Određeni integrali

Osobine određenih integrala su:

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

2. $$\int_a^a f(x)\,dx = 0$$

3. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4. $$\int_a^b [f_1(x) + f_2(x) - f_3(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx - \int_a^b f_3(x)\,dx$$

5. $$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx,$$ gdje je α konstanta

Određene integrale ćemo računati pomoću Njutn-Lejbicove formule

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big|_a^b = F(x)\Big|_a^b = F(b) - F(a)$$

gdje je $F'(x) = f(x)$

(#) Izračunati integrale

a) $\int\limits_2^3 3x^2\,dx$; b) $\int\limits_0^4 \left(1+e^{\frac{x}{4}}\right)dx$; c) $\int\limits_{-1}^{7} \frac{dt}{\sqrt{3t+4}}$;

d) $\int\limits_0^{\pi/2a} (x+3)\sin ax$

Rj.

a) $$\int\limits_2^3 3x^2\,dx = 3\int\limits_2^3 x^2\,dx = 3\cdot\frac{x^3}{3}\Big|_2^3 = x^3\Big|_2^3 = 3^3-2^3 = 27-8=19$$

b) $$\int\limits_0^4 \left(1+e^{\frac{x}{4}}\right)dx = \int\limits_0^4 dx + \int\limits_0^4 e^{\frac{x}{4}}\,dx = \int\limits_0^4 dx + 4\int\limits_0^4 e^{\frac{x}{4}}\,d\left(\frac{x}{4}\right) =$$

$$= x\Big|_0^4 + 4\,e^{\frac{x}{4}}\Big|_0^4 = (4-0)+4\left(e^{\frac{4}{4}}-e^{\frac{0}{4}}\right) = 4+4e-4=4e$$

c) $$\int\limits_{-1}^{7}\frac{dt}{\sqrt{3t+4}} = \int\limits_{-1}^{7}(3t+4)^{-\frac{1}{2}}\,dt = \left|\begin{array}{l} d(3t+4)=3\,dt \\ dt=\frac{1}{3}\,d(3t+4)\end{array}\right| =$$

$$= \frac{1}{3}\int\limits_{-1}^{7}(3t+4)^{-\frac{1}{2}}\,d(3t+4) = \frac{2}{3}(3t+4)^{\frac{1}{2}}\Big|_{-1}^{7} = \frac{2}{3}\left(\sqrt{25}-\sqrt{1}\right) = \frac{8}{3}$$

d) $$\int\limits_0^{\frac{\pi}{2a}}(x+3)\sin ax\,dx = \left|\begin{array}{ll} u=x+3 & dv=\sin ax\,dx \\ du=dx & v=\frac{1}{a}\int\sin ax\,d(ax) = \frac{-1}{a}\cos ax\end{array}\right| =$$

$$= -\frac{1}{a}(x+3)\cos ax\Big|_0^{\frac{\pi}{2a}} + \frac{1}{a}\int\limits_0^{\pi/2a}\cos ax\,dx = -\frac{1}{a}\left[\left(\frac{\pi}{2a}+3\right)\underbrace{\cos\frac{\pi}{2}}_{=0} - 3\underbrace{\cos 0}_{=1}\right] +$$

$$+\frac{1}{a}\cdot\frac{1}{a}\int\limits_0^{\pi/2a}\cos ax\,d(ax) = \frac{3}{a}+\frac{1}{a^2}\underbrace{\sin ax\Big|_0^{\frac{\pi}{2a}}}_{\sin\frac{\pi}{2}-\sin 0} = \frac{3}{a}+\frac{1}{a^2} = \frac{1+3a}{a^2}$$

Zadaci za vežbu

1. $\int_1^5 \frac{dx}{3x-2}$

2. $\int_0^1 \frac{dz}{(2z+1)^3}$

3. $\int_1^2 \frac{dt}{t^2+5t+4}$

4. $\int_0^2 \frac{x+3}{x^2+4}\,dx$

5. $\int_{-a}^{a} x\cos\frac{x}{a}\,dx$

6. $\int_0^{\pi} \cos\frac{x}{2}\cos\frac{3x}{2}\,dx$

7.* $\int_{-\pi}^{\pi} x\sin x\cos x\,dx$

8. $\int_1^e (1+\ln y)^2\,dy$

Rešenja:

1. $\frac{\ln 13}{3}$ 2. $\frac{2}{9}$ 3. $\frac{1}{3}\ln\frac{5}{4}$ 4. $\frac{3\pi}{8}+\frac{\ln 2}{2}$

5. 0 6. 0 7. $-\frac{\pi}{2}$ 8. $2e-1$

Zamena promenljivih u određenom integralu

$$\int_a^b f(x)\,dx = \left| \begin{matrix} x=\varphi(t) & x=a \Rightarrow a=\varphi(\alpha) \Rightarrow t=\alpha \\ dx=\varphi'(t)\,dt & x=b \Rightarrow b=\varphi(\beta) \Rightarrow t=\beta \end{matrix} \right|$$

$$= \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt = \int_\alpha^\beta F(t)\,dt$$

Izračunati integrale

a) $\displaystyle\int_0^5 \frac{x\,dx}{\sqrt{1+3x}}$; b) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{dx}{e^x-e^{-x}}$; c) $\displaystyle\int_1^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}}$; d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

Rj.

a) $$\int_0^5 \frac{x\,dx}{\sqrt{1+3x}} = \left| \begin{matrix} 1+3x=t^2 & 3\,dx=2t\,dt \\ \sqrt{1+3x}=t & dx=\frac{2}{3}t\,dt \\ 3x=t^2-1 & \\ x=\frac{t^2-1}{3} & x\Big|_0^5 \Rightarrow t\Big|_1^4 \end{matrix} \right| = \int_1^4 \frac{\frac{t^2-1}{3}\cdot\frac{2}{3}t\,dt}{t} =$$

$$= \frac{2}{9}\int_1^4 (t^2-1)\,dt = \frac{2}{9}\left(\frac{t^3}{3}\Big|_1^4 - t\Big|_1^4\right) = \frac{2}{9}\left(\frac{1}{3}(64-1)-(4-1)\right) =$$

$$= \frac{2}{9}\left(\frac{63}{3}-3\right) = \frac{2}{9}(21-3) = 4$$

b) $$\int_{\ln 2}^{\ln 3} \frac{dx}{e^x-e^{-x}} = \left| \begin{matrix} e^x=t & e^{-x}=t^{-1}=\frac{1}{t} \\ x=\ln t & \\ dx=\frac{dt}{t} & x\Big|_{\ln 2}^{\ln 3} \Rightarrow t\Big|_2^3 \end{matrix} \right| = \int_2^3 \frac{\frac{dt}{t}}{t-\frac{1}{t}} = \int_2^3 \frac{\frac{dt}{t}}{\frac{t^2-1}{t}} =$$

$$= \int_2^3 \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\Bigg|_2^3 = \frac{1}{2}\left(\ln\frac{2}{4}-\ln\frac{1}{3}\right) = \frac{1}{2}\cdot\ln\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\ln\frac{3}{2}}{2}$$

c)
$$\int_{1}^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}} = \left|\begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \quad x\Big|_{1}^{\sqrt{3}} \Rightarrow t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ x^3 = 8\sin^3 t \\ \sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = \sqrt{4(1-\sin^2 t)} \end{array}\right| =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(8\sin^3 t+1)\,2\cos t\,dt}{4\sin^2 t\sqrt{4\cos^2 t}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin^3 t+1}{4\sin^2 t}\,dt =$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin t\,dt + \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dt}{\sin^2 t} = -2\cos t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \frac{1}{4}\,\mathrm{ctg}\,t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$= -2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) - \frac{1}{4}\left(\frac{\sqrt{3}}{3} - \sqrt{3}\right) = \frac{7}{2\sqrt{3}} - 1$$

d)
$$\int_{0}^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \left|\begin{array}{ll} \mathrm{tg}\,\frac{x}{2} = z & dx = \frac{2dz}{1+z^2} \\ \cos x = \frac{1-z^2}{1+z^2} & x\Big|_{0}^{\frac{\pi}{2}} \Rightarrow z\Big|_{0}^{1} \end{array}\right| =$$

$$= \int_{0}^{1} \frac{\frac{2dz}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} = 2\int_{0}^{1} \frac{dz}{z^2+3} = \frac{2}{\sqrt{3}}\,\mathrm{arc\,tg}\,\frac{z}{\sqrt{3}}\Big|_{0}^{1} =$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

(#) Dokazati da za parnu f-ju $f(x)$ vrijedi

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

dok za neparnu f-ju $f(x)$ vrijedi $\int_{-a}^{a} f(x)\,dx = 0$.

Rj. Prvo rastavimo interval $[-a,a]$ na dva dijela $[-a,0]$ i $[0,a]$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \dots (*)$$

Posmatrajmo sad $\int_{-a}^{0} f(x)\,dx$. Ako uvedemo smjenu $x=-z$ imamo da je $dx=-dz$ i $z_1=a$ za $x_1=-a$, $z_2=0$ za $x_2=0$

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-z)\,dz = \int_{0}^{a} f(-z)\,dz = \left|\begin{matrix}\text{nova prom.}\\ x=z\end{matrix}\right| = \int_{0}^{a} f(-x)\,dx$$

Prema tome $(*)$ sad postaje

$$I = \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} [f(x)+f(-x)]\,dx$$

Za parnu f-ju znamo da $f(-x)=f(x)$ dok je za neparnu f-ju $f(-x)=-f(x)$. Prema tome

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx, & \text{ako je } f(x) \text{ parna f-ja} \\ 0, & \text{ako je } f(x) \text{ neparna f-ja} \end{cases}$$

(7) Znamo da za parnu f-ju $f(x)$ vrijedi

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx,$$

dok za neparnu f-ju $f(x)$ vrijedi: $\int_{-a}^{a} f(x)\,dx = 0$.

Iskoristiti ovu osobinu i izračunati sljedeće integrale:

a) $\int_{-\sqrt{5}}^{\sqrt{5}} (3x-2x^5)\,dx$ b) $\int_{-\pi}^{\pi} \sin^7 2x\,dx$ c) $\int_{3}^{-3} t^8 \arcsin t\,dt$

d) $\int_{-2}^{2} \frac{x^5+7x^4+x^3-5x^2-2}{x^3+x}\,dx$

Rj:

a) $f(x) = 3x-2x^5$

$f(-x) = 3(-x)-2(-x)^5 = -3x+2x^5 = -(3x-2x^5) = -f(x)$

$$\int_{-\sqrt{5}}^{\sqrt{5}} (3x-2x^5)\,dx = \left| \begin{array}{l} \text{primijetimo} \\ \text{da je} \\ 3x-2x^5 \text{ neparna} \\ \text{f-ja} \end{array} \right| = 0$$

b) $f(x) = \sin^7 2x \Rightarrow f(-x) = (\sin 2(-x))^7 = (-\sin 2x)^7 = -\sin^7 2x = -f(x)$

Kako je $\sin^7 2x$ neparna f-ja $\int_{-\pi}^{\pi} \sin^7 2x\,dx = 0$

c) $\int_{3}^{-3} t^8 \arcsin t\,dt = 0$ ZAŠTO? OBJASNITI!

d) $$\int_{-2}^{2} \frac{x^5+7x^4+x^3-5x^2-2}{x^3+x}\,dx = \int_{-2}^{2} \frac{\overbrace{x^5+x^3}^{x^2(x^3+x)}}{x^3+x}\,dx + \int_{-2}^{2} \frac{7x^4-5x^2-2}{x^3-x}\,dx =$$

$$= \int_{-2}^{2} x^2\,dx + 0 = 2\int_{0}^{2} x^2\,dx = 2\,\frac{x^3}{3}\Big|_0^2 = \frac{16}{3}$$

Zadaci za vježbu

Izračunati integrale

1. $$\int_0^1 \frac{x^2\,dx}{(x+1)^4}$$ uvođenjem smjene $x+1=z$.

2. $$\int_0^{\ln 2} \sqrt{e^x-1}\,dx$$ uvođenjem smjene $\sqrt{e^x-1}=t$.

3. $$\int_{\sqrt{3}}^{\sqrt{7}} \frac{x^3\,dx}{\sqrt[3]{(x^2+1)^2}}$$ uvođenjem smjene $z=x^2+1$.

4. $$\int_1^e \frac{\sqrt[4]{1+\ln x}}{x}\,dx$$ uvođenjem smjene $t=1+\ln x$.

