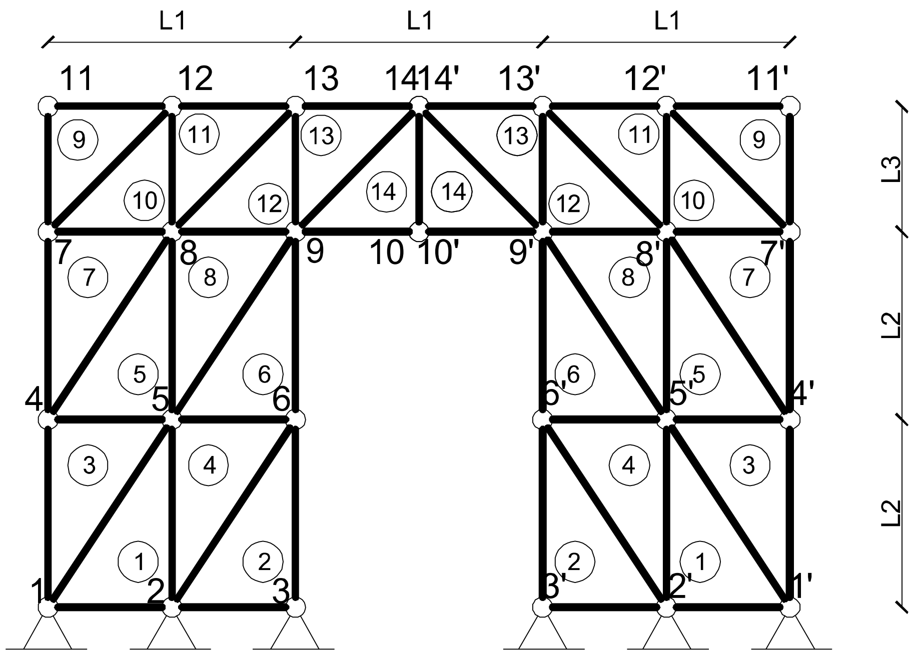

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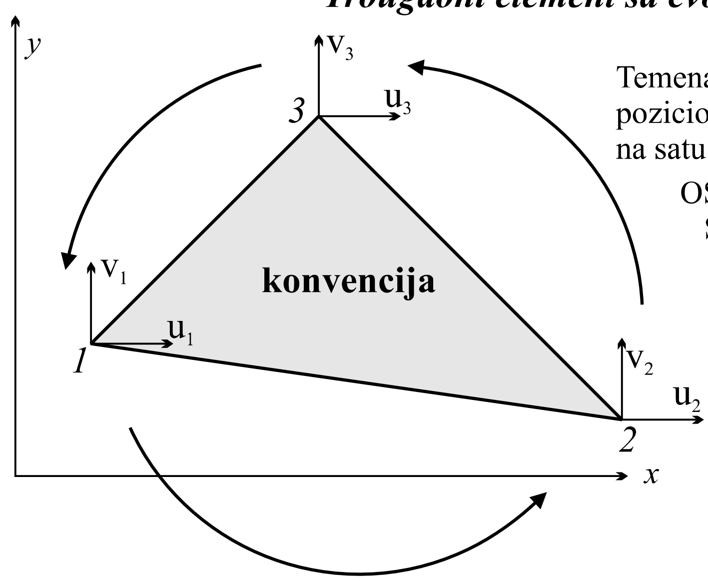
Trougaoni element sa čvorovima u temenima trougla



Temena su uvek obeležena tako da se indeksi 1,2 i 3 pozicioniraju u smeru suprotnom od kretanja kazaljke na satu

OSNOVNE NEPOZNATE U ČVOROVIMA SU KOMPONENTE POMERANJA ***u, v***

$$q_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix}$$ vektor nepoznatih u čvoru *i* 1

odnosno

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$ vektor pometanja u svim čvorovima trougla 2

Konačni element oblika trougla sa čvorovima u temenima ima 6 spoljašnjih stepeni slobode.

ZA KOMPONENTE POMERANJA ***u, v*** U ELEMENTU MOŽE SE PRETPOSTAVITI LINEARNA PROMENA, ODNOSNO U OBLIKU POLINOMA SA UKUPNO 6 NEPOZNATIH KOEFICIJENATA

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \beta_1 + \beta_2 x + \beta_3 y$$ 3

u matričnom obliku:

$$\mathbf{u} = \mathbf{A}\alpha \;\Rightarrow\; \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$ broj parametara jednak broju stepeni slobode 4,5

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odnosno, ako se prikažu za sve tri tačke:

$$\mathbf{q} = \mathbf{C}\alpha \quad \Rightarrow \quad \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \qquad \text{...... 6,7}$$

iz jednačina 6,7 mogu se odrediti nepoznati koeficijenti α_i i β_i (i= 1,2,3) u zavisnosti od pomeranja u čvorovima

$\alpha = \mathbf{C}^{-1}q$ gde je inverzna matrica $\mathbf{C}^{-1}$

$$\mathbf{C}^{-1} = \frac{1}{2\Delta} \begin{bmatrix} a_1 & 0 & a_2 & 0 & a_3 & 0 \\ b_1 & 0 & b_2 & 0 & b_3 & 0 \\ c_1 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & a_1 & 0 & a_2 & 0 & a_3 \\ 0 & b_1 & 0 & b_2 & 0 & b_3 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \end{bmatrix} \qquad \text{...... 8}$$

gde su:

