1}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} f(x) \frac{e^x-1}{e^x+1} = \lim_{x\to+\infty} \frac{e^x\left(1-\frac{1}{e^x}\right)}{e^x\left(1+\frac{1}{e^x}\right)} = 1$$

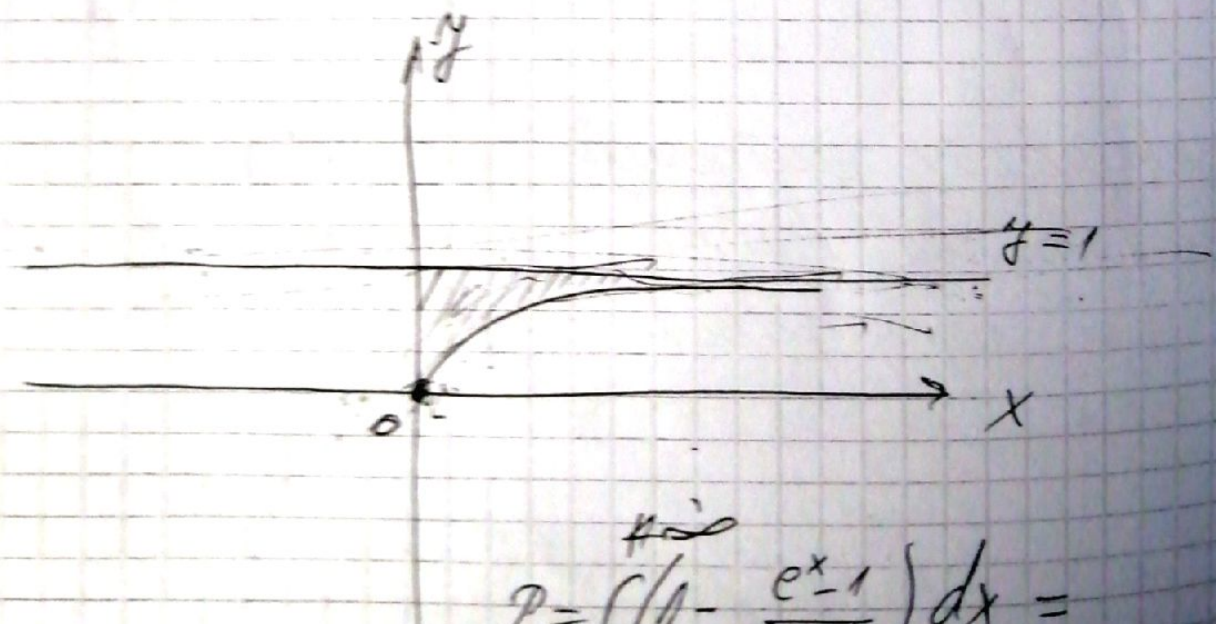
$y=1$ Х.А. $x \to +\infty$

$$1 - f(x) = 1 - \frac{e^x-1}{e^x+1} = \frac{e^x+1-e^x+1}{e^x+1} = \frac{2}{e^x+1} > 0$$

$\forall x \in R \Rightarrow$ график је испод ~~дх~~ осе

па и за $x \geq 0$

$\forall x \in [0,+\infty) \Rightarrow$ посматрамо 1. квадрант



$$P = \int_0^{+\infty} \left(1 - \frac{e^x-1}{e^x+1}\right) dx =$$

$f(0) = \frac{0}{2} = 0.$

$= \ln 2$

Испитати конвергенцију интеграла

а) $\int\limits_{-\infty}^{+\infty} \frac{dx}{x^2+2x+1}$

б) $\int\limits_{1}^{\infty} \frac{x+1}{(x^2+1)\cdot\sqrt[3]{x+2}}\,dx$

а) $f(x) = \frac{1}{x^2+x+1}$

Нека је $g(x) = \frac{1}{x^2+1}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\frac{1}{x^2+2x+1}}{\frac{1}{x^2+1}} = \lim_{x\to\infty}\frac{x^2+1}{x^2+2x+1} = 1 > 0 \Rightarrow$$

$\Rightarrow$ интеграл $\int\limits_{-\infty}^{\infty} f(x)\,dx$ и $\int\limits_{-\infty}^{\infty} g(x)\,dx$ - су еквиконвергентни

Показује се да је $\int\limits_{-\infty}^{\infty} g(x)\,dx$ конвергентан интервал, а одатле следи да је $\int\limits_{-\infty}^{\infty} f(x)\,dx$ конвергентан

$$\int\limits_{-\infty}^{\infty} g(x)\,dx = \int\limits_{-\infty}^{\infty}\frac{1}{x^2}\,dx = \lim_{\substack{a\to-\infty\\ b\to+\infty}}\int\limits_{a}^{b}\frac{1}{x^2}\,dx = \dots = \pi.$$

② Испитати конвергенцију интеграла $\int\limits_{1}^{+\infty} f(x)\,dx$

б) $f(x) = \frac{x+1}{(x^2+1)\sqrt[3]{x+2}}$

$$f(x) = \frac{x\left(1+\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x^2}\right)\cdot\sqrt[3]{x\left(1+\frac{2}{x}\right)}} = \frac{x\cdot\left(1+\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x^2}\right)\cdot\sqrt[3]{x}\cdot\sqrt[3]{1+\frac{2}{x}}}$$

$$f(x) = \frac{1+\frac{1}{x}}{x\cdot\left(1+\frac{1}{x^2}\right)\cdot\sqrt[3]{x}\cdot\sqrt[3]{1+\frac{2}{x}}} \sim \frac{1}{x^{\frac{4}{3}}}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\dfrac{1+\frac{1}{x}}{x(1+\frac{1}{x^2})\cdot\sqrt[3]{x}\cdot\sqrt[3]{1+\frac{2}{x}}}}{\dfrac{1}{x^{\frac{4}{3}}}} =$$

$$\lim_{x\to\infty} \frac{1+\cancel{\frac{1}{x}}^{\to 0}}{(1+\cancel{\frac{1}{x^2}})\cdot\sqrt[3]{1+\frac{2}{x}}} = 1 > 0 \Rightarrow$$

интеграл $\int_1^\infty f(x)dx$ и $\int_1^\infty g(x)dx$

су еквиконвергентни

- Интеграл $\int_1^\infty g(x)\cdot dx$, тј. $\int_1^\infty \frac{1}{x^{\frac{4}{3}}} dx$

је конвергентан ($\int_1^\infty \frac{dx}{x^\alpha}$ конв. ако $\alpha > 1$)

па одатле следи да је конвергентан

и $\int_1^\infty f(x)dx$