5. $$\int_{-3}^{3} x^2\sqrt{9-x^2}\,dx$$ uvođenjem smjene $x=3\cos\varphi$

6. $$\int_5^1 \frac{t\,dt}{\sqrt{5+4t}}$$

7. $$\int_0^{\frac{\pi}{4}} \frac{1+\operatorname{tg}^2\varphi}{1+\operatorname{tg}\varphi}\,d\varphi$$

8. $$\int_{\ln 3}^{0} \frac{1-e^x}{1+e^x}\,dx$$

9. $$\int_{-1}^{0} \frac{dx}{1+\sqrt[3]{x+1}}$$

10. * $$\int_0^8 \sqrt{\frac{x}{6-x}}\,dx$$

11. * $$\int_0^{\frac{\pi}{2}} \sin^3\varphi\sqrt{\cos\varphi}\,d\varphi$$

Rješenja:

1. $\frac{1}{24}$ 2. $\frac{4-\pi}{2}$ 3. 3 4. $0{,}8(2\sqrt[4]{2}-1)$ 5. $\frac{81\pi}{8}$ 6. $-\frac{17}{6}$

7. $\ln 2$ 8. $\ln\frac{4}{3}$ 9. $\frac{3}{2}(\ln 4-1)$ 10. $\frac{3(\pi-2)}{2}$ (uvodimo smjenu $x=6\sin^2 t$)

11. $8/21$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Računje određenih integrali i Smjena promjenjivih u određenim integralima)

1. $$\int_{\pi/6}^{\pi/3} \frac{dx}{\cos^2 x} = tg\, x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = tg\frac{\pi}{3} - tg\frac{\pi}{6} = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

2. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin x\, dx = -\cos x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{4}\right) = -\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) = -\frac{1-\sqrt{2}}{2}$$

3. $$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \arcsin\frac{\sqrt{3}}{2} - \arcsin\frac{1}{2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

4. $$\int_a^b x^m dx = \frac{x^{m+1}}{m+1}\Big|_a^b = \frac{1}{m+1}\left(b^{m+1} - a^{m+1}\right)$$

5. $$\int_0^1 (e^x-1)^4 e^x\, dx = \left| \begin{array}{l} e^x - 1 = t \\ e^x dx = dt \\ x=0 \Rightarrow t=0 \\ x=1 \Rightarrow t=e-1 \end{array} \right| = \int_0^{e-1} t^4 dt = \frac{t^5}{5}\Big|_0^{e-1} = \frac{1}{5}(e-1)^5$$

6. $$\int_2^9 \sqrt[3]{x-1}\, dx = \left| \begin{array}{l} x-1 = t^3 \\ dx = 3t^2 dt \\ x=2 \Rightarrow t=1 \\ x=9 \Rightarrow t=2 \end{array} \right| = \int_1^2 \sqrt[3]{t^3}\cdot 3t^2 dt = 3\int_1^2 t^3 dt = \frac{3}{4}t^4\Big|_1^2 =$$
$$= \frac{3}{4}(16-1) = \frac{45}{4}$$

7. $$\int_0^{\ln 5} \frac{\sqrt{e^x-1}}{1+3e^{-x}}\, dx \frac{\cdot e^x}{\cdot e^x} = \int_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\, dx = \left| \begin{array}{l} e^x - 1 = t^2 \\ e^x dx = 2t\,dt \\ x=0 \Rightarrow t=0 \\ x=\ln 5 \Rightarrow t=2 \\ e^x = t^2+1 \end{array} \right| = \int_0^2 \frac{\sqrt{t^2}\cdot 2t}{t^2+1+3}\, dt$$
$$= 2\int_0^2 \frac{t^2 +4-4}{t^2+4}\, dt = 2\int_0^2 dt - 2\int_0^2 \frac{4}{t^2+4}\, dt = 2t\Big|_0^2 - 8\cdot\frac{1}{2}\operatorname{arctg}\frac{t}{2}\Big|_0^2 =$$
$$= 4 - 4\left(\operatorname{arctg} 1 - \operatorname{arctg} 0\right) = 4 - 4\cdot\frac{\pi}{4} = 4 - \pi$$

Osobine određenih integrala

a) $\int_a^a f(x)\,dx = 0$

b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

c) $\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$

d) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

e) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall x$

8. $$\int_0^{\sqrt{7}} \frac{dx}{7+x^2} = \frac{1}{\sqrt{7}}\,\mathrm{arctg}\,\frac{x}{\sqrt{7}} = \frac{1}{\sqrt{7}}\left(\mathrm{arctg}\,\frac{\sqrt{7}}{\sqrt{7}} - \mathrm{arctg}\,\frac{0}{\sqrt{7}}\right) = \frac{1}{\sqrt{7}}\cdot\frac{\pi}{4} = \frac{\sqrt{7}\pi}{28}$$

9. $$\int_0^{1/2} \sqrt{1-x^2}\,dx = \left| \begin{array}{l} x = \sin t \\ dx = \cos t\,dt \\ x=0 \Rightarrow \sin t = 0 \Rightarrow t=0 \\ x=\frac{1}{2} \Rightarrow \sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6} \end{array} \right| = \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2 t}\,\cos t\,dt =$$

kako je $\sin^2 t + \cos^2 t = 1$

$$= \int_0^{\pi/6} \sqrt{\cos^2 t}\,\cos t\,dt = \int_0^{\pi/6} \cos^2 t\,dt = \frac{1}{2}\int_0^{\pi/6} (1+\cos 2t)\,dt = \frac{1}{2}t\Big|_0^{\frac{\pi}{6}} + \frac{1}{4}\sin 2t\Big|_0^{\frac{\pi}{6}}$$

$1 = \cos^2 t + \sin^2 t$
$\cos 2t = \cos^2 t - \sin^2 t$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} = \frac{\pi}{12} + \frac{\sqrt{3}}{8} = \frac{2\pi + 3\sqrt{3}}{24}$$

10. $$\int_0^4 \frac{dx}{1+\sqrt{2x+1}}$$ uputa: smjena $2x+1 = t^2$

rj. $2 - \ln 2$

11. $$\int_0^1 \frac{dx}{\sqrt{2-x^2+x}}$$ uputa: $-x^2+x+2 = \ldots$

$\ldots = \frac{9}{4} - \left(x - \frac{1}{2}\right)^2$

$x - 1 = \frac{3}{2}t$

rj. $2\arcsin\frac{1}{3}$

12. $$\int_1^e x\ln x\,dx = \left| \begin{array}{ll} u = \ln x & dv = x\,dx \\ du = \frac{dx}{x} & v = \frac{x^2}{2} \end{array} \right| = \frac{1}{2}x^2 \ln x\Big|_1^e - \frac{1}{2}\int_1^e x^2 \frac{dx}{x} =$$

$$= \frac{1}{2}\left(e^2 \ln e - 1^2 \ln 1\right) - \frac{1}{2}\int_1^e x\,dx = \frac{1}{2}e^2 - \frac{1}{2}\cdot\frac{1}{2}x^2\Big|_1^e =$$

$$= \frac{1}{2}e^2 - \frac{1}{4}\left(e^2 - 1\right) = \frac{1}{4}e^2 + \frac{1}{4}$$

(#) Izračunati integral $\displaystyle\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5}\,dx$.

Rj. $x-4\sqrt{x}+5 = x-2\cdot\sqrt{x}\cdot 2+4+1 = (\sqrt{x}-2)^2+1$

$$\int_1^4 \frac{\sqrt{x}+2}{(\sqrt{x}-2)^2+1}\,dx = \left| \begin{array}{l} x=t^2 \\ dx=2t\,dt \\ x=1 \Rightarrow t=1 \\ x=4 \Rightarrow t=2 \end{array} \right| = \int_1^2 \frac{t+2}{(t-2)^2+1}\cdot 2t\,dt =$$

$$=2\int_1^2 \frac{t^2+2t}{t^2-4t+5}\,dt = 2\int_1^2 \frac{t^2+2t-6t+6t+5-5}{t^2-4t+5}\,dt = 2\int_1^2 \frac{t^2-4t+5}{t^2-4t+5}\,dt +$$

$$+2\int_1^2 \frac{6t-5}{t^2-4t+5}\,dt = 2\int_1^2 dt + 2\cdot 3\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}$$

$$\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}\,dt = \int_1^2 \frac{2t-4+4-\frac{5}{3}}{t^2-4t+5}\,dt = \int_1^2 \frac{2t-4}{t^2-4t+5}\,dt + \frac{7}{3}\int_1^2 \frac{dt}{t^2-4t+5}$$

$$\int_1^2 dt = t\Big|_1^2 = 2-1 = 1\,, \quad \int_1^2 \frac{2t-4}{t^2-4t+5}\,dt = \left| \begin{array}{l} t^2-4t+5=s \\ (2t-4)dt=ds \\ t=1 \Rightarrow s=2 \\ t=2 \Rightarrow s=1 \end{array} \right| = \int_2^1 \frac{ds}{s} = \ln|s|\Big|_2^1$$

$$= \ln 1 - \ln 2 = -\ln 2$$

$$\int_1^2 \frac{dt}{t^2-4t+5} = \int_1^2 \frac{dt}{(t-2)^2+1} = \left| \begin{array}{l} t-2=s \\ dt=ds \\ t=1 \Rightarrow s=-1 \\ t=2 \Rightarrow s=0 \end{array} \right| = \int_{-1}^0 \frac{ds}{s^2+1} = \operatorname{arctg} s\Big|_{-1}^0 = 0-\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5} = 2\cdot 1 + 6\left(-\ln 2 + \frac{7}{3}\cdot\frac{\pi}{4}\right) = 2 - 6\ln 2 + \frac{7\pi}{2} \approx 8{,}8367$$

traženo rješenje

(#) Izračunati integral $\int_0^{\pi/4} \sin^5 x \cos^7 x \, dx$

Rj. $\int_0^{\frac{\pi}{4}} \sin^5 x \cdot \cos^7 x \, dx = \int_0^{\frac{\pi}{4}} \sin^5 x \cdot \cos^6 x \cdot \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \\ x=0 \Rightarrow t=0 \\ x=\frac{\pi}{4} \Rightarrow t=\frac{\sqrt{2}}{2} \\ \cos^6 x = (\cos^2 x)^3 = \\ = (1-\sin^2 x)^3 = (1-t^2)^3 \end{array} \right| =$

$$= \int_0^{\frac{\sqrt{2}}{2}} t^5 (1-t^2)^3 \, dt =$$

1
1 1
1 2 1
1 3 3 1

$$= \int_0^{\frac{\sqrt{2}}{2}} t^5 (1-3t^2+3t^4-t^6) \, dt = \int_0^{\frac{\sqrt{2}}{2}} (t^5-3t^7+3t^9-t^{11}) \, dt =$$

$$= \frac{1}{6} t^6 \Big|_0^{\frac{\sqrt{2}}{2}} - \frac{3}{8} t^8 \Big|_0^{\frac{\sqrt{2}}{2}} + \frac{3}{10} t^{10} \Big|_0^{\frac{\sqrt{2}}{2}} - \frac{1}{12} t^{12} \Big|_0^{\frac{\sqrt{2}}{2}} =$$

$$= \frac{1}{\cancel{6}_3} \cdot \frac{\cancel{8}^{\cancel{4}\,1}}{\cancel{64}_{16}} - \frac{3}{\cancel{8}_1} \cdot \frac{\cancel{16}^{\cancel{2}\,1}}{\cancel{256}_{128}} + \frac{3}{\cancel{10}_5} \cdot \frac{\cancel{32}^{\cancel{16}\,1}}{\cancel{1024}_{64}} - \frac{1}{\cancel{12}_3} \cdot \frac{\cancel{64}^{\cancel{16}\,1}}{\cancel{4096}_{256}} =$$

$$= \frac{1}{\underbrace{3 \cdot 16}_{3 \cdot 2^4}} - \frac{3}{\underbrace{128}_{2^7}} + \frac{3}{\underbrace{5 \cdot 64}_{5 \cdot 2^6}} - \frac{1}{\underbrace{3 \cdot 256}_{3 \cdot 2^8}} = \frac{5 \cdot 2^4 - 3 \cdot 3 \cdot 5 \cdot 2 + 3 \cdot 3 \cdot 2^2 - 5}{3 \cdot 5 \cdot 2^8}$$

$$= \frac{80 - 90 + 36 - 5}{3 \cdot 5 \cdot 2^8} = \frac{\cancel{21}^{7}}{\cancel{3}_1 \cdot 5 \cdot 2^8} = \frac{7}{1280}$$ traženo rješenje

(#) Izračunati: $\displaystyle\int_3^4 \frac{6x+8}{x^2+x-6}\,dx$

Rj.

$$\frac{6x+8}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3} \quad /(x-2)(x+3)$$

$D = 1+24 = 25$

$x_{1,2} = \frac{-1\pm 5}{2}$

$x_1 = -3 \quad x_2 = 2$

$x^2+x-6 = (x-2)(x+3)$

$$6x+8 = A(x+3)+B(x-2)$$

$$6x+8 = (A+B)x + (3A-2B)$$

$$\begin{aligned} A+B&=6 \quad /\cdot 2\\ 3A-2B&=8 \end{aligned}$$

$$\begin{aligned} 2A+2B&=12\\ +\ 3A-2B&=8 \\ \hline 5A&=20\\ A&=4 \end{aligned}$$

$A+B=6$

$4+B=6$

$B=2$

$$\int \frac{6x+8}{x^2+x-6}\,dx = \int\left(\frac{4}{x-2}+\frac{2}{x+3}\right)dx = 4\int\frac{dx}{x-2} + 2\int\frac{dx}{x+3} =$$

$$= 4\ln|x-2| + 2\ln|x+3| + C$$

$$\int_3^4 \frac{6x+8}{x^2+x-6}\,dx = 4\ln|x-2|\Big|_3^4 + 2\ln|x+3|\Big|_3^4 = 4(\ln 2 - \underbrace{\ln 1}_{=0}) +$$

$$+ 2(\ln 7 - \ln 6) = 4\ln 2 + 2\ln\frac{7}{6} = \ln 2^4 + \ln\left(\frac{7}{6}\right)^2$$

$$= \ln\frac{7^2}{2^2\cdot 3^2}\cdot 2^4 = \ln\frac{49\cdot 4}{9} = \ln\frac{196}{9}$$

$$\int_3^4 \frac{6x+8}{x^2+x-6}\,dx = \ln\frac{196}{9}$$ traženo rješenje

(#) Izračunati integral $I=\int_0^1 \arcsin\frac{x}{2}\,dx$.