2Δ - determinanta matrice **C**

$$2\Delta = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

koja je jednaka dvostrukoj površini trougla

$$a_1 = x_2 y_3 - x_3 y_2 \qquad b_1 = y_2 - y_3 \qquad c_1 = x_3 - x_2 \qquad \text{...... 9,10}$$

Ostali koeficijenti $\mathbf{C}^{-1}$ dobijaju se cikličnom permutacijom indeksa 1,2 i 3

Uvođenjem izraza $\alpha = \mathbf{C}^{-1}q$ u jednačinu $\mathbf{u} = \mathbf{A}\alpha$ dobija se $\mathbf{u} = \mathbf{A}\mathbf{C}^{-1}\,\mathbf{q}$, odnosno u razvijenom obliku:

$$\mathbf{u} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \frac{1}{2\Delta} \begin{bmatrix} a_1 & 0 & a_2 & 0 & a_3 & 0 \\ b_1 & 0 & b_2 & 0 & b_3 & 0 \\ c_1 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & a_1 & 0 & a_2 & 0 & a_3 \\ 0 & b_1 & 0 & b_2 & 0 & b_3 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

ili

$$\mathbf{u} = \frac{1}{2\Delta} \begin{bmatrix} a_1 + b_1x + c_1y & 0 & a_2 + b_2x + c_2y & 0 & a_3 + b_3x + c_3y & 0 \\ 0 & a_1 + b_1x + c_1y & 0 & a_2 + b_2x + c_2y & 0 & a_3 + b_3x + c_3y \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} \qquad 11$$

Prikazani izraz najčešće se sreće u obliku $\mathbf{u} = \mathbf{N}\,\mathbf{q}$ 12

N - MATRICA INTERPOLACIONIH FUNKCIJA

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

sa elementima

$$N_i = \frac{1}{2\Delta}(a_i + b_i x + c_i y) \qquad i = 1, 2, 3 \qquad \text{..... 13,14}$$

Dakle FUNKCIJE Ni PREDSTAVLJAJU LINEARNU INTERPOLACIJU POLJA POMERANJA

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Smenom izraza $\mathbf{u = N\,q}$ u veze između komponenata deformacija i pomeranja $\varepsilon = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$ dobija se:

$$\epsilon = \underbrace{\begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{\mathbf{q}} \quad \Rightarrow \quad \epsilon = \mathbf{B\,q} \qquad \text{...... 15}$$

gde je:

B - matrica veze deformacija i generalisanih pomeranja

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

Obzirom da su elementi matrice **B** konstantni, sledi da su komponente deformacije u elementu konstantne.

Zbog toga se **trougaoni element sa čvorovima u temenima** naziva *ELEMENT SA KONSTANTNIM DEFORMACIJAMA.*

Vraćajući se na definiciju matrice krutosti, za trougaoni element sa konstantnom debljinom h, dobija se:

$$\mathbf{k} = \int_V \mathbf{B}^T\mathbf{D}\mathbf{B}\, dV = h\int \mathbf{B}^T\mathbf{D}\mathbf{B}\, dx dy = h\,\Delta\, \mathbf{B}^T\mathbf{D}\mathbf{B} \qquad \text{...... 16}$$

Matrica **D** ***kojom se uspostavlja veza između komponenata napona i komponenata deformacija,*** za izotropna tela

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \alpha t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \qquad \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} - \frac{E\,\alpha\, t}{1-\nu^2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

ima strukturu

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

Radi jednostavnijeg sračunavanja matricu **k** dekomponujemo na blokove 2x2

$$\mathbf{k} = h\,\Delta\,\mathbf{B}^T\mathbf{D}\mathbf{B} = \frac{h}{4\Delta}\begin{bmatrix} b_1 & 0 & c_1 \\ 0 & c_1 & b_1 \\ b_2 & 0 & c_2 \\ 0 & c_2 & b_2 \\ b_3 & 0 & c_3 \\ 0 & c_3 & b_3 \end{bmatrix}\begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} & \mathbf{k}_{13} \\ \mathbf{k}_{21} & \mathbf{k}_{22} & \mathbf{k}_{23} \\ \mathbf{k}_{31} & \mathbf{k}_{32} & \mathbf{k}_{33} \end{bmatrix}}_{\text{matrica krutosti elementa } \mathbf{k}}$$

pa ako i matricu **B** pogodno prikažemo u obliku,

matrica krutosti elementa **k**

$$\mathbf{B} = \frac{1}{2\Delta}\left[\begin{matrix} b_1 & 0 \\ 0 & c_1 \\ c_1 & b_1 \end{matrix}\quad \begin{matrix} b_2 & 0 \\ 0 & c_2 \\ c_2 & b_2 \end{matrix}\quad \begin{matrix} b_3 & 0 \\ 0 & c_3 \\ c_3 & b_3 \end{matrix}\right] = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \end{bmatrix}$$ gde su očigledno blokovi $$\mathbf{B}_i = \frac{1}{2\Delta}\begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix}_{i=1,2,3}$$

blokove matrice krutosti **k** definišemo kao $\mathbf{k}_{ij} = \mathbf{B}_i^T\,\mathbf{D}\,\mathbf{B}_j\, h\Delta \quad i, j = 1, 2, 3$

