Određeni integrali

75. $$\int_0^8 \left(1+\sqrt{2x}+\sqrt[3]{x}\right)dx = \int_0^8 dx + \sqrt{2}\int_0^8 \sqrt{x}\,dx + \int_0^8 \sqrt[3]{x}\,dx =$$

$$= x\Big|_0^8 + \sqrt{2}\cdot\frac{2}{3}\sqrt{x^3}\Big|_0^8 + \frac{3}{4}\sqrt[3]{x^4}\Big|_0^8 =$$

$$= 8-0 + \frac{2\sqrt{2}}{3}\left(\sqrt{2}\cdot 2^4-0\right) + \frac{3}{4}\left(2^4-0\right) =$$

$$= 8 + \frac{64}{3} + 12 = \frac{124}{3}$$

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(76) $$\int_{-1}^{1} \frac{x}{\sqrt{5-4x}}\,dx = \left[\begin{array}{l|c|c} 5-4x=t & x & -1 \quad 1 \\ dx=-\frac{1}{4}dt & t & 9 \quad 1 \end{array}\right] =$$

$$= -\frac{1}{4}\int_{9}^{1} \frac{\frac{5-t}{4}}{\sqrt{t}}\,dt = -\frac{1}{16}\int_{9}^{1} \frac{5-t}{\sqrt{t}}\,dt =$$

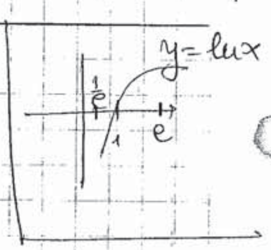
$$= \frac{1}{16}\int_{1}^{9} \left(5t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)dt = \frac{5}{16}\,2\sqrt{t}\,\Big|_{1}^{9} - \frac{1}{16}\,\frac{2}{3}\sqrt{t^3}\,\Big|_{1}^{9} = \frac{1}{6}$$

(77) $$\int_{0}^{2} f(x)\,dx, \quad f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \end{cases}$$

$$\int_{0}^{2} f(x)\,dx = \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx = \int_{0}^{1} x^2\,dx + \int_{1}^{2} (2-x)\,dx =$$

$$= \frac{x^3}{3}\Big|_{0}^{1} + 2x\Big|_{1}^{2} - \frac{x^2}{2}\Big|_{1}^{2} = \frac{5}{6}$$

(78) $$\int_{\frac{1}{e}}^{e} |\ln x|\,dx = \int_{\frac{1}{e}}^{1} |\ln x|\,dx + \int_{1}^{e} |\ln x|\,dx =$$



$$= \int_{\frac{1}{e}}^{1} (-\ln x)\,dx + \int_{1}^{e} \ln x\,dx = \left[\begin{array}{l} U = \ln x \Rightarrow dU = \frac{1}{x}dx \\ V = \int dx = x \end{array}\right] =$$

$$= -\left(x\ln x\,\Big|_{\frac{1}{e}}^{1} - \int_{\frac{1}{e}}^{1} dx\right) + x\ln x\,\Big|_{1}^{e} - \int_{1}^{e} dx =$$

$$= -x\ln x\,\Big|_{\frac{1}{e}}^{1} + x\Big|_{\frac{1}{e}}^{1} + e\ln e - 1\ln 1 - x\Big|_{1}^{e} =$$

$$= -\frac{1}{e} + 1 - \frac{1}{e} + e - e + 1 = 2 - \frac{2}{e}$$

$$\boxed{\int R(x, \sqrt{a^2-x^2})\, dx \to x = a\sin t \text{ ili } x = a\cos t}$$

(79) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \left| \begin{array}{l} x = 2\sin t \\ dx = 2\cos t\, dt \\ \sin t = \frac{x}{2} \\ t = \arcsin\frac{x}{2} \end{array} \quad \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 0 & \frac{\pi}{6} \end{array} \right| =$

$$= \int_0^{\frac{\pi}{6}} \frac{4\sin^2 t}{\sqrt{4-4\sin^2 t}}\, 2\cos t\, dt = 4\int_0^{\frac{\pi}{6}} \frac{\sin^2 t \cos t}{|\cos t|}\, dt =$$