Rj.

$$\int \arcsin\frac{x}{2}\,dx = \left|\begin{array}{l} \frac{x}{2}=t \\ \frac{1}{2}dx=dt \\ dx=2dt \end{array}\right| = 2\int \arcsin t\,dt = \left|\begin{array}{ll} u=\arcsin t & dv=dt \\ du=\frac{dt}{\sqrt{1-t^2}} & v=t \end{array}\right| =$$

$$= 2\left(t\arcsin t - \int \frac{t}{\sqrt{1-t^2}}dt\right) = 2t\arcsin t - \int \frac{2t\,dt}{\sqrt{1-t^2}} \underline{\underline{(**)}}$$

$$\int \frac{-2t\,dt}{\sqrt{1-t^2}} = \left|\begin{array}{l} 1-t^2=s \\ -2t\,dt=ds \end{array}\right| = \int \frac{ds}{\sqrt{s}} = \int s^{-\frac{1}{2}}ds = \frac{s^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{s}+c =$$

$$= 2\sqrt{1-t^2}+c$$

$$\underline{\underline{(**)}}\ 2t\arcsin t + 2\sqrt{1-t^2}+c = x\arcsin\frac{x}{2} + 2\sqrt{1-\frac{x^2}{4}} + c$$

$$\int_0^1 \arcsin\frac{x}{2}\,dx = x\arcsin\frac{x}{2}\Big|_0^1 + 2\sqrt{1-\frac{x^2}{4}}\Big|_0^1 = \arcsin\frac{1}{2} + \left(2\sqrt{1-\frac{1}{4}} - 2\right) =$$

$$= \frac{\pi}{6} + \frac{2\sqrt{3}}{2} - 2 = \frac{\pi}{6} + \sqrt{3} - 2$$

Zadaci za vježbu

Integrali s beskonačnim granicama

U zadacima 2366 2385 izračunati donje nesvojstvene integrale (ili ustanoviti njihovu divergenciju).

2366. $\int\limits_1^\infty \frac{dx}{x^4}$. **2367.** $\int\limits_1^\infty \frac{dx}{\sqrt{x}}$. **2368.** $\int\limits_0^\infty e^{-ax}\,dx \quad (a>0)$.

2369. $\int\limits_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$. **2370.** $\int\limits_{-\infty}^\infty \frac{dx}{x^2+2x+2}$. **2371.** $\int\limits_2^\infty \frac{\ln x}{x}\,dx$.

2372. $\int\limits_1^\infty \frac{dx}{x^2(x+1)}$. **2373.** $\int\limits_0^\infty \frac{x}{(1+x)^3}\,dx$ **2374.** $\int\limits_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$.

2375. $\int\limits_{a^2}^\infty \frac{dx}{x\sqrt{1+x^2}}$. **2376.** $\int\limits_0^\infty xe^{-x^2}\,dx$. **2377.** $\int\limits_0^\infty x^3e^{-x^2}\,dx$.

2378. $\int\limits_0^\infty x\sin x\,dx$. **2379.** $\int\limits_0^\infty e^{-\sqrt{x}}\,dx$. **2380.** $\int\limits_0^\infty e^{-x}\sin x\,dx$.

2381. $\int\limits_0^\infty e^{-ax}\cos bx\,dx$. **2382.** $\int\limits_1^\infty \frac{\operatorname{arctg} x}{x^2}\,dx$. **2383.** $\int\limits_0^\infty \frac{dx}{1+x^3}$.

2384. $\int\limits_{-\infty}^\infty \frac{dx}{(x^2+1)^2}$. **2385** $\int\limits_1^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx$.

U zadacima 2386 — 2393 ispitati da li su dati integrali konvergentni.

2386. $\int\limits_0^\infty \frac{x}{x^3+1}\,dx$. 2387. $\int\limits_1^\infty \frac{x^3+1}{x^4}\,dx$.

2388. $\int\limits_0^\infty \frac{x^{13}}{(x^5+x^3+1)^3}\,dx$. 2389. $\int\limits_1^\infty \frac{\ln(x^2+1)}{x}\,dx$.

2390. $\int\limits_0^\infty \sqrt{xe^{-x}}\,dx$. 2391. $\int\limits_0^\infty \frac{x\operatorname{arctg} x}{\sqrt[3]{1+x^4}}\,dx$.

2392. $\int\limits_{e^2}^\infty \frac{dx}{x\ln\ln x}$. 2393. $\int\limits_e^\infty \frac{dx}{x(\ln x)^{\frac{3}{2}}}$.

Rješenja

2366. $\frac{1}{3}$.

2367. Divergira. **2368.** $\frac{1}{a}$.

2369. Divergira. **2370.** π.

2371. Divergira. **2372.** $1-\ln 2$.

2373. $\frac{1}{2}$. **2374.** $\frac{\pi}{4}$.

2375. $\ln\frac{\sqrt{a^4+1}+1}{a^2}$. **2376.** $\frac{1}{2}$.

2377. $\frac{1}{2}$. **2378.** Divergira.

2379. 2. **2380.** $\frac{1}{2}$.

2381. $\frac{a}{a^2+b^2}$, ako je $a>0$, a ne postoji ako je $a<0$.

2382. $\frac{\pi}{4}+\frac{1}{2}\ln 2$. **2383.** $\frac{2\pi}{3\sqrt{3}}$.

2384. $\frac{\pi}{2}$. **2385.** $\frac{1}{2}+\frac{\pi}{4}$.

2386. Konvergira. **2387.** Divergira.

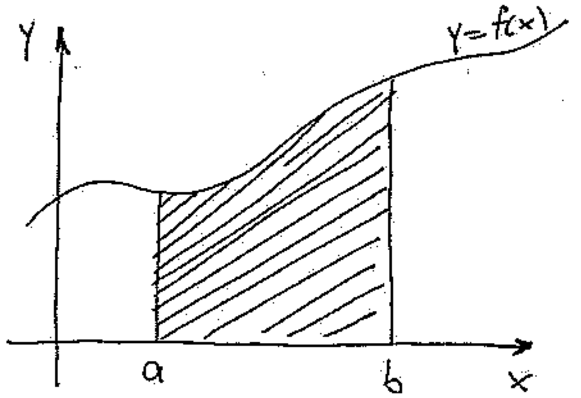
2388. Konvergira. **2389.** Divergira.

2390. Konvergira. **2391.** Divergira.

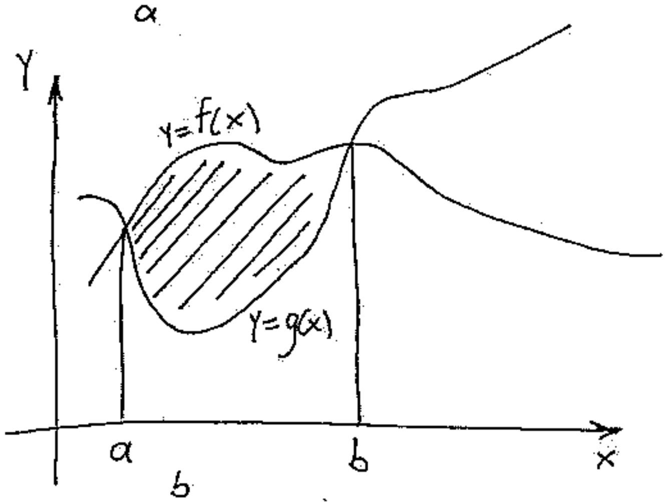
2392. Divergira. **2393.** Konvergira.

Primjena određenog integrala

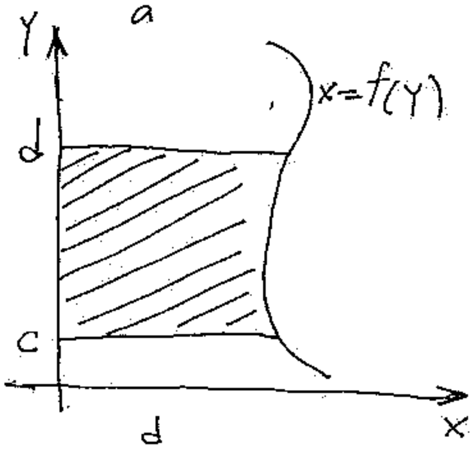
I Izračunavanje površine ravne figure



$$P=\int_a^b f(x)\,dx$$

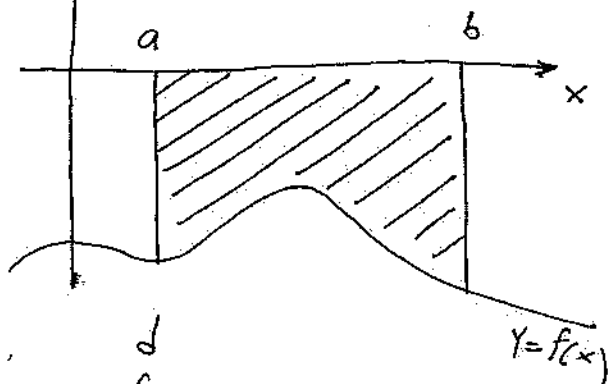


$$P=\int_a^b [f(x)-g(x)]\,dx$$

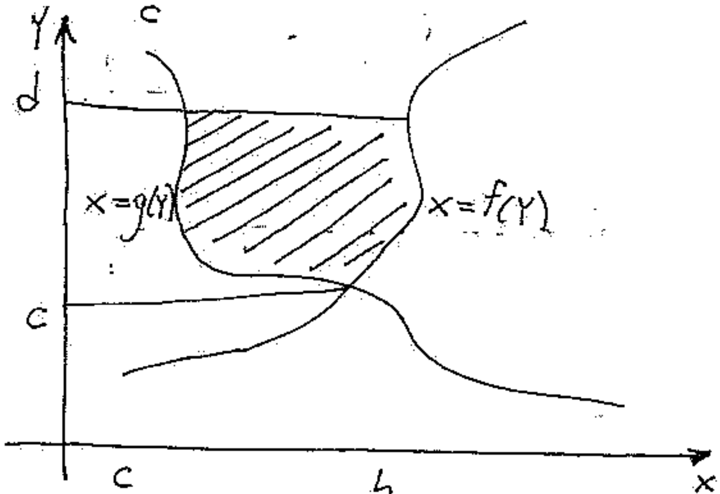


$$P=\int_c^d f(y)\,dy$$

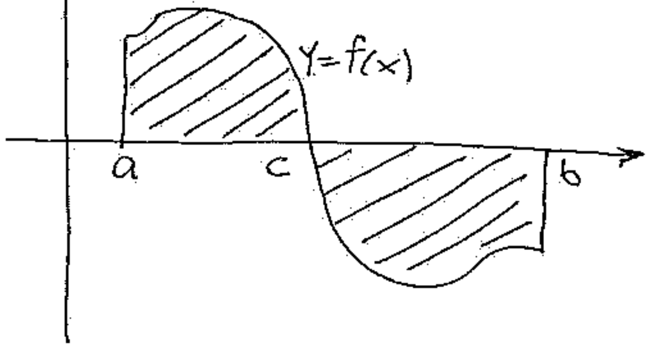
$$P=-\int_a^b f(x)\,dx$$



$$P=\int_c^d [f(y)-g(y)]\,dy$$



$$P=\int_a^c f(x)\,dx+\left|\int_c^b f(x)\,dx\right|$$



1. Izračunati površinu ravne figure koja je ograničena linijama $y = 4-(x-2)^2$ i $y=0$.

Rj.

$y = 4-(x-2)^2$

$y = 4-(x^2-4x+4)$

$y = -x^2+4x$

$y = -x(x-4)$

Kriva $y = ax^2+bx+c$ ima grafik u obliku parabole.