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smenom

$$\mathbf{k_{ij}} = \mathbf{B_i^T}\,\mathbf{D}\,\mathbf{B_j}\,h\Delta = \frac{1}{2\Delta}\begin{bmatrix} b_i & 0 & c_i \\ 0 & c_i & b_i \end{bmatrix}\begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}\begin{bmatrix} b_j & 0 \\ 0 & c_j \\ c_j & b_j \end{bmatrix}\frac{1}{2\Delta}\;h\,\Delta \qquad i, j = 1, 2, 3$$

posle izvršenog množenja dobija se

$$\mathbf{k_{ij}} = \frac{h}{4\Delta}\begin{bmatrix} b_i & 0 & c_i \\ 0 & c_i & b_i \end{bmatrix}\begin{bmatrix} d_{11}\,b_j & d_{12}\,c_j \\ d_{21}\,b_j & d_{22}\,c_j \\ d_{33}\,c_j & d_{33}\,b_j \end{bmatrix} = \frac{h}{4\Delta}\begin{bmatrix} b_i\,d_{11}\,b_j + c_i\,d_{33}\,c_j & b_i\,d_{12}\,c_j + c_i\,d_{33}\,b_j \\ c_i\,d_{21}\,b_j + b_i\,d_{33}\,c_j & c_i\,d_{22}\,c_j + b_i\,d_{33}\,b_j \end{bmatrix} \qquad i, j = 1, 2, 3$$

Na kraju, za usvojeni slučaj ravnog stanja napona i izotropnog materijala, kada je matrica **D** u obliku

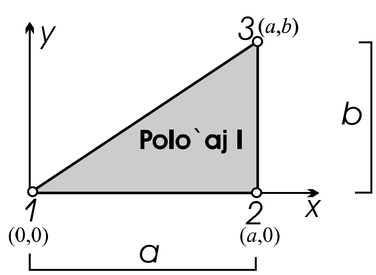
$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} = \frac{E}{1 - \nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \begin{matrix} d_{11} = d_{22} = 1 \\ d_{12} = d_{12} = \nu \end{matrix} \qquad d_{33} = \frac{1-\nu}{2}$$

izveden je KONAČAN izraz za članove matrice krutosti elementa

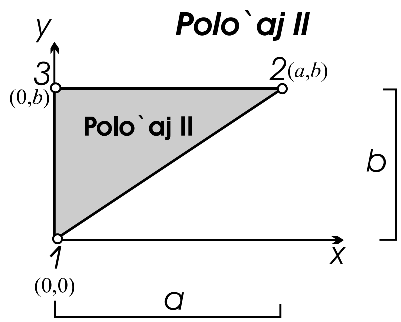
$$\mathbf{k_{ij}} = \frac{Eh}{4\Delta(1 - \nu^2)}\begin{bmatrix} b_i\,b_j + \beta\,c_i\,c_j & \nu\,b_i\,c_j + \beta\,c_i\,b_j \\ \nu\,c_i\,b_j + \beta\,b_i\,c_j & c_i\,c_j + \beta\,b_i\,b_j \end{bmatrix} \qquad \beta = \frac{1 - \nu}{2} \qquad i, j = 1, 2, 3$$

U mreži konačnih elemenata nalazi se 8 trougaonih elemenata koji imaju jedan od dva sledeća položaja:

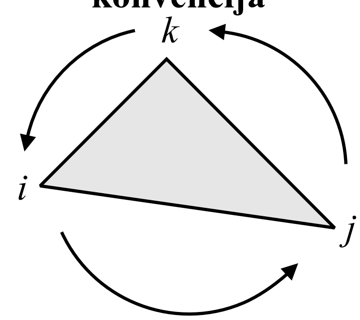
Polo`aj I



Polo`aj II



konvencija



$$\boldsymbol{a_i = x_j y_k - x_k y_j}$$
$$\boldsymbol{b_i = y_j - y_k}$$
$$\boldsymbol{c_i = x_k - x_j}$$

Polo`aj I

$a_1 = x_2 y_3 - x_3\,y_2 = ab$
$a_2 = x_3 y_1 - x_1\,y_3 = 0$
$a_3 = x_1 y_2 - x_2\,y_1 = 0$

$b_1 = y_2 - y_3 = -b$
$b_2 = y_3 - y_1 = \;b$
$b_3 = y_1 - y_2 = \;0$

$c_1 = x_3 - x_2 = \;0$
$c_2 = x_1 - x_3 = -a$
$c_3 = x_2 - x_1 = \;a$

Polo`aj II

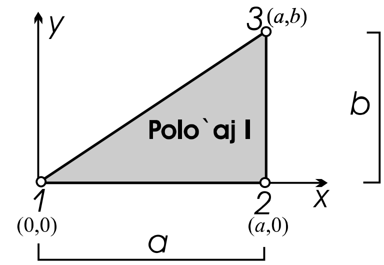
$a_1 = x_2 y_3 - x_3\,y_2 = ab$
$a_2 = x_3 y_1 - x_1\,y_3 = 0$
$a_3 = x_1 y_2 - x_2\,y_1 = 0$