→ oslobađamo se lako jer $0-\pi/6>0$, ali PAŽLJIVO!

$$= 4 \cdot \int_0^{\frac{\pi}{6}} \frac{\sin^2 t\, \cancel{\cos t}}{\cancel{\cos t}}\, dt = 4\int_0^{\frac{\pi}{6}} \frac{1-\cos 2t}{2}\, dt =$$

$$= 2t\Big|_0^{\frac{\pi}{6}} - 2\cdot\frac{1}{2}\sin 2t\Big|_0^{\frac{\pi}{6}} = 2\cdot\frac{\pi}{6} - \frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

---

(80) a) Dokazati da je: $\displaystyle\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

- Neka je $F(x)$ primitivna funkcija funkcije $f(x)$.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \frac{d}{dx}\left(F(t)\Big|_a^x\right) = \frac{d}{dx}\left(F(x) - F(a)\right) =$$

izvod const = 0

$$= F'(x) = f(x)$$

ućemo u zagradu sa $\frac{d}{dx}$

b) Izračunati $\displaystyle\frac{d}{dx}\left(\int_x^b \sin t^2\, dt\right)$

Neka je $f(t) = \sin t^2$ i neka je F primitivna funkcija funkcije f.

$$\frac{d}{dx}\left(\int_x^b \sin t^2\, dt\right) = \frac{d}{dx}\left(\int_x^b f(t)\,dt\right) =$$

$$= \frac{d}{dx}\left(F(t)\Big|_x^b\right) = \frac{d}{dx}\left(F(b) - F(x)\right) =$$

$$= 0 - F'(x) = -f(x) = -\sin x^2$$

Površina ravnog lika

(81) Izračunati površinu figure ograničene krivom $y = x^2 + 2x + 2$ i pravama $y=0$, $x=2$ i $x=-3$.

$y = x^2 + 2x + 2 \rightarrow a = 1;\ b = 2;\ c = 2$

$D = b^2 - 4ac = 4 - 4\cdot 2\cdot 1 = -4 < 0 \rightarrow$ kriva nema presjeka sa x osom

Tjeme $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ $T\left(-\frac{2}{2}, -\frac{-4}{4}\right)$

$\boxed{T(-1, 1)}$

Presjek parabole i y ose $(x=0)$

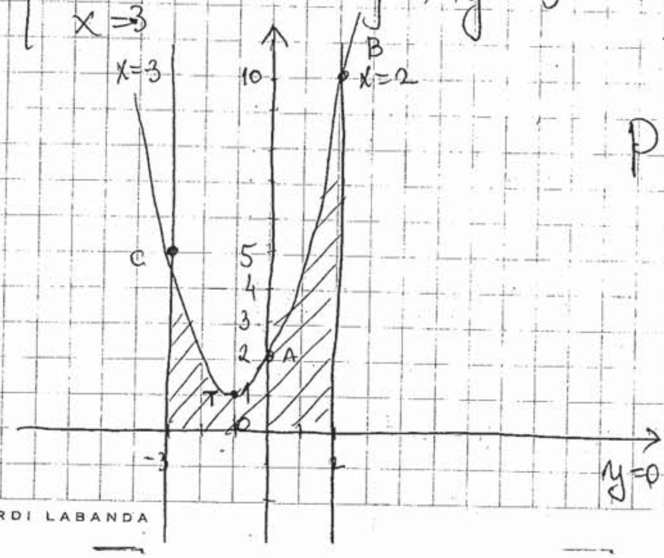
$\left\{ \begin{matrix} y = x^2 + 2x + 2 \\ x = 0 \end{matrix} \right\} \Rightarrow y = 2 \rightarrow \boxed{A(0, 2)}$

Presjek parabole i prave $x = 2$

$\left\{ \begin{matrix} y = x^2 + 2x + 2 \\ x = 2 \end{matrix} \right\} \Rightarrow y = 10 \rightarrow \boxed{B(2, 10)}$