Tjeme parabole $T\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

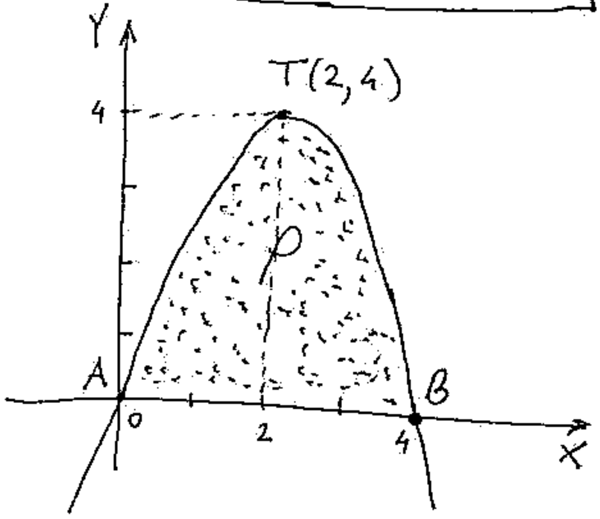
za $a>0$  za $a<0$ 

Nule $A(0,0)$ i $B(4,0)$

$-\frac{b}{2a} = -\frac{4}{2\cdot(-1)} = 2$

$\frac{4ac-b^2}{4a} = \frac{0-16}{-4} = 4$

Tjeme parabole $y=4-(x-2)^2$ je u tački $(2,4)$.



$$P = \int_0^4 (-x^2+4x)\,dx = \int_0^4 (-x^2)\,dx + \int_0^4 4x\,dx = -\frac{x^3}{3}\Big|_0^4 + 4\cdot\frac{x^2}{2}\Big|_0^4 = -\frac{1}{3}(4^3-0^3)$$

$$+2(4^2-0^2) = -\frac{1}{3}\cdot 64 + 32 = \frac{96}{3} - \frac{64}{3} = \frac{32}{3}$$

2. Izračunati površinu ravne figure koja je ograničena krivom $y=x^2-4x+3$ i pravama $y=0$, $x=0$ i $x=2$.

Rj. $y = x^2-4x+3$

$D = 16-12 = 4$

$x_{1,2} = \frac{4\pm 2}{2}$

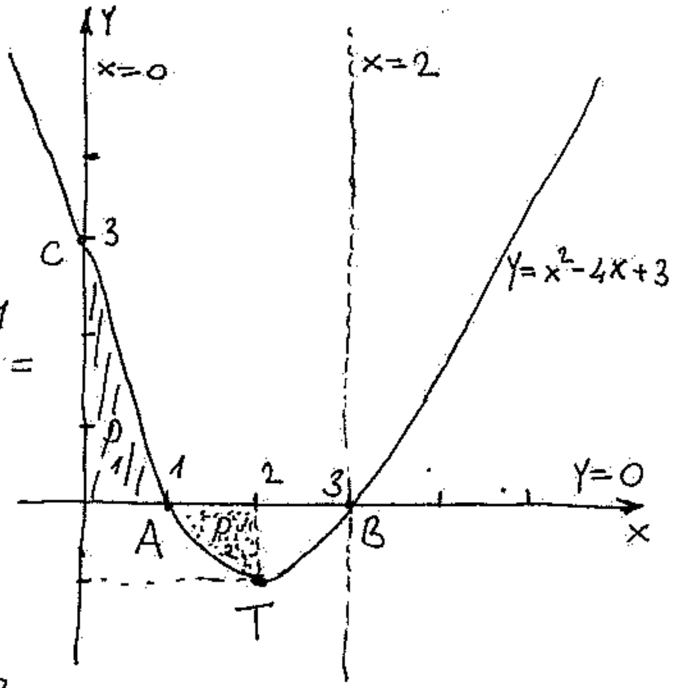
Nule krive $A(1,0)$ i $B(3,0)$

$-\frac{b}{2a} = -\frac{-4}{2} = 2$

$\frac{4ac-b^2}{4a} = \frac{12-16}{4} = -1$

Tjeme krive $y=x^2-4x+3$ je u tački $T(2,-1)$.

$C(0,3)$ je tačka presjeka krive sa Y-osom



$$P = P_1 + P_2$$

$$P_1 = \int_0^1 (x^2-4x+3)\,dx = \frac{x^3}{3}\Big|_0^1 - 4\cdot\frac{x^2}{2}\Big|_0^1 + 3x\Big|_0^1 =$$

$$= \frac{1}{3}(1-0) - 2(1-0) + 3(1-0) =$$

$$= \frac{1}{3} + 1 = \frac{4}{3}$$

$$P_2 = -\int_1^2 (x^2-4x+3)\,dx = -\left(\frac{x^3}{3}\Big|_1^2 - 4\cdot\frac{x^2}{2}\Big|_1^2 + 3x\Big|_1^2\right) = -\left(\frac{1}{3}(8-1) - 2(4-1) + 3\cdot 1\right)$$

$$= -\left(\frac{7}{3} - 6 + 3\right) = -\left(-\frac{2}{3}\right) = \frac{2}{3}$$

$P = \frac{4}{3} + \frac{2}{3} = 2$ tražena površina ravne figure.

(3.) Izračunati površinu ravne figure kojeg čine parabola $Y = x^2 - 2x + 2$ i prava $x + 2Y - 9 = 0$.

Rj.

prava $x + 2Y - 9 = 0$

$$2Y = -x + 9$$

$$Y = -\frac{1}{2}x + \frac{9}{2}$$

prava prolazi kroz tačke $A(0, \frac{9}{2})$ i $B(9, 0)$.

$Y = x^2 - 2x + 2$

$D = 4 - 8 = -4 < 0$

kriva ne siječe x-o
– nema nula

$x = 0 \Rightarrow Y = 2$

$C(0,2)$ je presjek krive sa Y-osom

$$-\frac{b}{2a} = -\frac{-2}{2} = 1$$

$$\frac{4ac - b^2}{4a} = \frac{8-4}{4} = 1$$

$T(1,1)$ je tjeme parabole

Trebamo naći još tačke presjeka prave i parabole.

$$Y = x^2 - 2x + 2$$
$$\underline{x + 2Y - 9 = 0}$$

$$Y = x^2 - 2x + 2$$
$$\underline{x = -2Y + 9}$$

$$Y = (-2Y+9)^2 - 2(-2Y+9) + 2$$

$$Y = 4Y^2 \underline{-36Y} + 81 \underline{+4Y} - 18 + 2$$

$$4Y^2 - 33Y + 65 = 0$$

$$D = 33^2 - 16 \cdot 65 = 49$$

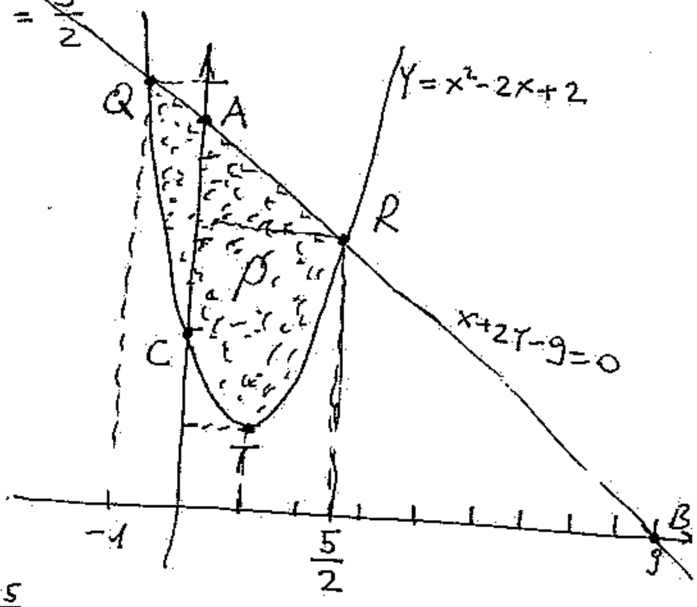
$$Y_{1,2} = \frac{33 \pm 7}{8} \qquad Y_1 = \frac{26}{8} = \frac{13}{4}$$

$$Y_2 = 5$$

$$Y_1 = \frac{13}{4} \Rightarrow x = -2 \cdot \frac{13}{4} + 9 = -\frac{13}{2} + \frac{18}{2} = \frac{5}{2}$$

$$Y_2 = 5 \Rightarrow x = -2 \cdot 5 + 9 = -1$$

Tačke presjeka prave i parabole su $R\left(\frac{5}{2}, \frac{13}{4}\right)$ i $Q(-1, 5)$



$$P = \int_{-1}^{\frac{5}{2}} \left[\left(-\frac{1}{2}x + \frac{9}{2}\right) - \left(x^2 - 2x + 2\right)\right] dx$$

$$\int_{-1}^{\frac{5}{2}} \left(-\frac{1}{2}x + \frac{9}{2}\right) dx = -\frac{1}{2} \cdot \frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}} + \frac{9}{2}x\Big|_{-1}^{\frac{5}{2}} = -\frac{1}{4}\left(\frac{25}{4} - 1\right) + \frac{9}{2}\left(\frac{5}{2} - (-1)\right)$$

$$= -\frac{1}{4} \cdot \frac{21}{4} + \frac{9}{2} \cdot \frac{7}{2} = \frac{231}{16}$$

$$\int_{-1}^{\frac{5}{2}} (x^2 - 2x + 2)\, dx = \frac{x^3}{3}\Big|_{-1}^{\frac{5}{2}} - 2\frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}} + 2x\Big|_{-1}^{\frac{5}{2}} = \frac{1}{3}\left(\frac{125}{8} - (-1)\right) - \left(\frac{25}{4} - 1\right) +$$

$$+ 2\left(\frac{5}{2} - (-1)\right) = \frac{1}{3} \cdot \frac{133}{8} - \frac{21}{4} + 2 \cdot \frac{7}{2} = \frac{133}{24} + \frac{7}{4} = \frac{175}{24}$$

$$P = \frac{231}{16} - \frac{175}{24} = \frac{231}{4 \cdot 4} - \frac{175}{6 \cdot 4} = \frac{686}{96} = \frac{343}{48}$$

4. Izračunati površinu ravne figure koja je ograničena krivom $Y^2 = 2x + 1$ i pravom $Y = 2x - 1$.

Rj. prava $Y = 2x - 1$ prolazi kroz tačke $A(0, -1)$ i $B\left(\frac{1}{2}, 0\right)$.

$y^2 = 2x + 1$

$2x = y^2 - 1$

$x = \frac{1}{2}y^2 - \frac{1}{2}$

$x = 0 \Rightarrow y^2 = 1$

$A(0,-1)$; $B(0,1)$ su tačke presjeka f-je sa y-osom

$C(-\frac{1}{2}, 0)$ je nula f-je

Kriva oblika $x = ay^2 + bx + c$ ima grafik u obliku parabole

$\supset$ ili $\subset$

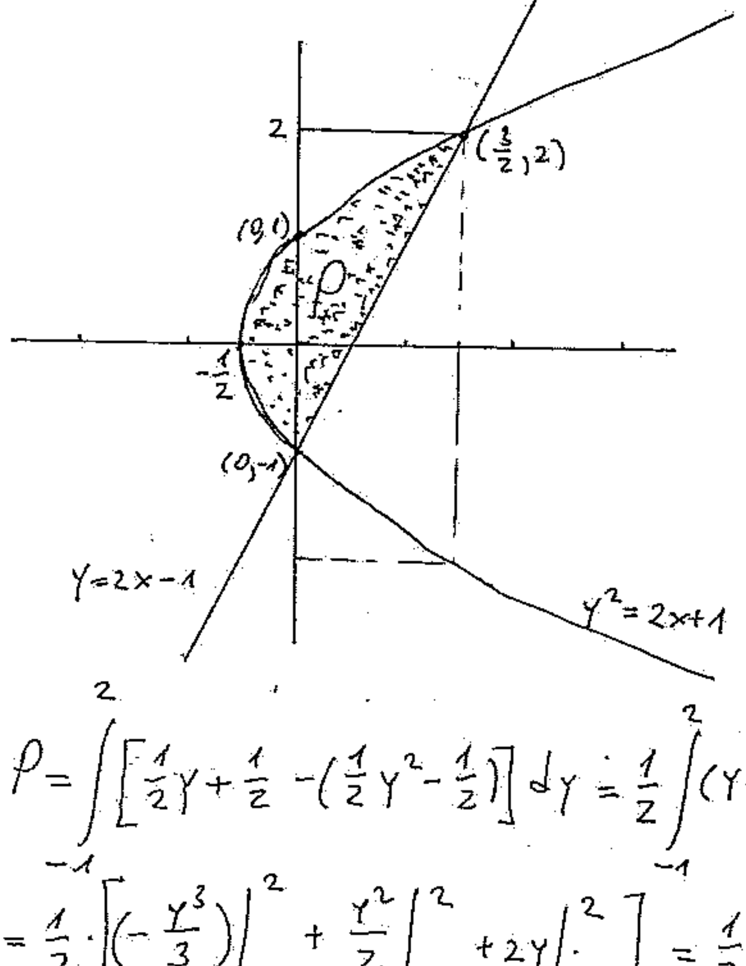
$a < 0$ $\qquad$ $a > 0$

Tjeme krive se traži po formuli $T\left(-\frac{D}{4a}, -\frac{b}{2a}\right)$

$D = 1 \qquad -\frac{D}{4a} = -\frac{1}{4 \cdot \frac{1}{2}}$

$-\frac{b}{2a} = -\frac{0}{2 \cdot \frac{1}{2}} = 0$

$T(-\frac{1}{2}, 0)$ je tjeme parabole



Tražimo još tačke presjeka krive i prave

za $x=0 \Rightarrow y=-1$

$y = 2x - 1$

$y^2 = 2x + 1$

$(2x-1)^2 = 2x + 1$

$4x^2 - 4x + 1 - 2x - 1 = 0$

$4x^2 - 6x = 0$

$2x(2x-3) = 0$

$x = 0 \quad \vee \quad x = \frac{3}{2}$

$D(0,-1)$ i $E(\frac{3}{2}, 2)$ su tačke presjeka krive i prave

$y = 2x - 1 \Rightarrow x = \frac{1}{2}y + \frac{1}{2}$

$y^2 = 2x + 1 \Rightarrow x = \frac{1}{2}y^2 - \frac{1}{2}$

$$P = \int_{-1}^{2} \left[\frac{1}{2}y + \frac{1}{2} - \left(\frac{1}{2}y^2 - \frac{1}{2}\right)\right] dy = \frac{1}{2}\int_{-1}^{2} (y + 1 - y^2 + 1)\, dy = \frac{1}{2}\int_{-1}^{2} (-y^2 + y + 2)\, dy =$$

$$= \frac{1}{2} \cdot \left[\left(-\frac{y^3}{3}\right)\Big|_{-1}^{2} + \frac{y^2}{2}\Big|_{-1}^{2} + 2y\Big|_{-1}^{2}\right] = \frac{1}{2}\left[-\frac{1}{3}(8+1) + \frac{1}{2}(4-1) + 2(2+1)\right] =$$

$$= \frac{1}{2}\left(-3 + \frac{3}{2} + 6\right) = \frac{1}{2} \cdot \frac{9}{2} = \frac{9}{4}$$ tražena površina

(#) Na parabolu $y=1-x^2$ povučena je normala u tački presjeka parabole i pozitivnog dijela x-ose. Odrediti površinu figure koju čine data parabola, povučena normala i y-osa.