$b_1 = y_2 - y_3 = \;0$
$b_2 = y_3 - y_1 = \;b$
$b_3 = y_1 - y_2 = -b$

$c_1 = x_3 - x_2 = -a$
$c_2 = x_1 - x_3 = \;0$
$c_3 = x_2 - x_1 = \;a$

MATRICA KRUTOSTI ELEMENTA

Položaj I



$$b_1 = -b \quad c_1 = 0$$
$$b_2 = b \quad c_2 = -a \qquad \beta = \frac{1-\nu}{2}$$
$$b_3 = 0 \quad c_3 = a$$

submatrice matrice krutosti

$$\mathbf{k}_{ij} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} b_i b_j + \beta\, c_i c_j & \nu\, b_i c_j + \beta\, c_i b_j \\ \hline \nu\, c_i b_j + \beta\, b_i c_j & c_i c_j + \beta\, b_i b_j \end{array} \right] \qquad i, j = 1, 2, 3$$

$$\mathbf{k}_{11} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} b^2 & 0 \\ \hline 0 & \beta b^2 \end{array} \right] \qquad \mathbf{k}_{22} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} b^2 + \beta a^2 & -\nu ab - \beta ab \\ \hline -\nu ab - \beta ab & a^2 + \beta b^2 \end{array} \right]$$

$$\mathbf{k}_{12} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} -b^2 & \nu ab \\ \hline \beta ab & -\beta b^2 \end{array} \right] = \mathbf{k}_{21} \qquad \mathbf{k}_{23} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} -\beta a^2 & \nu ab \\ \hline \beta ab & -a^2 \end{array} \right] = \mathbf{k}_{32}$$

$$\mathbf{k}_{13} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} 0 & -\nu ab \\ \hline -\beta ab & 0 \end{array} \right] = \mathbf{k}_{31} \qquad \mathbf{k}_{33} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} \beta a^2 & 0 \\ \hline 0 & a^2 \end{array} \right]$$

$$K^{I\ \text{polo`aj}} = \left[\begin{array}{c|c|c} \mathbf{k}_{11} & \mathbf{k}_{12} & \mathbf{k}_{13} \\ \hline \mathbf{k}_{21} & \mathbf{k}_{22} & \mathbf{k}_{23} \\ \hline \mathbf{k}_{31} & \mathbf{k}_{32} & \mathbf{k}_{33} \end{array} \right] = \frac{Eh}{4\,\Delta\,(1-\nu^2)} \left[\begin{array}{cc|cc|cc} b^2 & 0 & -b^2 & \nu ab & 0 & -\nu ab \\ & \beta b^2 & \beta ab & -\beta b^2 & -\beta ab & 0 \\ \hline & & b^2 + \beta a^2 & -ab(\nu+\beta) & -\beta a^2 & \nu ab \\ & & & a^2 + \beta b^2 & \beta ab & -a^2 \\ \hline & \text{simetri\sim no} & & & \beta a^2 & 0 \\ & & & & & a^2 \end{array} \right]$$

Ili, ako se izvuče veličina b^2

$$K^{I\ \text{polo`aj}} = \frac{Ehb^2}{4\,\Delta\,(1-\nu^2)} \left[\begin{array}{cc|cc|cc} 1 & 0 & -1 & \nu\frac{a}{b} & 0 & -\nu\frac{a}{b} \\ & \beta & \beta\frac{a}{b} & -\beta & -\beta\frac{a}{b} & 0 \\ \hline & & 1+\beta\left(\frac{a}{b}\right)^2 & -\frac{a}{b}(\nu+\beta) & -\beta\left(\frac{a}{b}\right)^2 & \nu\frac{a}{b} \\ & & & \left(\frac{a}{b}\right)^2 + \beta & \beta\frac{a}{b} & -\left(\frac{a}{b}\right)^2 \\ \hline & \text{simetri\sim no} & & & \beta\left(\frac{a}{b}\right)^2 & 0 \\ & & & & & \left(\frac{a}{b}\right)^2 \end{array} \right]$$

KONKRETNI PRIMERI

Slučaj 1 - odabrani primer a = b = 4 m (a/b = 1) ν = 1/3

površine trougla - $\Delta = 0.5\,ab = 0.5 \cdot 4 \cdot 4 = 8$

$$\frac{Ehb^2}{4\,\Delta\,(1-\nu^2)} = \frac{Eh \cdot 4^2}{4 \cdot 8\,(1-\frac{1}{9})} = \frac{9 \cdot Eh}{16}; \quad \beta = \frac{1-\nu}{2} = \frac{1}{3}$$

$K^{I\ polo`aj} = \frac{9 \cdot Eh}{16}$

1	0	-1	1/3	0	-1/3
	1/3	1/3	-1/3	-1/3	0
		1+1/3	-2/3	-1/3	1/3
			1+1/3	1/3	-1
	simetri~no			1/3	0
					1

Matrica krutosti elementa 1, za specijalni slučaj **a/b = 1 i β = ν= 1/3**, obično se u literaturi prikazuje u obliku