Presjek parabole i prave $x = -3$

$\left\{ \begin{matrix} y^2 = x^2 + 2x + 2 \\ x = -3 \end{matrix} \right\} \Rightarrow y = 5 \rightarrow \boxed{C(-3, 5)}$



$$P = \int_{-3}^{2} (x^2 + 2x + 2)\, dx =$$

$$= \frac{x^3}{3}\Big|_{-3}^{2} + \frac{x^2}{1}\Big|_{-3}^{2} + 2x\Big|_{-3}^{2} =$$

$$= \frac{50}{30} = \frac{5}{3}$$

81) Izračunati površinu ograničenu krivom $y = x^2 - 1$, osom Ox i pravama $x = 0$ i $x = 2$

$y = x^2 - 1 \rightarrow y = 0 \Leftrightarrow x = 1 \vee x = -1$

↳ presjek sa x osom ($\rightarrow y = 0$) $\rightarrow$ $\boxed{A_1(-1,0)}$ $\boxed{A_2(1,0)}$

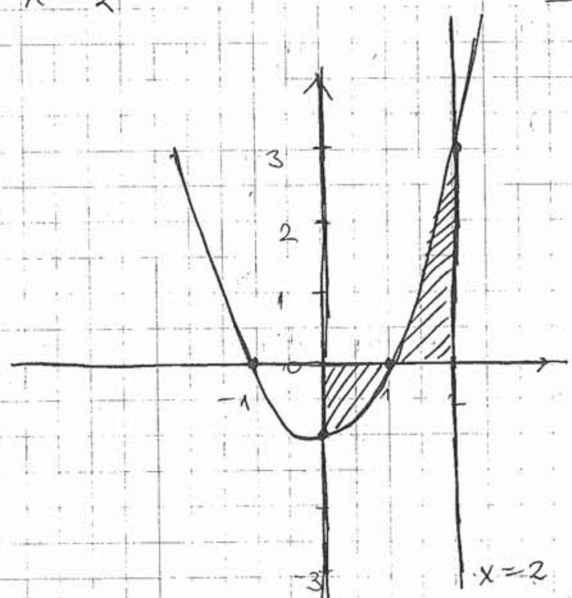
Presjek sa Oy osom ($x = 0$) → ovo će ujedno biti i tjeme

→ presjek sa Oy osom je ujedno i tjeme kada je $b = 0$

$$\begin{cases} y = x^2 - 1 \\ x = 0 \end{cases} \Rightarrow y = -1 \quad \boxed{T(0,-1)}$$

Presjek parabole i prave $x = 2$

$$\left\{ \begin{array}{l} y = x^2 - 1 \\ x = 2 \end{array} \right\} \Rightarrow y = 3 \quad \boxed{B(2,3)}$$



$$P = \int_0^2 |x^2 - 1|\, dx =$$

$$= -\int_0^1 (x^2 - 1)\,dx + \int_1^2 (x^2 - 1)\,dx$$

površina mora da bude pozitivna; $x^2 - 1$ na ovom djelu negativna f-ja

$$P = -\frac{x^3}{3}\Big|_0^1 + x\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 - x\Big|_1^2 = 2$$

(82) Izračunati površinu figure ograničene parabolom $y = x^2 + 4x$ i pravom $y = x + 4$.

Presjek parabole sa Ox osom $(y = 0)$

$y = x(x+4)$

$$\begin{cases} y = x^2 + 4x \\ y = 0 \end{cases} \qquad \begin{aligned} x_1 &= 0 \to y_1 = 0 \\ x_2 &= -4 \to y_2 = 0 \end{aligned} \qquad \boxed{A_1(0, 0)} \quad \boxed{A_2(-4, 0)}$$

Presjek parabole sa Oy osom $(x = 0)$ → tjeme

$$\text{Tjeme}\left(-\frac{b}{2a}, -\frac{D}{4a}\right) \qquad T\left(\frac{-4}{2}, -\frac{16}{4}\right)$$

$b = 4 \quad c = 0$

$a = 1 \quad D = b^2 - 4ac = b^2$

$$\boxed{T(-2, -4)}$$

Presjek parabole sa pravom $y = x + 4$

$$\begin{cases} y = x^2 + 4x \\ y = x + 4 \end{cases} \qquad \begin{aligned} x + 4 &= x^2 + 4x \\ x^2 + 3x - 4 &= 0 \end{aligned} \to x_{1,2} = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = \begin{cases} 1 \\ -4 \end{cases}$$