Rj.

$y=1-x^2$

$y(0)=1$

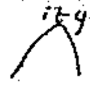
$(0,1)$ je presjek sa y-osom

$1-x^2=0$

$x^2=1$

$x_{1,2}=\pm 1$

$(-1,0)$ i $(1,0)$ su nule f-je

$y=-x^2+1$ parabola izgleda 

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$-\frac{b}{2a} = -\frac{0}{2\cdot(-1)} = 0$

$D = 0-4(-1)(1) = 4$

$-\frac{D}{4a} = -\frac{4}{4\cdot(-1)} = 1$

$T(0,1)$

$y-y_1 = y'(x_1)(x-x_1)$ jednačina tangente u tački (x_1,y_1)

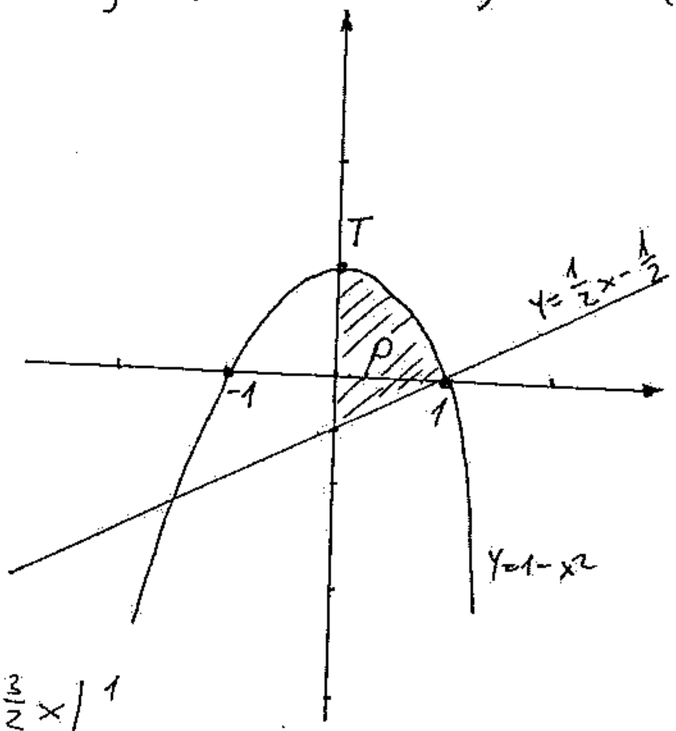
$y-y_1 = -\frac{1}{y'(x_1)}(x-x_1)$ jednačina normale u tački (x_1,y_1)

$y' = -2x$ presjek parabole i pozitivnog dijela x-ose je tačka $(1,0)$

$y'(1) = -2$

$y-0 = -\frac{1}{-2}(x-1)$

$y = \frac{1}{2}x - \frac{1}{2}$ jednačina normale u tački $(1,0)$



$$P = \int_0^1 \left[(1-x^2) - \left(\frac{1}{2}x - \frac{1}{2}\right)\right] dx =$$

$$= \int_0^1 \left(-x^2 - \frac{1}{2}x + \frac{3}{2}\right) dx = -\frac{1}{3}x^3\Big|_0^1 - \frac{1}{4}x^2\Big|_0^1 + \frac{3}{2}x\Big|_0^1$$

$$= -\frac{1\cdot 4}{3\cdot 4} - \frac{1\cdot 3}{4\cdot 3} + \frac{3}{2} = \frac{3\cdot 6}{2\cdot 6} - \frac{7}{12} = \frac{18-7}{12} = \frac{11}{12}$$

$P = \frac{11}{12}$ tražena površina

(#) Izračunati površinu koju pravi kriva $y = x^2 + x - 6$ zajedno sa svojim tangentama povučenim na tu krivu u nultačkama krive.

R: $y = x^2 + x - 6$

$D = 1 + 24 = 25$

$y = (x-2)(x+3)$

$x_1 = 2 \quad x_2 = -3$

$(2,0)$ i $(-3,0)$ su nule f-je

$f(0) = -6$

$(0,-6)$ je tačka presjeka f-je sa y-osom

$y' = 2x + 1$

$(-3,0)$, $y'(-3) = -5$

$y - 0 = -5(x+3)$

$y = -5x - 15$ jednačina tangente na krivu y u tački $(-3,0)$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ je tjeme f-je

$a > 0$ f-ja je $\cup$ oblika

$-\frac{b}{2a} = -\frac{1}{2}$, $-\frac{D}{4a} = -\frac{25}{4} = -6\frac{1}{4}$

$T\left(-\frac{1}{2}, -6\frac{1}{4}\right)$

$y - y_1 = k(x - x_1)$ jednačina prave kroz tačku (x_1, y_1) i koeficijentom k

u slučaju tangente $k = y'(x_1)$

$(2,0)$, $y'(2) = 5$

$y - 0 = 5(x-2)$

$y = 5x - 10$ jednačina tangente na krivu y u tački $(2,0)$

presjek pravih:

$y = -5x - 15 \quad (1)$

$y = 5x - 10 \quad (2)$

$(1)+(2)$: $2y = -25$

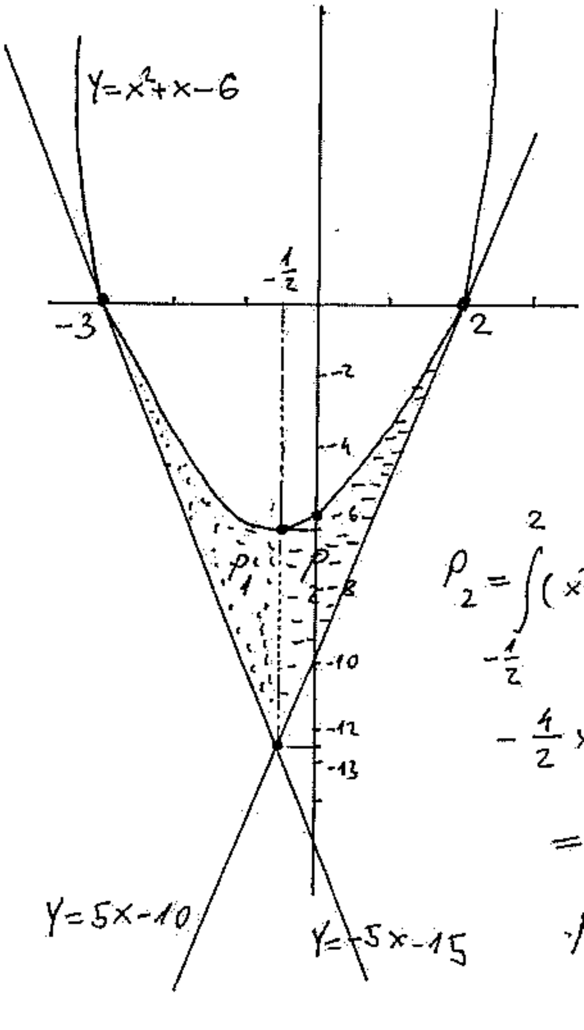
$y = -\frac{25}{2} = -12\frac{1}{2}$

$(1)-(2)$: $-10x - 5 = 0$

$-10x = 5$

$x = -\frac{1}{2}$

$\left(-\frac{1}{2}, -12\frac{1}{2}\right)$ je tačka presjeka pravih



$P = P_1 + P_2$

$$P_1 = \int_{-3}^{-\frac{1}{2}} \left(x^2 + x - 6 - (-5x - 15)\right)dx = \int_{-3}^{-\frac{1}{2}} (x^2 + 6x + 9)\,dx =$$

$$= \frac{1}{3}x^3\Big|_{-3}^{-\frac{1}{2}} + \frac{6}{2}x^2\Big|_{-3}^{-\frac{1}{2}} + 9x\Big|_{-3}^{-\frac{1}{2}} = \frac{1}{3}\left(-\frac{1}{8} + 27\right)$$

$$+ 3\left(\frac{1}{4} - 9\right) + 9\left(-\frac{1}{2} + 3\right) = \frac{1}{3}\cdot\frac{215}{8} + 3\cdot\frac{-35}{4}$$

$$+ 9\cdot\frac{5}{2} = \frac{215}{24} - \frac{630}{24} + \frac{540}{24} = \frac{125}{24}$$

$$P_2 = \int_{-\frac{1}{2}}^{2} \left(x^2 + x - 6 - (5x - 10)\right)dx = \int_{-\frac{1}{2}}^{2} (x^2 - 4x + 4)\,dx = \frac{1}{3}x^3\Big|_{-\frac{1}{2}}^{2} -$$

$$- \frac{4}{2}x^2\Big|_{-\frac{1}{2}}^{2} + 4x\Big|_{-\frac{1}{2}}^{2} = \frac{1}{3}\left(8 + \frac{1}{8}\right) - 2\left(4 - \frac{1}{4}\right) + 4\left(2 + \frac{1}{2}\right) =$$

$$= \frac{1}{3}\cdot\frac{65}{8} - 2\cdot\frac{15}{4} + 4\cdot\frac{5}{2} = \frac{65}{24} - \frac{180}{24} + \frac{240}{24} = \frac{125}{24}$$

$P = P_1 + P_2 = \frac{125}{24} + \frac{125}{24} = \frac{125}{12}$ tražena površina

Izračunati površinu figure koju ograničavaju linije

$x=y^2-2y-3$ i $y=3-3x$

$$y=3-3x$$
$$3x=3-y$$
$$x=1-\frac{1}{3}y$$

Rj. Nađimo presječnu tačku ovih linija

$$x=y^2-2y-3$$
$$y=3-3x$$

$$x=(3-3x)^2-2(3-3x)-3$$
$$x=9-18x+9x^2-6+6x-3$$
$$9x^2-13x=0$$
$$x(9x-13)=0$$
$$x=0 \text{ ili } 9x=13$$
$$x=\frac{13}{9}$$

$$x=0 \Rightarrow y=3$$
$$x=\frac{13}{9} \Rightarrow y=3-\cancel{3}\cdot\frac{13}{\cancel{9}\,3}=\frac{9}{3}-\frac{13}{3}=-\frac{4}{3}$$

$A(0,3)$ i $B(\frac{13}{9},-\frac{4}{3})$ su presječne tačke linija

$x=y^2-2y-3$ je kriva oblika parabole $\subset$ čije je tjeme $T(-\frac{D}{4a},-\frac{b}{2a})$

$$-\frac{b}{2a}=-\frac{-2}{2}=1,\quad D=4+12=16 \qquad -\frac{D}{4a}=-\frac{16}{4}=-4$$

$$T(1,-4)$$

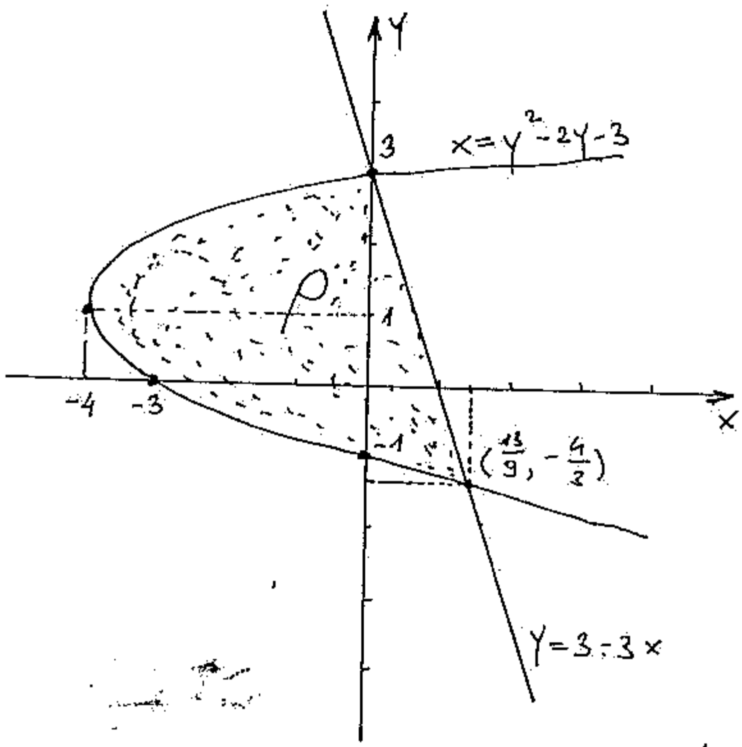
$$y_{1,2}=\frac{2\pm4}{2}$$
$$y_1=\frac{-2}{2}=-1 \quad y_2=\frac{6}{2}=3$$

$M_1(0,-1)$ i $M_2(0,3)$ su presjek parabole sa y-osom

$x=y^2-2y-3$

$y=0 \Rightarrow x=-3$

$(-3,0)$ je presjek krive sa x-osom



$$P=\int_{-\frac{4}{3}}^{3}\left[\left(1-\frac{1}{3}y\right)-\left(y^2-2y-3\right)\right]dy=$$
$$=\int_{-\frac{4}{3}}^{3}\left(-y^2+\frac{5}{3}y+4\right)dy=$$
$$=-\frac{1}{3}y^3\Big|_{-\frac{4}{3}}^{3}+\frac{5}{3}\cdot\frac{1}{2}y^2\Big|_{-\frac{4}{3}}^{3}+4y\Big|_{-\frac{4}{3}}^{3}=$$
$$=-\frac{1}{3}\left(27+\frac{64}{27}\right)+\frac{5}{6}\left(9-\frac{16}{9}\right)+4\left(3+\frac{4}{3}\right)$$
$$=-\frac{1}{3}\cdot\frac{793}{27}+\frac{5}{6}\cdot\frac{65}{9}+4\cdot\frac{13}{3}=$$
$$=-\frac{793}{81}+\frac{325}{54}+\frac{52}{3}=\frac{-793\cdot2+325\cdot3+52\cdot54}{162}=\frac{-1586+975+2808}{162}$$

$$P=\frac{2197}{162}=13\frac{91}{162}$$ tražena površina

(#) Izračunati površinu ravne figure koja je ograničena parabolama $y=-x^2-4x$ i $y=x^2+2x$.