$K^{I\ polo`aj} = \frac{3 \cdot Eh}{16}$

3	0	-3	1	0	-1
	1	1	-1	-1	0
		4	-2	-1	1
			4	1	-3
	simetri~no			1	0
					3

Slučaj 2 - primer a = 4, b = 5 m (a/b = 0.8), ν = 1/4 (varijanta za grafički rad)

površine trougla - $\Delta = 0.5\,ab = 0.5 \cdot 4 \cdot 5 = 10$

$$\frac{Ehb^2}{4\,\Delta\,(1-\nu^2)} = \frac{Eh \cdot 5^2}{4 \cdot 10\,(1-\frac{1}{16})} = \frac{4 \cdot Eh}{3}; \quad \beta = \frac{1-\nu}{2} = \frac{3}{8}$$

$K^{I\ polo`aj} = \frac{4 \cdot Eh}{3}$

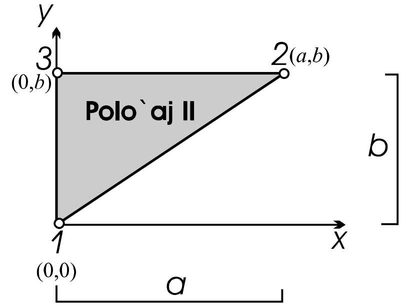
1	0	-1	1/5	0	-1/5
	3/8	3/10	-3/8	-3/10	0
		31/25	-1/2	-6/25	1/5
			203/200	3/10	-16/25
	simetri~no			6/25	0
					16/25

Dakle, kod svih ostalih varijanti, kod kojih je odnos a/b ≠ 1, nema nekog praktičnog smisla izdvajati veličinu ν Možda je jedino pogodnije matricu prikazati u obliku:

$K^{I\ polo`aj} = \frac{4 \cdot Eh}{3}$

1	0	-1	0.2	0	-0.2
	0.375	0.3	-0.375	-0.3	0
		1.24	-0.5	-0.24	0.2
			1.015	0.3	-0.64
	simetri~no			0.24	0

MATRICA KRUTOSTI ELEMENTA



Položaj II

$$b_1 = 0 \qquad c_1 = -a$$
$$b_2 = b \qquad c_2 = 0 \qquad \beta = \frac{1-\nu}{2}$$
$$b_3 = -b \qquad c_3 = a$$

submatrice matrice krutosti

$$\mathbf{k}_{ij} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} b_i b_j + \beta c_i c_j & \nu b_i c_j + \beta c_i b_j \\ \hline \nu c_i b_j + \beta b_i c_j & c_i c_j + \beta b_i b_j \end{array} \right] \qquad i, j = 1, 2, 3$$

$$\mathbf{k}_{11} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} \beta a^2 & 0 \\ \hline 0 & a^2 \end{array} \right] \qquad \mathbf{k}_{22} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} b^2 & 0 \\ \hline 0 & \beta b^2 \end{array} \right]$$

$$\mathbf{k}_{12} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} 0 & -\beta ab \\ \hline -\nu ab & 0 \end{array} \right] = \mathbf{k}_{21} \qquad \mathbf{k}_{23} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} -b^2 & \nu ab \\ \hline \beta ab & -\beta b^2 \end{array} \right] = \mathbf{k}_{32}$$

$$\mathbf{k}_{13} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} -\beta a^2 & \beta ab \\ \hline \nu ab & -a^2 \end{array} \right] = \mathbf{k}_{31} \qquad \mathbf{k}_{33} = \frac{Eh}{4\Delta(1-\nu^2)} \left[\begin{array}{c|c} b^2 + \beta a^2 & -\nu ab - \beta ab \\ \hline -\nu ab - \beta ab & a^2 + \beta b^2 \end{array} \right]$$

$$K^{II\ polo`aj} = \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} & \mathbf{k}_{13} \\ \mathbf{k}_{21} & \mathbf{k}_{22} & \mathbf{k}_{23} \\ \mathbf{k}_{31} & \mathbf{k}_{32} & \mathbf{k}_{33} \end{bmatrix} = \frac{Eh}{4\Delta(1-\nu^2)} \begin{bmatrix} \beta a^2 & 0 & 0 & -\beta ab & -\beta a^2 & \beta ab \\ & a^2 & -\nu ab & 0 & \nu ab & -a^2 \\ & & b^2 & 0 & -b^2 & \nu ab \\ & & & \beta b^2 & \beta ab & -\beta b^2 \\ & \text{simetri\~no} & & & b^2 + \beta a^2 & -ab(\nu+\beta) \\ & & & & & a^2 + \beta b^2 \end{bmatrix}$$

Ili, ako se izvuče veličina b^2

$$K^{II\ polo`aj} = \frac{Ehb^2}{4\Delta(1-\nu^2)} \begin{bmatrix} \beta\left(\frac{a}{b}\right)^2 & 0 & 0 & -\beta\frac{a}{b} & -\beta\left(\frac{a}{b}\right)^2 & \beta\frac{a}{b} \\ & \left(\frac{a}{b}\right)^2 & -\nu\frac{a}{b} & 0 & \nu\frac{a}{b} & -\left(\frac{a}{b}\right)^2 \\ & & 1 & 0 & -1 & \nu\frac{a}{b} \\ & & & \beta & \beta\frac{a}{b} & -\beta \\ & \text{simetri\~no} & & & 1+\beta\left(\frac{a}{b}\right)^2 & -\frac{a}{b}(\nu+\beta) \\ & & & & & \left(\frac{a}{b}\right)^2 + \beta \end{bmatrix}$$