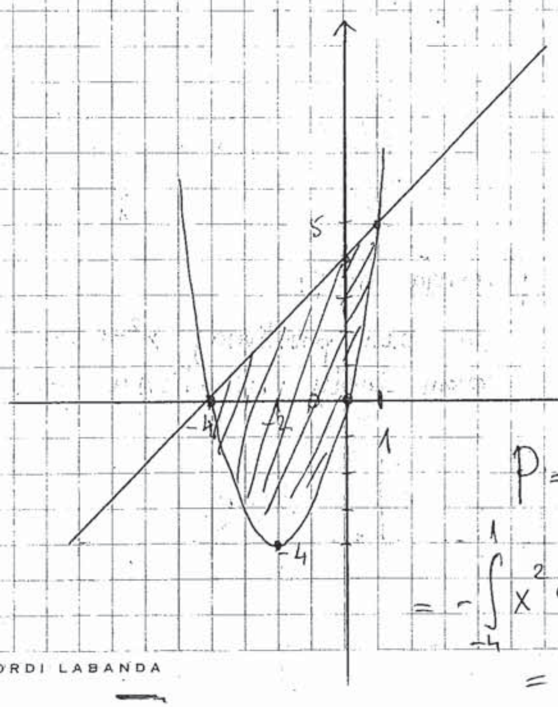
$y_1 = 5 \quad y_2 = 0$

$$\boxed{B_1(-4, 0)} \quad \boxed{B_2(1, 5)}$$

$y = x + 4$

$y = 0 \Rightarrow x = -4$

$x = 0 \Rightarrow y = 4$



→ na intervalu -4 do 1 → prava iznad parabole

$$\Rightarrow P = \int_{-4}^{1} \text{prava} - \text{parabola}$$

$$P = \int_{-4}^{1} (x + 4 - x^2 - 4x)\,dx = \int_{-4}^{1} (-x^2 - 3x + 4)\,dx =$$

$$= -\int_{-4}^{1} x^2\,dx - 3\int_{-4}^{1} x\,dx + \int_{-4}^{1} dx = -\frac{x^3}{3}\Big|_{-4}^{1} - \frac{3}{2}x^2\Big|_{-4}^{1} + 4x\Big|_{-4}^{1}$$

$$= -\frac{1}{3} - \frac{64}{3} - \frac{3}{2}(1 - 16) + 4 + 16 = \frac{125}{6}$$

83) Izračunati površinu ograničenu krivom

$y^2 = 2x+1$ i pravom $y = x-1 \to \boxed{x = y+1}$

$y^2 = 2x+1$

$2x = y^2 - 1$

$\boxed{x = \frac{1}{2}y^2 - \frac{1}{2}}$

$a = \frac{1}{2}$

$b = 0$

$c = -\frac{1}{2}$

$x = ay^2 + by + c$

$T = \left(-\frac{D}{4a}, -\frac{b}{2a}\right)$

$D = b^2 - 4ac \to$ ili D ili c

$\to$ pitamo da li siječe y osu

Presjek sa Oy osom ($x=0$)

$x = 0 \iff \frac{1}{2}y^2 - \frac{1}{2} = 0 \to \frac{1}{2}y^2 = \frac{1}{2} \to y = \pm 1$