Rj:

Za parabolu $y=-x^2-4x$ znamo da je $\cap$ oblika.
Vidimo da x-osu sječe u tačkama -4 i 0.

$y'=-2x-4$ Tjeme ove parabole je $T(-2,4)$

$-2x-4=0$

$x=-2$

Za parabolu $y=x^2+2x$ znamo da je $\cup$ oblika.
Vidimo da x-osu sječe u tačkama -2 i 0.

$y'=2x+2$ Tjeme ove parabole je $T(-1,1)$

$2x+2=0$

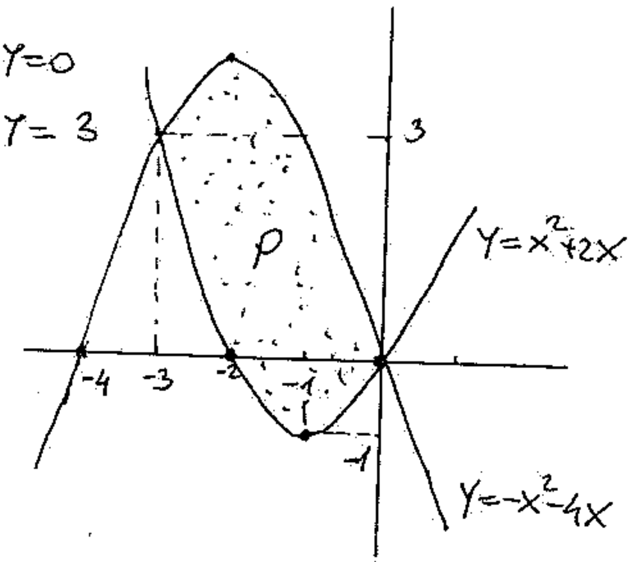
$x=-1$

Pronađimo još presječne tačke dvije date parabole.

$$
\begin{aligned}
y &= -x^2-4x \\
y &= x^2+2x \\
\hline
-2x^2-6x &= 0 \quad /:(-2) \\
x^2+3x &= 0 \\
x(x+3) &= 0
\end{aligned}
$$

$x=0 \Rightarrow y=0$

$x=-3 \Rightarrow y=3$



$$P=\int_{-3}^{0}\left[(-x^2-4x)-(x^2+2x)\right]dx=\int_{-3}^{0}(-2x^2-6x)\,dx=-\frac{2}{3}x^3\Big|_{-3}^{0}-6\cdot\frac{1}{2}x^2\Big|_{-3}^{0}=$$

$$=-\frac{2}{3}(0-(-27))-3(0-9)=-18+27=9$$ vrijednost tražene površine

(#) Izračunati površinu ravne figure koja je ograničena krivim linijama $x=y^2-1$ i $x=-y^2-2y+3$

Rj: Za krivu $x=y^2-1$ vidimo da je sljedećeg oblika $\subset$ y-osu sijeće u tačkama -1 i 1

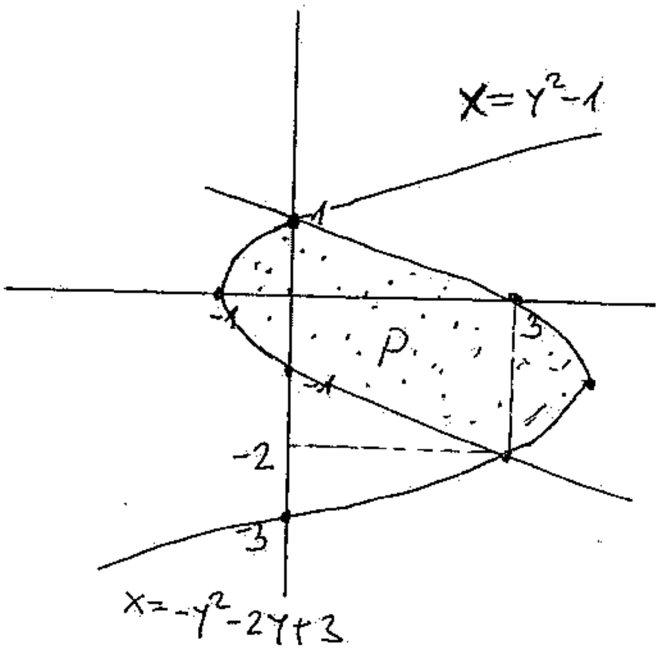
Kriva $x=-y^2-2y+3$ je oblika $\supset$. y-osu sijeće u tačkama -3 i 1.

Pronađimo presječne tačke dvije date krive.

$$
\begin{array}{l}
\; x = y^2-1 \\
-\; x = -y^2-2y+3 \\
\hline
2y^2+2y-4=0 \quad /:2 \\
y^2+y-2=0 \\
(y+2)(y-1)=0
\end{array}
$$

$y=-2 \Rightarrow x=3$

$y=1 \Rightarrow x=0$



$$P=\int_{-2}^{1}\left[(-y^2-2y+3)-(y^2-1)\right]dy=$$

$$=\int_{-2}^{1}(-2y^2-2y+4)\,dy=-\frac{2}{3}y^3\Big|_{-2}^{1}-y^2\Big|_{-2}^{1}+4y\Big|_{-2}^{1}=$$

$24\cdot 3=72$

$$=-\frac{2}{3}(1-(-8))-(1-4)+4(1-(-2))=$$

$$=-6+3+12=9 \text{ tražena površina}$$

(#) Izračunati površinu ravne figure koja je ograničena parabolama $x=y^2-4y+3$ i $x=-y^2+2y+3$.

Rj.

Parabola $x=y^2-4y+3$ je C oblika. $x=(y-3)(y-1)$

y-osu siječe u tačkama 1 i 3.

$$x'=2y-4 \qquad 2y-4=0 \qquad y=2$$

$x=4-8+3=-1$ Tjeme ove parabole je $T(-1,2)$

Parabola $x=-y^2+2y+2$ je Ɔ oblika. $x=-(y+1)(y-3)$

y-osu siječe u tačkama -1 i 3.

$$x'=-2y+2 \qquad -2y+2=0 \qquad y=1$$

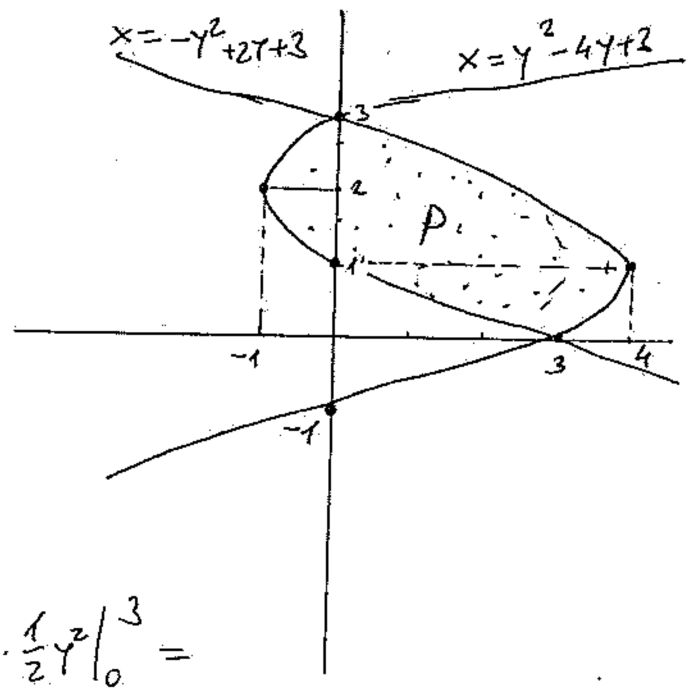
$x=-1+2+3=4$ Tjeme ove parabole je $T(4,1)$

Pronađimo još presječne tačke dvije date parabole

$$\begin{array}{r} x=y^2-4y+3 \\ -\quad x=-y^2+2y+3 \\ \hline 2y^2-6y=0 \\ 2y(y-3)=0 \end{array}$$

$y=0 \Rightarrow x=3$

$y=3 \Rightarrow x=0$



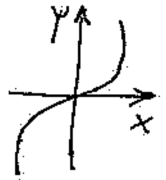
$$P=\int_0^3\left[(-y^2+2y+3)-(y^2-4y+3)\right]dy=$$

$$=\int_0^3(-2y^2+6y)\,dy=-\frac{2}{3}y^3\Big|_0^3+6\cdot\frac{1}{2}y^2\Big|_0^3=$$

$$=-2\cdot 9+3\cdot 9=9$$ tražena površina

7) Izračunati površinu figure koja je određena linijama $y=-x$, $y=\sqrt[3]{x}$, $y=3x-2$.

R: Grafički nije teško predstaviti prave $y=-x$ i $y=3x-2$. Problem predstavlja kriva $y=\sqrt[3]{x}$.

Ako znamo da kriva $y=x^3$ izgleda ovako 

Onda nije teško nacrtati krivu $x=y^3$ što je ekvivalentno sa $y=\sqrt[3]{x}$.

Pronađimo tačke presjeka datih krivih.

$$\begin{aligned} y&=-x\\ y&=3x-2\\ \hline -x&=3x-2\\ -4x&=-2\\ x&=\tfrac{1}{2} \Rightarrow y=-\tfrac{1}{2}\end{aligned}$$

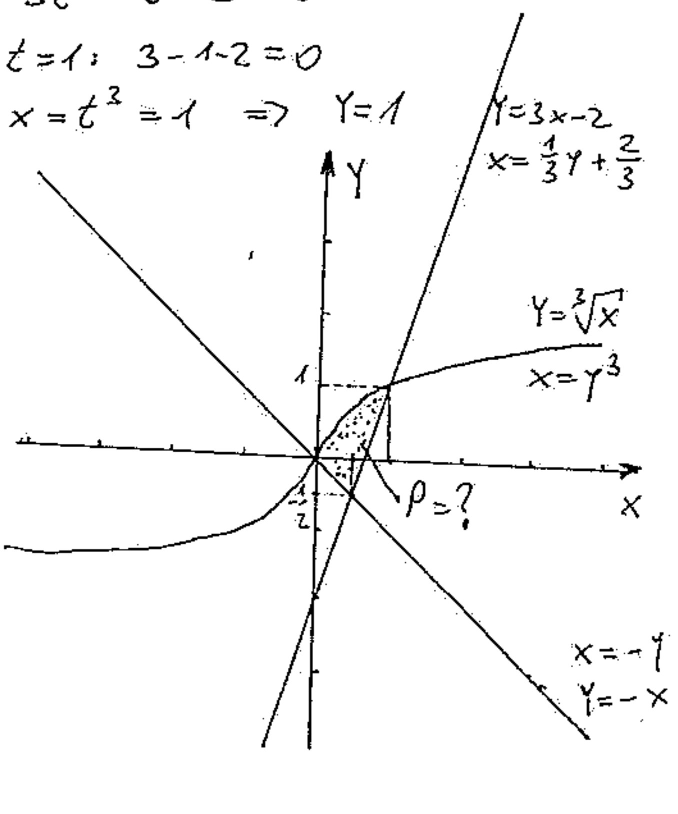
$$\begin{aligned} y&=-x\\ y&=\sqrt[3]{x}\\ \hline y&=-x\\ y^3&=x\\ \hline -x^3&=x\\ x^3+x&=0\\ x(x^2+1)&=0\\ x=0 &\Rightarrow y=0\end{aligned}$$

$$\begin{aligned} y&=3x-2\\ y&=\sqrt[3]{x}\\ \hline \sqrt[3]{x}&=3x-2\\ (3x-2)^3&=x\\ 27x^3-3\cdot(3x)^2\cdot 2 &+\\ +3\cdot 3x\cdot(-2)^2+(-2)^3&=x\\ 27x^3-54x^2+36x-8&=x\\ 27x^3-54x^2+35x-8&=0\end{aligned}$$

pokušajmo riješiti problem na drugi način

$$\sqrt[3]{x}=3x-2$$
$$x=t^3$$
$$3t^3-2=t$$
$$3t^3-t-2=0$$
$$t=1:\ 3-1-2=0$$
$$x=t^3=1 \Rightarrow y=1$$

$\boxed{3x=y+2}$



$$P=\int_{-\frac{1}{2}}^{0}\left[\left(\frac{1}{3}y+\frac{2}{3}\right)-(-y)\right]dy+\int_{0}^{1}\left[\left(\frac{1}{3}y+\frac{2}{3}\right)-y^3\right]dy=$$

$$=\int_{-\frac{1}{2}}^{0}\left(\frac{4}{3}y+\frac{2}{3}\right)dy+\int_{0}^{1}\left(-y^3+\frac{1}{3}y+\frac{2}{3}\right)dy=$$

$$=\frac{4}{3}\cdot\frac{1}{2}y^2\Big|_{-\frac{1}{2}}^{0}+\frac{2}{3}y\Big|_{-\frac{1}{2}}^{0}-\frac{1}{4}y^4\Big|_{0}^{1}+\frac{1}{3}\cdot\frac{1}{2}y^2\Big|_{0}^{1}$$

$$+\frac{2}{3}y\Big|_{0}^{1}=\frac{2}{3}\cdot\left(-\frac{1}{4}\right)+\frac{2}{3}\cdot\frac{1}{2}-\frac{1}{4}+\frac{1}{6}+$$

$$+\frac{2}{3}=-\cancel{\frac{1}{6}}+\frac{1}{3}-\frac{1}{4}+\cancel{\frac{1}{6}}+\frac{2}{3}=\frac{3}{4}$$

(#) Izračunati površinu figure koja je određena linijama $y=-2$, $y=x^3+x$, $x+y=3$.