KONKRETNI PRIMERI

Slučaj 1 **-** odabrani primer a = b = 4 m (a/b = 1) ν = 1/3

površine trougla **-** $\Delta = 0.5\ ab = 0.5 \cdot 4 \cdot 4 = 8$

$$\frac{Ehb^2}{4\,\Delta\,(1-\nu^2)} = \frac{Eh \cdot 4^2}{4 \cdot 8\,(1-\frac{1}{9})} = \frac{9 \cdot Eh}{16}; \quad \beta = \frac{1-\nu}{2} = \frac{1}{3}$$

$K^{II\ polo`aj} = \frac{9 \cdot Eh}{16}$

1/3	0	0	-1/3	-1/3	1/3
	1	-1/3	0	1/3	-1
		1	0	-1	1/3
			1/3	1/3	-1/3
	simetri~no			1+1/3	-2/3
					1+1/3

Matrica krutosti elementa 2, za specijalni slučaj **a/b = 1 i** β **=** ν**= 1/3**, obično se u literaturi prikazuje u obliku

$K^{II\ polo`aj} = \frac{3 \cdot Eh}{16}$

1	0	0	-1	-1	1
	3	-1	0	1	-3
		3	0	-3	1
			1	1	-1
	simetri~no			4	-2
					4

Slučaj 2 **-** primer a = 4, b = 5 m (a/b = 0.8), ν = 1/4 (varijanta za grafički rad)

površine trougla **-** $\Delta = 0.5\ ab = 0.5 \cdot 4 \cdot 5 = 10$

$$\frac{Ehb^2}{4\,\Delta\,(1-\nu^2)} = \frac{Eh \cdot 5^2}{4 \cdot 10\,(1-\frac{1}{16})} = \frac{4 \cdot Eh}{3}; \quad \beta = \frac{1-\nu}{2} = \frac{3}{8}$$

$K^{II\ polo`aj} = \frac{4 \cdot Eh}{3}$

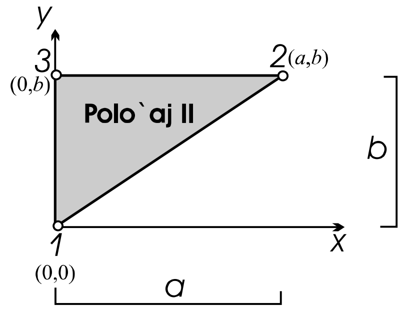
6/25	0	0	-3/10	-6/25	3/10
	16/25	-1/5	0	1/5	-16/25
		1	0	-1	1/5
			3/8	3/10	-3/8
	simetri~no			31/25	-1/2
					203/200

Dakle, kod svih ostalih varijanti, kod kojih je odnos a/b ≠ 1, nema nekog praktičnog smisla izdvajati veličinu ν Možda je jedino pogodnije matricu prikazati u obliku:

$K^{II\ polo`aj} = \frac{4 \cdot Eh}{3}$

0.24	0	0	-0.3	-0.24	0.3
	0.64	-0.2	0	0.2	-0.64
		1	0	-1	0.2
			0.375	0.3	-0.375
	simetri~no			1.24	-0.5

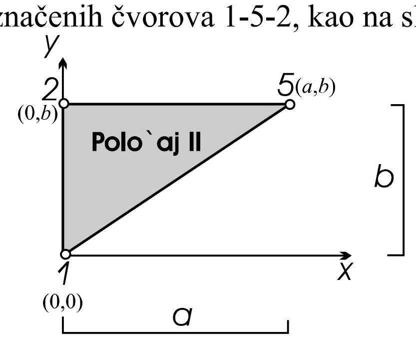
Položaj II



Matrica krutosti elementa 2, za specijalni slučaj **a/b = 1 i** β = ν= **1/3**, u slučaju čvoriva 1-2-3

$$K^{II\ polo`aj} = \frac{3\cdot Eh}{16}\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ & k_{22} & k_{23} \\ \text{simetri~no} & & k_{33} \end{bmatrix} = \frac{3\cdot Eh}{16}\left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & -1 & -1 & 1 \\ & 3 & -1 & 0 & 1 & -3 \\ \hline & & 3 & 0 & -3 & 1 \\ & & & 1 & 1 & -1 \\ \hline & \text{simetri~no} & & & 4 & -2 \\ & & & & & 4 \end{array}\right]$$

Međutim, ako je redosled označenih čvorova 1-5-2, kao na slici,



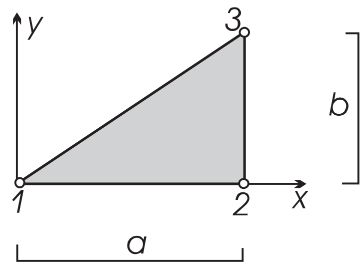
zbog usvojene konvencije položaj karakterističnih submatrica je sledeći:

$$K^{II\ polo`aj} = \frac{3\cdot Eh}{16}\begin{bmatrix} k_{11} & k_{15} & k_{12} \\ & k_{55} & k_{52} \\ \text{simetri~no} & & k_{22} \end{bmatrix} = \frac{3\cdot Eh}{16}\left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & -1 & -1 & 1 \\ & 3 & -1 & 0 & 1 & -3 \\ \hline & & 3 & 0 & -3 & 1 \\ & & & 1 & 1 & -1 \\ \hline & \text{simetri~no} & & & 4 & -2 \\ & & & & & 4 \end{array}\right]$$