$\boxed{A_1(0,1)}$ $\boxed{A_2(0,-1)}$

Presjek sa Ox osom $\to y = 0$

$y^2 = 2x+1 \to 2x = -1 \Rightarrow x = -\frac{1}{2}$

$T\left(-\frac{1}{2}, 0\right)$

$x = \frac{1}{2}y^2 - \frac{1}{2}$; $y = 0 \Rightarrow x = -\frac{1}{2}$

nema b $\Rightarrow$ tjeme presjek sa Ox osom

Presjek parabole i prave $y = x-1$

$\begin{cases} y^2 = 2x+1 \\ y = x-1 \end{cases}$

$(x-1)^2 = 2x+1$

$x^2 - 2x + \cancel{1} = 2x + \cancel{1}$

$x^2 - 4x = 0$

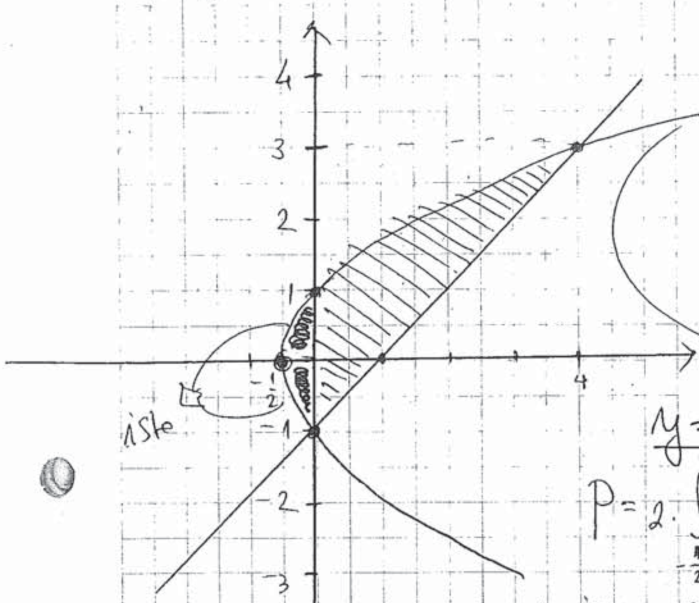
$x(x-4) = 0 \iff x = 0 \lor x = 4$

$y = -1$ $y = 3$

$B_1(0,-1)$

$B_2(4,3)$

$x = f(y)$



$P = \int_{-1}^{3} \left(y + 1 - \frac{1}{2}y^2 + \frac{1}{2}\right) dy =$

$\frac{3}{2}$

$= -\frac{1}{2}\frac{y^3}{3}\Big|_{-1}^{3} + \frac{y^2}{2}\Big|_{-1}^{3} + \frac{3}{2}y\Big|_{-1}^{3} = \frac{16}{3}$

$y = f(x)$

$P = 2 \cdot \int_{-\frac{1}{2}}^{0} \sqrt{2x+1}\, dx + \int_{0}^{4} \left(\sqrt{2x+1} - (x-1)\right) dx$

smjena $2x+1 = t$

(84.) U presječenim tačkama prave $x - y + 1 = 0$ i parabole $y = x^2 - 4x + 5$ povučene su tangente na parabolu. Izračunati površinu figure ograničene parabolom i tangentama.

$y = x^2 - 4x + 5$ → $a = 1$, $b = -4$, $c = 5$ $\quad D = b^2 - 4ac = -4 < 0$

⇒ nema presjeka sa Oy osom

→ Tjeme $\left(-\frac{b}{2a}, -\frac{D}{4a}\right) \rightarrow \boxed{T(2, 1)}$

* Presjek sa Oy osom ⇒ $x = 0$

$$\begin{cases} y = x^2 - 4x + 5 \\ x = 0 \end{cases} \Rightarrow y = 5 \rightarrow \boxed{A(0,5)}$$

* Presjek parabole i prave

$$\begin{cases} y = x^2 - 4x + 5 \\ y = x + 1 \end{cases} \rightarrow x + 1 = x^2 - 4x + 5$$