Rj. $y=-2$, $x+y=3$ su prave linije i njih nije teško nacrtati. Problem za crtanje predstavlja kriva $y=x^3+x$.

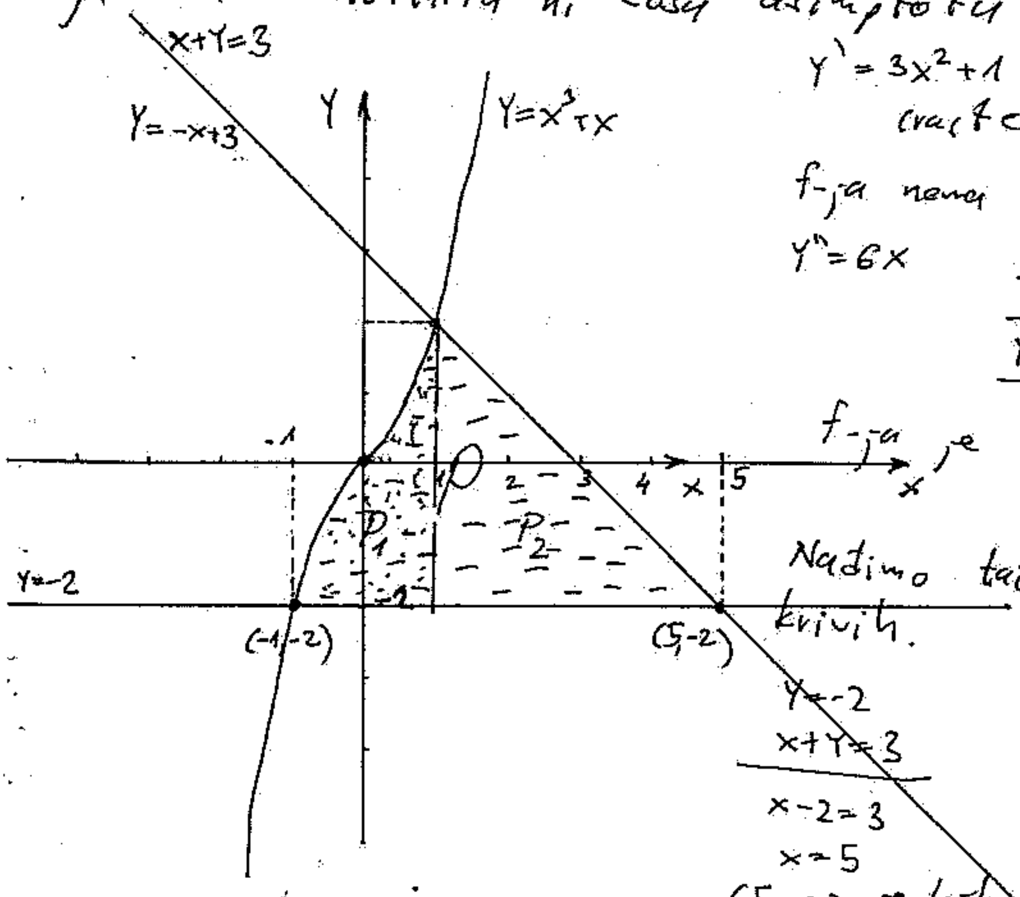
Ispitajmo f-ju $y=x^3+x$. $\mathcal{D}: x\in\mathbb{R}$

$f(-x)=-x^3-x=-(x^3+x)$ f-ja je neparna

$A(0,0)$ je nula f-je i presjek sa y-osom

f-ja nema prekida ⇒ f-ja nema vertikalnu asimptotu

f-ja nema horizontalnu ni kosu asimptotu



$y'=3x^2+1$ f-ja je uvijek pozitivna (raste za svako x)

f-ja nema ekstrem

$y''=6x$

x	$(0,+\infty)$
y''	+
y	∪

$(0,0)$ je prevojna tačka

f-ja je ovog oblika 

Nađimo tačke presjeka datih krivih.

$y=-2$
$x+y=3$
$x-2=3$
$x=5$
$(5,-2)$ je tačka presjeka

$y=-2$
$y=x^3+x$
$-2=x^3+x$
$x^3+x+2=0$
$x=-1$: $-1-1+2=0$

$(x^3+x+2):(x+1)=x^2-x+2$
$-\ x^3+x^2$
$-x^2+x+2$
$-\ -x^2-x$
$2x+2$
$2x+2$
// //

$x^3+x+2=(x+1)\underbrace{(x^2-x+2)}_{>0\ \forall x}$

Rješenje jednačine $x^3+x+2=0$ je $x=-1$.

$(-1,-2)$ je tačka presjeka datih krivih

$$\begin{cases} y = x^3 + x \\ x + y = 3 \end{cases}$$

$$\begin{cases} y = x^3 + x \\ y = -x + 3 \end{cases}$$

$-x+3 = x^3 + x$

$x^3 + 2x - 3 = 0$

$x = 1: \quad 1^3 + 2\cdot 1 - 3 = 3 - 3 = 0$

$(x^3 + 2x - 3) : (x-1) = x^2 + x + 3$

$\underline{x^3 - x^2}$

$x^2 + 2x - 3$

$\underline{-\ x^2 - x}$

$3x - 3$

$\underline{-\ 3x - 3}$

$= \quad =$

$x^3 + 2x - 3 = \underbrace{(x^2 + x + 3)}_{>0\ \forall x}(x-1)$

$(1, 2)$ je presječna tačka krivih

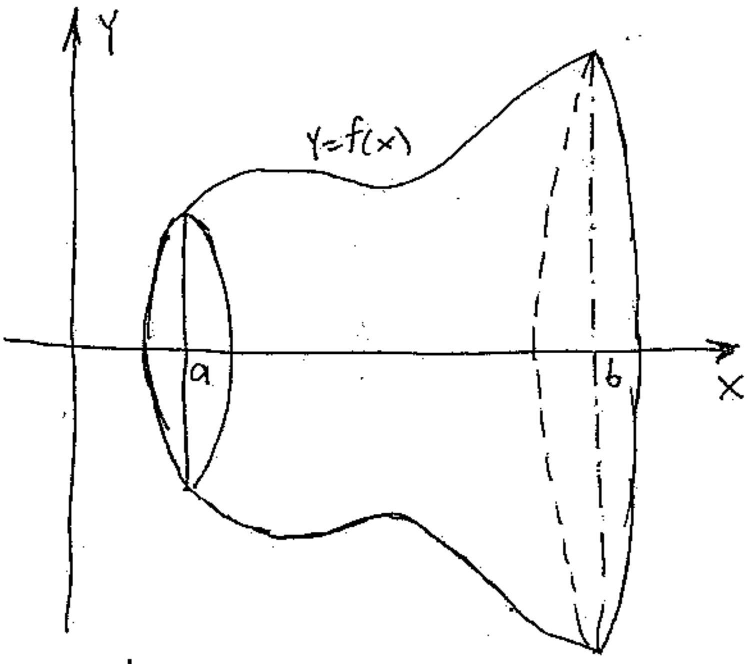
$$P_1 = \int_{-1}^{1} \left[(x^3 + x) - (-2)\right] dx = \int_{-1}^{1} (x^3 + x + 2)\,dx = \frac{1}{4}x^4\Big|_{-1}^{1} + \frac{1}{2}x^2\Big|_{-1}^{1} + 2x\Big|_{-1}^{1} = 4$$

$$P_2 = \int_{1}^{5} \left[(-x+3) - (-2)\right] dx = \int_{1}^{5} (-x + 5)\,dx = -\frac{x^2}{2}\Big|_{1}^{5} + 5x\Big|_{1}^{5} =$$

$$= -\frac{1}{2}(25 - 1) + 5\cdot 4 = -\frac{1}{2}\cdot 24 + 20 = 20 - 12 = 8$$

$P = P_1 + P_2 = 8 + 4 = 12$. površina figure

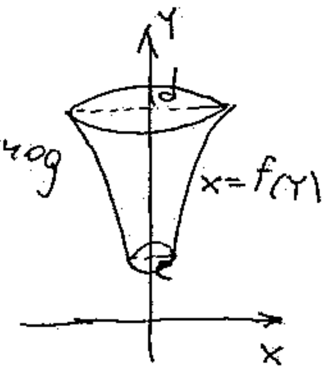
II Zapremina rotacionog tijela



zapremina tijela dobijenog rotacijom dijela krive $y=f(x)$ oko x-ose

$$V_x = \pi \cdot \int_a^b [f(x)]^2 \, dx$$

$$V_y = \pi \int_c^d [f(y)]^2 \, dy$$ - zapremina tijela dobijenog rotacijom dijela krive $x=f(y)$ oko y-ose



Ako je kriva data u parametarskom obliku:

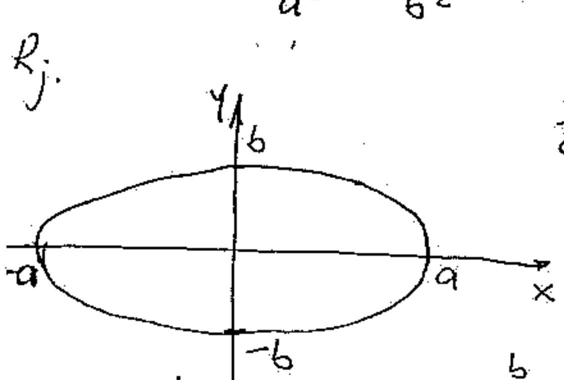
$x = \alpha(t)$
$y = \beta(t)$
$t_1 \le t \le t_2$

$$V_x = \pi \int_{t_1}^{t_2} [\beta(t)]^2 \, |\alpha'(t)| \, dt$$

$$V_y = \pi \int_{t_1}^{t_2} [\alpha(t)]^2 \, |\beta'(t)| \, dt$$

1. Izračunati zapreminu tijela koje nastaje rotacijom krive $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ oko y-ose.

Rj.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 = a^2 \cdot \left(1 - \frac{y^2}{b^2}\right)$$

$$x^2 = a^2 \cdot \frac{b^2 - y^2}{b^2}$$

$$x^2 = \frac{a^2}{b^2} \cdot (b^2 - y^2)$$

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2}$$

$$V_y = \pi \int_{-b}^{b} [f(y)]^2 dy = \pi \int_{-b}^{b} \frac{a^2}{b^2}(b^2 - y^2)\, dy = 2\pi \frac{a^2}{b^2} \int_0^b (b^2 - y^2)\, dy =$$

Parna f-ja (simetrična u odnosu na y-osu)

$$= 2\pi \frac{a^2}{b^2}\left(b^2 y\Big|_0^b - \frac{y^3}{3}\Big|_0^b\right) = 2\pi \frac{a^2}{b^2}\left(b^3 - \frac{1}{3}b^3\right) = 2\pi \frac{a^2}{b^2}\cdot\frac{2}{3}b^3 = \frac{4\pi a^2 b}{3}.$$

2. Figura u ravni ograničena parabolom $y = 4 - x^2$ i pravama $y \geq 3x$, $y \geq 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

Rj.