Na kraju se matrica elementa, u clju jednostavnijeg pozicioniranja unutar matrice krutosti celokupnog sistema, može u ovom slučaju prikazati u obliku:

$$K^{II\ polo`aj} = \frac{3\cdot Eh}{16}\begin{bmatrix} k_{11} & k_{12} & k_{15} \\ & k_{22} & k_{25} \\ \text{simetri~no} & & k_{55} \end{bmatrix} = \frac{3\cdot Eh}{16}\left[\begin{array}{cc|cc|cc} 1 & 0 & -1 & 1 & 0 & -1 \\ & 3 & 1 & -3 & -1 & 0 \\ \hline & & 4 & -2 & -3 & 1 \\ & & & 4 & 1 & -1 \\ \hline & \text{simetri~no} & & & 3 & 0 \\ & & & & & 1 \end{array}\right]$$

Polozaj I



Matrice krutosti

$$K^{I\ polozaj} = \frac{Ehb^2}{4\,\Delta\,(1-\nu^2)} \begin{bmatrix} 1 & 0 & -1 & \nu\frac{a}{b} & 0 & -\nu\frac{a}{b} \\ & \beta & \beta\frac{a}{b} & -\beta & -\beta\frac{a}{b} & 0 \\ & & 1+\beta\left(\frac{a}{b}\right)^2 & -\frac{a}{b}(\nu+\beta) & -\beta\left(\frac{a}{b}\right)^2 & \nu\frac{a}{b} \\ & & & \left(\frac{a}{b}\right)^2+\beta & \beta\frac{a}{b} & -\left(\frac{a}{b}\right)^2 \\ & \text{simetri\textasciitilde no} & & & \beta\left(\frac{a}{b}\right)^2 & 0 \\ & & & & & \left(\frac{a}{b}\right)^2 \end{bmatrix}$$

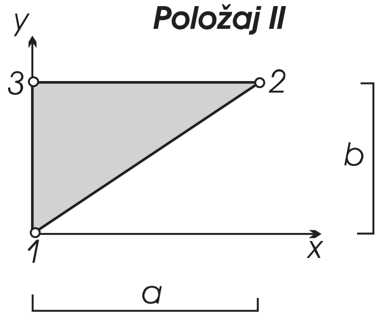
$a =$ $\qquad b =$ $\qquad \nu =$ $\qquad \beta =$

$$\frac{b^2}{4\,\Delta\,(1-\nu^2)} =$$

$$K^{I\ polozaj} = \quad Eh \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & \text{simetri\textasciitilde no} & & & & \\ & & & & & \end{bmatrix}$$

	Seminarski rad **METOD KONAČNIH ELEMENATA**	**ZAD. BR.** **LIST BR.**

Položaj II



Matrice krutosti

$$K^{II\ polozaj} = \frac{Ehb^2}{4\,\Delta\,(1-\nu^2)}$$

$\beta\left(\frac{a}{b}\right)^2$	0	0	$-\beta\,\frac{a}{b}$	$-\beta\left(\frac{a}{b}\right)^2$	$\beta\,\frac{a}{b}$
	$\left(\frac{a}{b}\right)^2$	$-\nu\frac{a}{b}$	0	$\nu\frac{a}{b}$	$-\left(\frac{a}{b}\right)^2$
		1	0	-1	$\nu\frac{a}{b}$
			β	$\beta\,\frac{a}{b}$	$-\beta$
	simetri~no			$1+\beta\left(\frac{a}{b}\right)^2$	$-\frac{a}{b}(\nu+\beta)$
					$\left(\frac{a}{b}\right)^2+\beta$

$a =$ $\quad b =$ $\quad \nu =$ $\quad \beta =$

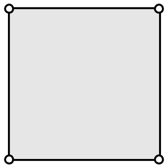
$$\frac{b^2}{4\,\Delta\,(1-\nu^2)} =$$

$$K^{II\ polozaj} = \quad Eh$$

	simetri~no				

DATUM	**IZRADIO**

Element



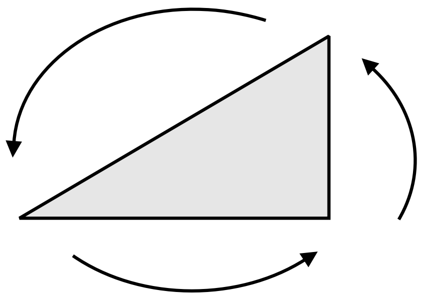
$K = \frac{Eh}{180(1 - \nu^2)}$ [ ] $K = \frac{Eh}{}$ [ ]