$x^2 - 5x + 4 = 0 \rightarrow x_{1,2} = \frac{5 \pm \sqrt{25-16}}{2}$

$x_1 = 4$; $x_2 = 1$

$y_1 = 5$ $\quad y_2 = 2$

$\boxed{B_1(4,5)}$ $\quad \boxed{B_2(1,2)}$

* J-na tangente u tački $B_1(4,5)$

$t_1: y - 5 = y'(4)(x-4)$ → $t_1: y = 4x - 11$

$y' = 2x - 4$; $y'(4) = 4$

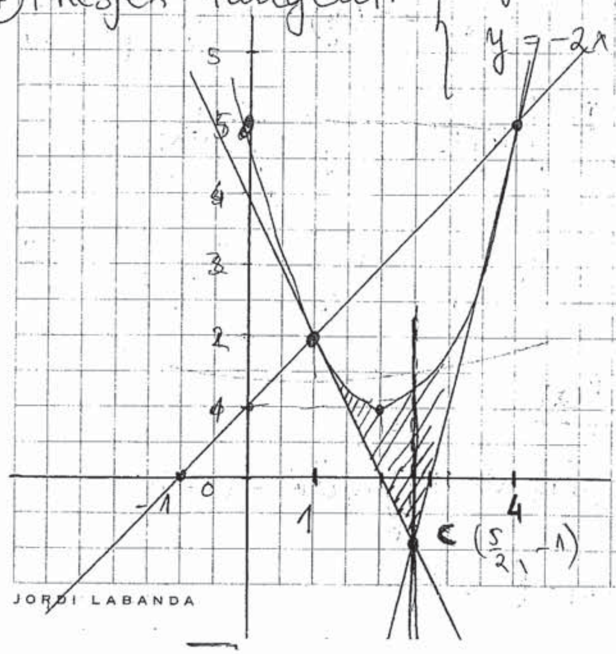
* J-na tangente u $B_2(1,2)$

$t_2: y - 2 = y'(1)(x-1)$

$t_2: y = -2x + 4$

* Presjek tangenti

$$\begin{cases} y = 4x - 11 \\ y = -2x + 4 \end{cases} \rightarrow \boxed{C\left(\frac{5}{2}, -1\right)}$$



$y = 0 \Rightarrow$
$-2x = -4$
$x = 2$

$y = 4x - 11$

$x = 0 \Rightarrow y = -11$

$y = 0 \Rightarrow 4x = 11 \rightarrow x =$

$$P = \int_1^{\frac{5}{2}} \text{parabola} - t_2 + \int_{5/2}^{4} \text{parab} - t_1$$

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85. Izračunati površinu ograničenu sa $y = 2 - |x|$ i $y = x^2$.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$2 - |x| = \begin{cases} 2 - x, & x \geq 0 \\ 2 + x, & x < 0 \end{cases}$$

Presjek parabole $y = x^2$ i prave $y = 2 - x$, za $x \geq 0$

$$\begin{cases} y = x^2 \\ y = 2 - x \\ x \geq 0 \end{cases} \qquad x^2 = 2 - x \qquad x^2 + x - 2 = 0 \qquad x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$

$\cancel{x_1 = -2}$ nije ≥ 0 ; $x_2 = 1 \Rightarrow y_2 = 1 \rightarrow \boxed{A(1,1)}$

Presjek parabole $y = x^2$ i prave $y = 2 + x$, za $x < 0$

$$\begin{cases} y = x^2 \\ y = 2 + x \\ x < 0 \end{cases} \qquad x^2 = 2 + x \qquad x^2 - x - 2 = 0 \qquad x_{1,2} = \frac{+1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