$y = 4 - x^2$
$y = 3x$

$3x = 4 - x^2$

$x^2 + 3x - 4 = 0$

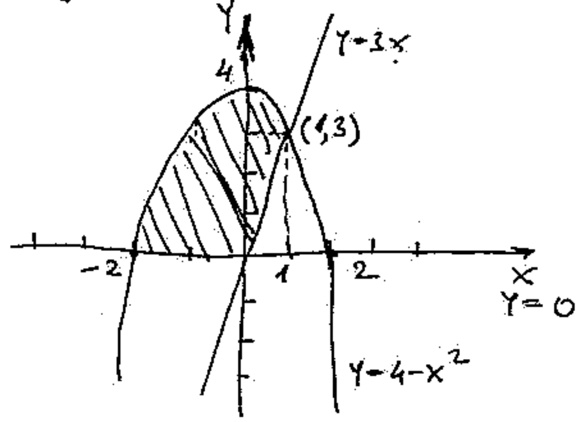
$D = 9 + 16 = 25$

$x_{1,2} = \frac{-3 \pm 5}{2}$

$x_1 = -4 \Rightarrow y_1 = -12$

$x_2 = 1 \Rightarrow y_2 = 3$

$A(-4, -12)$ i $B(1,3)$ su tačke presjeka prave i parabole



$$V_x = V_1 - V_2, \quad V_1 = \pi \int_{-2}^{1} (4-x^2)^2 dx, \quad V_2 = \pi \int_0^1 (3x)^2 dx$$

$$V_1 = \pi \int_{-2}^{1} (16 - 8x^2 + x^4)\, dx = \pi \left(16x\Big|_{-2}^{1} - 8\frac{x^3}{3}\Big|_{-2}^{1} + \frac{x^5}{5}\Big|_{-2}^{1} \right) = \pi \left(16\cdot 3 - \frac{8}{3}\cdot 9 + \frac{1}{5}\cdot 33\right)$$

$1-(-2)$ $\quad 1^3-(-2)^3$ $\quad 1^5-(-2)^5$

$$= \pi \left(48 - 24 + \frac{33}{5}\right) = \frac{158}{5}\pi$$

$$V_2 = \pi \cdot 9 \int_0^1 x^2 dx = 9\pi \frac{x^3}{3}\Big|_0^1 = 3\pi(1-0) = 3\pi$$

$$V = V_1 - V_2 = \frac{158}{5}\pi - 3\pi = \frac{158\pi - 15\pi}{5} = \frac{138}{5}\pi$$

3. Izračunati zapreminu tijela nastalog obrtanjem oko x-ose figure omeđenu krivom $y = \arcsin x$ i pravama $x = 1$ i $y = 0$. Uputa: parcijalna integracija 2x

4. Izračunati zapreminu tijela koje nastaje rotacijom ravne figure ograničene parabolom $y = 6 - x - x^2$ i prave $y = 0$ oko x-ose.

(#) Izračunati zapreminu tijela koje nastaje rotacijom figure određene parabolom $y^2 = 9-3x$, tangentom na tu parabolu u tački $A(0,3)$ i x-osom oko x-ose.

Rj.

$y^2 = 9-3x$

$3x = -y^2+9$

$x = -\frac{1}{3}y^2+3$

f-ja je ovog oblika

$x' = -\frac{2}{3}y$

$x' = 0$ akko $y = 0$

$T(3,0)$ je tjeme f-je

$x = 0 \Rightarrow y = \pm 3$

Koeficijent pravca tangente

$k = x'(A) = -\frac{2}{3}\cdot 3 = -2$

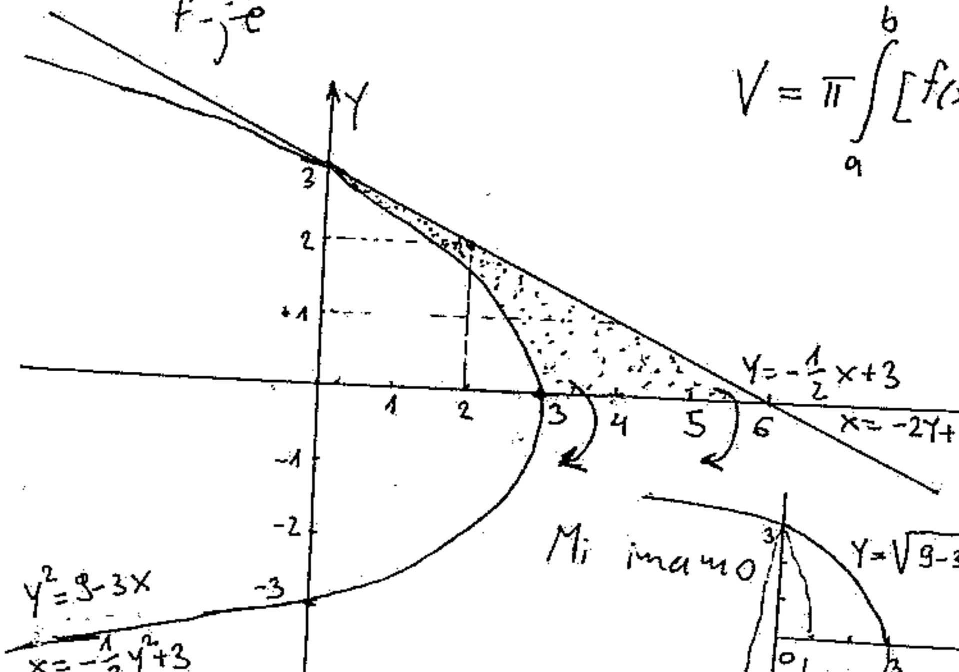
$x - x_1 = k(y-y_1)$

$x - 0 = (-2)(y-3)$

$x = -2y+6$

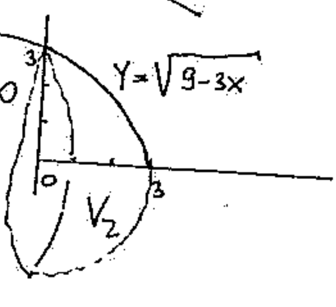
$-2y = x-6$

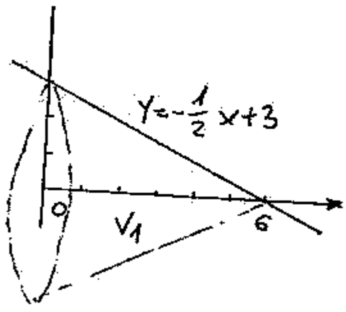
$y = -\frac{1}{2}x+3$



$V = \pi\int_a^b [f(x)]^2 dx$ je zapremina tijela dobijena rotacijom dijela krive $y=f(x)$ oko x-ose

Mi imamo





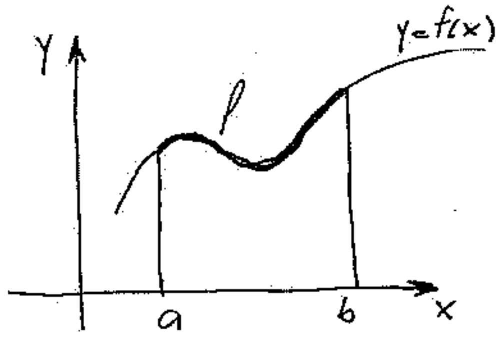
Zapremina našeg tijela se računa po formuli

$$V = V_1 - V_2 = \pi\int_0^6 \left(-\frac{1}{2}x+3\right)^2 dx - \pi\int_0^3 \left(\sqrt{9-3x}\right)^2 dx \overset{(1)\,i\,(2)}{=} \frac{9}{2}\pi$$

$$V_1 = \pi\int_0^6 \left(\frac{1}{4}x^2-3x+9\right)dx = \pi\left(\frac{1}{4}\cdot\frac{1}{3}x^3\Big|_0^6 - 3\cdot\frac{1}{2}x^2\Big|_0^6 + 9x\Big|_0^6\right) = \pi(18-54+54) = 18\pi \quad \dots(1)$$

$$V_2 = \pi\int_0^3 (9-3x)dx = \pi\left(9x\Big|_0^3 - 3\cdot\frac{1}{2}x^2\Big|_0^3\right) = \pi\left(27-\frac{27}{2}\right) = \frac{27}{2}\pi \quad \dots(2)$$

III Dužina luka krive



$$l=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

dužina luka krive $y=f(x)$

$$l=\int_c^d \sqrt{1+[f'(y)]^2}\,dy$$

dužina luka krive $x=f(y)$

Ako je kriva data u parametarskom obliku:

$$\begin{aligned} x&=x(t)\\ y&=y(t)\\ t_1&\le t\le t_2 \end{aligned} \quad\Rightarrow\quad l=\int_{t_1}^{t_2}\sqrt{(\dot{x})^2+(\dot{y})^2}\,dt$$

gdje je $\dot{x}=\frac{dx}{dt}$

i $\dot{y}=\frac{dy}{dt}$ (izvod po t)

1° Izračunati dužinu luka krive $y=\frac{x^2}{2}-\frac{\ln x}{4}$ ako je $1\le x\le 3$.

Rj. $l=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$, $\quad y=f(x)=\frac{x^2}{2}-\frac{\ln x}{4}=\frac{1}{2}x^2-\frac{1}{4}\ln x$

$$y'=\frac{1}{2}\cdot 2x-\frac{1}{4}\cdot\frac{1}{x}=x-\frac{1}{4x}$$

$$l=\int_1^3\sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx=\int_1^3\sqrt{1+x^2-\cancel{2x}\cdot\frac{1}{\underset{2}{\cancel{4x}}}+\frac{1}{16x^2}}\,dx=\int_1^3\sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}$$

$$=\int_1^3\sqrt{\left(x+\frac{1}{4x}\right)^2}\,dx=\int_1^3\left(x+\frac{1}{4x}\right)dx=\frac{x^2}{2}\Big|_1^3+\frac{1}{4}\ln x\Big|_1^3=\frac{1}{2}(9-1)+\frac{1}{4}(\ln 3-\ln 1)$$

$$=4+\frac{1}{4}\ln 3=4+\ln\sqrt[4]{3}. \qquad \left[\frac{1}{4}\ln 3=\ln 3^{\frac{1}{4}}\right]$$

2. Naći dužinu luka kojeg na paraboli $y^2=2x+1$ odsjeca prava $x-y=1$.

Rj.

$$\begin{array}{l} y^2=2x+1 \\ \underline{x-y=1} \\ y^2=2x+1 \\ \underline{y=x-1} \\ (x-1)^2=2x+1 \end{array}$$

$x^2-2x+1=2x+1$

$x^2-4x=0$

$x(x-4)=0$

$x_1=0 \Rightarrow y_1=-1$

$x_2=4 \Rightarrow y_2=3$

$A(0,-1)$; $B(4,3)$ su tačke presjeka parabole i prave

$y^2=2x+1$ $\quad x=0 \Rightarrow y=\pm 1$

$2x=y^2-1$

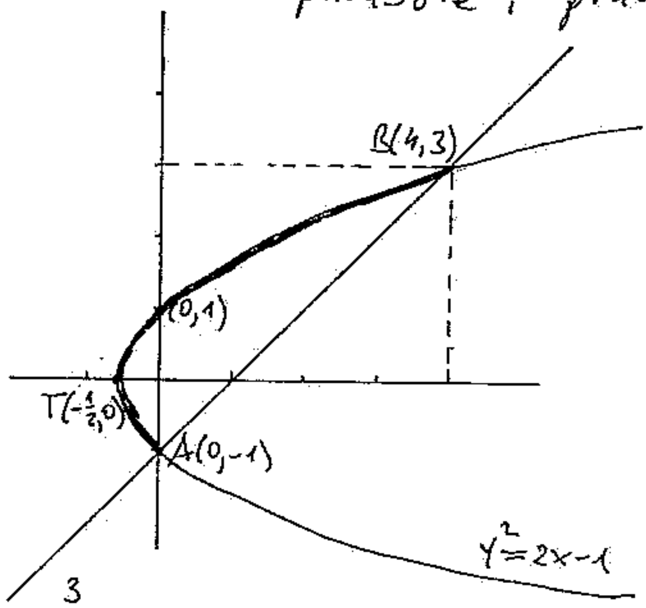
$x=\frac{1}{2}y^2-\frac{1}{2}$

$a>0$

$T\left(-\frac{1}{2},0\right)$

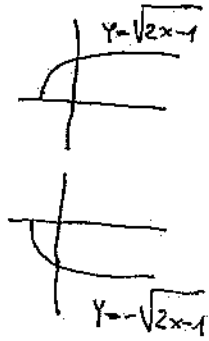
$D=1$

$-\frac{D}{4a}=-\frac{1}{4\cdot\frac{1}{2}}=-\frac{1}{2}$, $-\frac{b}{2a}=0$



$y^2=2x+1$

$y=\pm\sqrt{2x-1}$



$y^2=2x+1$

$x=\frac{1}{2}y^2-\frac{1}{2}$

$x'=\frac{1}{2}\cdot 2y=y$ tj. $x'_y=y$

$$l=\int_{-1}^{3}\sqrt{1+y^2}\,dy$$

integral $\int\sqrt{1+y^2}\,dy$ smo uradili Metodom Ostrogradskog, 3 zadatak na 63 strani u skripti (umjesto y imali smo x)

$$l=\int_{-1}^{3}\sqrt{1+y^2}\,dy=\frac{1}{2}y\sqrt{y^2+1}\Big|_{-1}^{3}+\frac{1}{2}\ln\left|y+\sqrt{y^2+1}\right|\Big|_{-1}^{3}=$$

$$=\frac{1}{2}\left(3\sqrt{10}-(-1)\sqrt{2}\right)+\frac{1}{2}\left(\ln|3+\sqrt{10}|-\ln|-1+\sqrt{2}|\right)=$$

$$=\frac{3\sqrt{10}+\sqrt{2}}{2}+\frac{1}{2}\ln\left|\frac{3+\sqrt{10}}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}\right|=\frac{3\sqrt{10}+\sqrt{2}}{2}+\frac{1}{2}\ln\left|(3+\sqrt{10})(\sqrt{2}+1)\right|$$

$$=\frac{3\sqrt{10}-\sqrt{2}}{2}+\ln\sqrt{(3+\sqrt{10})(\sqrt{2}+1)}$$