O…
subm…

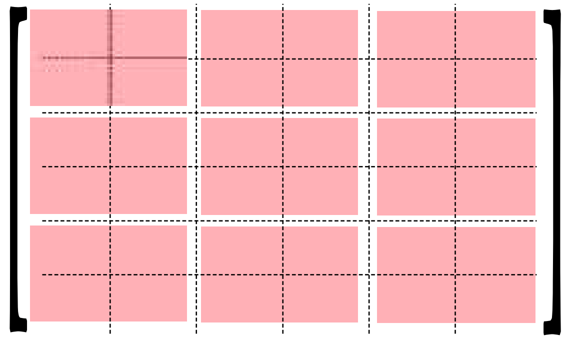
Element u matrici krutosti sistema

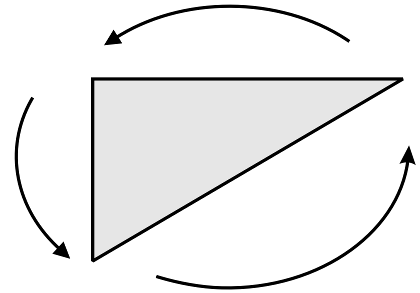
$K = \frac{Eh}{}$

1	2	3	4	5	6	7	8	9	1…



$K_i^* =$



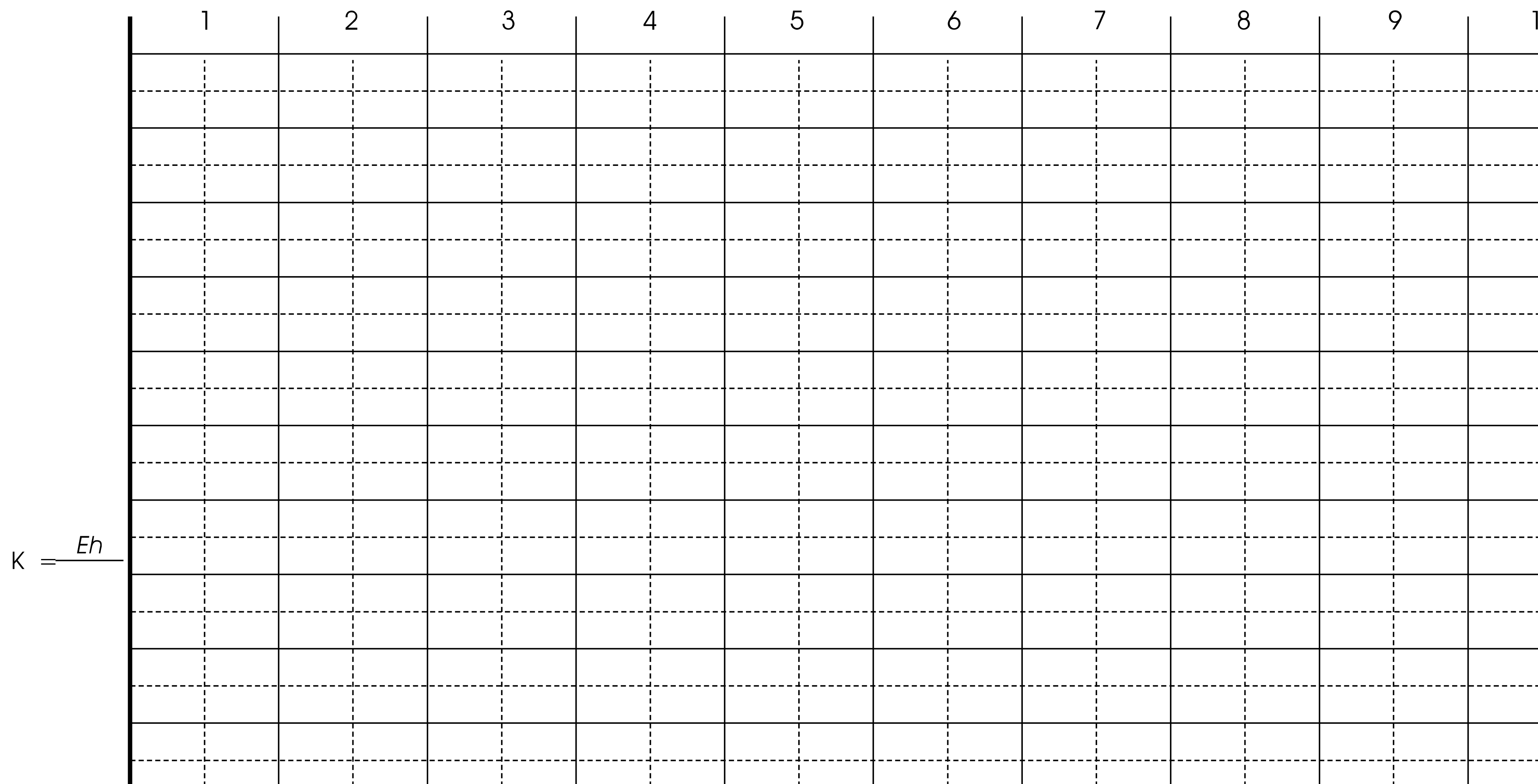


$K_i^* =$

Element u matrici krutosti sistema

$K = \frac{Eh}{\quad}$

1	2	3	4	5	6	7	8	9	1


1
2
3
4
5
6
7
8
9
K = Eh/