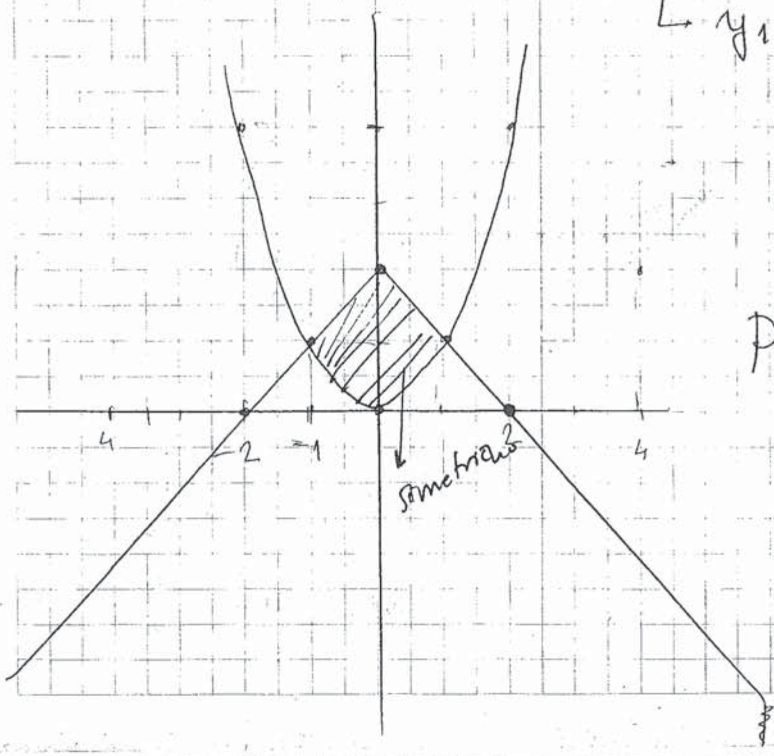
$x_1 = -1$ $\cancel{x_2 = 2}$ nije < 0

$\rightarrow y_1 = 1 \rightarrow \boxed{B(-1,1)}$

$y = 0 \Rightarrow x = 2$

$x = 0 \quad y = 2$

$y = 0 \rightarrow x = -2$



$$P = 2 \int_{-1}^{0} (2 + x - x^2)\, dx =$$

$$= 2 \int_{0}^{1} (2 - x - x^2)\, dx =$$

$= \ldots$

86. Izračunati površinu figure ograničene krivom $y=\frac{1}{x}$, njenom tangentom u tački $M(1,1)$ i pravom $x=3$.

Presjek krive $y=\frac{1}{x}$ i prave $x=3$

$$\begin{cases} y=\frac{1}{x} \\ x=3 \end{cases} \longrightarrow \boxed{A\left(3,\frac{1}{3}\right)}$$

J-na tangente u tački $M(1,1)$ je:

$$t:\quad y-1=y'(1)(x-1) \longrightarrow y-1=-1(x-1)$$

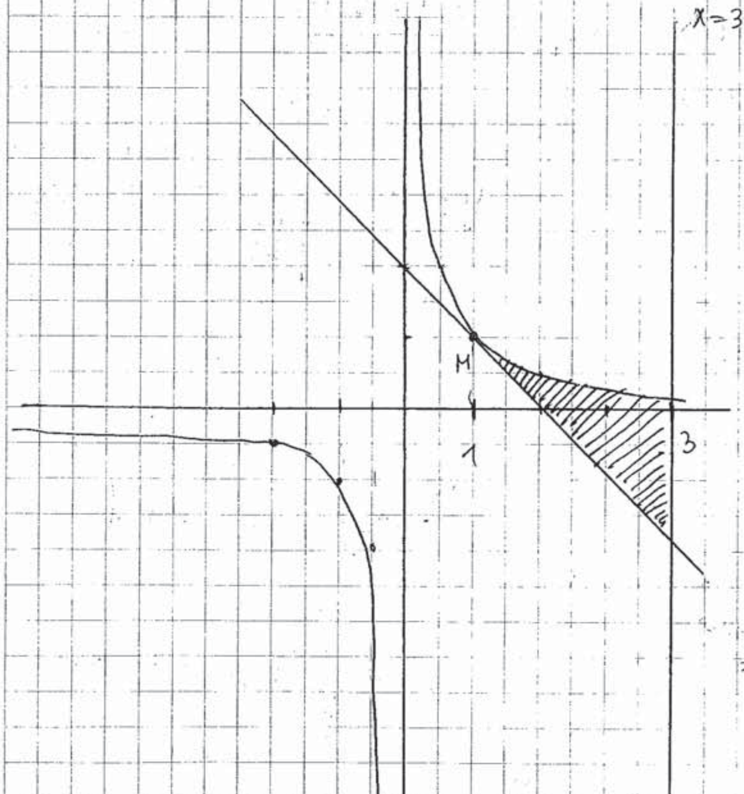
$$y'=-\frac{1}{x^2}$$

$$y'(1)=-1$$

$$t:\ y=-x+2$$

$y=0 \Rightarrow x=2$

$x=0 \Rightarrow 2$



kriva-tangenta

$$P=\int_1^3\left(\frac{1}{x}+x-2\right)dx=$$

$$=\int_1^3\left(\frac{1}{x}+x-2\right)dx=$$

$$=\int_1^3\frac{1}{x}dx+\int_1^3 x\,dx-2\int_1^3 dx=$$

$$=\ln x\Big|_1^3+\frac{x^2}{2}\Big|_1^3-2x\Big|_1^3=$$

$$=\ln 3-\ln 1+\frac{3^2}{2}-\frac{1}{2}-6+2=$$