Прва недеља 15.02.2021. : Неодређени интеграл. Метод смјене. Метод парцијалне интеграције

Интегралом функције $f(x)$ називамо функцију $F(x)$ ако је $F'(x)=f(x)$ и пишемо $\int f(x)dx = F(x)+C$. $F(x)$ називамо примитивна функција.

Особине

$$\left(\int f(x)dx\right)' = f(x)$$

$$\int f'(x)dx = f(x)+C$$

$$\int af(x)dx = a\int f(x)dx \quad a \neq 0$$

$$\int \left(f_1(x)+f_2(x)\right)dx = \int f_1(x)dx + \int f_2(x)dx$$

Таблица интеграла

1. $\int x^n dx = \frac{x^{n+1}}{n+1}+C, \quad n \neq -1$
2. $\int \frac{dx}{x} = \ln|x|+C$
3. $\int a^x dx = \frac{a^x}{\ln a}+C, \quad \int e^x dx = e^x + C$
4. $\int \sin x dx = -\cos x + C$
5. $\int \cos x dx = \sin x + C$
6. $\int \frac{dx}{\cos^2 x} = tgx + C$
7. $\int \frac{dx}{\sin^2 x} = -ctgx + C$
8. $\int \frac{dx}{\sin x} = \ln\left|tg\frac{x}{2}\right| + C$
9. $\int \frac{dx}{\cos x} = \ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$
10. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}arctg\frac{x}{a}+C, \quad a \neq 0$
11. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C$
12. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+C, \quad |x|<a$
13. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|+C, \quad |x|>a$
14. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right|+C$

Непосредна интеграција

1. $\int \frac{dx}{x^2-x^4} = \int \frac{dx}{x^2(1-x^2)} = \int \frac{1-x^2+x^2}{x^2(1-x^2)}dx = \int \frac{dx}{x^2} + \int \frac{dx}{1-x^2} = -\frac{1}{x} + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$

2. $\int \frac{\left(\sqrt{a}+\sqrt{x}\right)^2}{\sqrt{ax}}dx = \int \frac{a+2\sqrt{ax}+x}{\sqrt{ax}}dx = \int \left(\frac{\sqrt{a}}{\sqrt{x}} + 2 + \frac{\sqrt{x}}{\sqrt{a}}\right)dx = \sqrt{a}2\sqrt{x} + 2x + \frac{1}{\sqrt{a}}\frac{2}{3}x^{\frac{3}{2}} + C$

3. $\int 2^x e^x dx = \int (2e)^x dx = \frac{(2e)^x}{\ln 2e} + C = \frac{(2e)^x}{\ln 2 + 1} + C$

4. $\int \frac{(1+x)^2 dx}{x(1+x^2)} = \int \frac{1+2x+x^2}{x(1+x^2)}dx = \int \frac{dx}{x} + \int \frac{2}{1+x^2}dx = \ln|x| + 2arctgx + C$

5. $\int \frac{\sqrt{x^2-3}-\sqrt{x^2+3}}{\sqrt{x^4-9}}dx = \int \frac{\sqrt{x^2-3}-\sqrt{x^2+3}}{\sqrt{x^2-3}\cdot\sqrt{x^2+3}}dx = \int \frac{dx}{\sqrt{x^2+3}} - \int \frac{dx}{\sqrt{x^2-3}} =$

$= \ln\left|x+\sqrt{x^2+3}\right| - \ln\left|x+\sqrt{x^2+3}\right| + C$

Метод смјене

Промјенљиву x можемо замијенити новом промјенљивом t користећи смјену $x=\varphi(t)$. Тада је $dx=\varphi'(t)dt$ и $\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt = \int F(t)dt$.

1. $\int \sin^3 x\cos x dx = \left\langle \begin{matrix} \sin x = t \\ \cos x dx = dt \end{matrix} \right\rangle = \int t^3 dt = \frac{t^4}{4} + C = \frac{\sin^4 x}{4} + C$

2. $\int \frac{2x+1}{x^2+x-3}dx = \int \frac{d(x^2+x-3)}{x^2+x-3} = \left\langle x^2+x-3=t \right\rangle = \int \frac{dt}{t} = \ln|t| + C = \ln\left|x^2+x-3\right| + C$

3. $\int \sin(\ln x)\frac{dx}{x} = \left\langle \begin{matrix} \ln x = t \\ \frac{dx}{x} = dt \end{matrix} \right\rangle = \int \sin t dt = -\cos t + C = -\cos(\ln|x|) + C$

4. $\int \frac{dx}{\cos\left(x-\frac{\pi}{4}\right)} = \left\langle \begin{matrix} x-\frac{\pi}{4} = t \\ dx = dt \end{matrix} \right\rangle = \int \frac{dt}{\cos(t)} = \ln\left|tg\left(\frac{t}{2}+\frac{\pi}{4}\right)\right| + C = \ln\left|tg\left(\frac{x}{2}-\frac{\pi}{8}+\frac{\pi}{4}\right)\right| + C$

$= \ln\left|tg\left(\frac{x}{2}+\frac{\pi}{8}\right)\right| + C$

5. $\int \frac{\sin x dx}{\sqrt{\cos^2 x + 4}} = \left\langle \begin{matrix} \cos x = t \\ -\sin x dx = dt \end{matrix} \right\rangle = -\int \frac{dt}{\sqrt{t^2+4}} = -\ln\left|t+\sqrt{t^2+4}\right| + C = \ln\left|\cos x + \sqrt{\cos^2 x + 4}\right| + C$

6. $\int \frac{e^x}{\left(7-e^x\right)^2}dx = \left\langle \begin{matrix} 7-e^x = t \\ -e^x dx = dt \end{matrix} \right\rangle = -\int \frac{dt}{t^2} = \frac{1}{t} + C = \frac{1}{7-e^x} + C$

7. $\int \frac{1+x}{1+\sqrt{x}}dx = \left\langle \begin{matrix} x = t^2 \\ dx = 2tdt \end{matrix} \right\rangle = \int \frac{1+t^2}{1+t}2tdt = 2\int \frac{t+t^3}{1+t}dt$

Како је послије дијељења $\frac{t^3+t}{t+1}=t^2-t+2+\frac{-2}{t+1}$ то је

$$2\int\frac{t+t^3}{1+t}dt=2\int(t^2-t+2)dt-4\int\frac{dt}{t+1}=2(\frac{t^3}{3}-\frac{t^2}{2}+2t)-4\ln|t+1|+C=$$

$$=2(\frac{x^{\frac{3}{2}}}{3}-\frac{x}{2}+2x^{\frac{1}{2}})-4\ln\left(\sqrt{x}+1\right)+C$$

8. $$\int\frac{e^{2x}}{e^x+1}dx=\left\langle\begin{matrix}e^x=t\\e^xdx=dt\end{matrix}\right\rangle=\int\frac{tdt}{t+1}=\int\frac{t+1-1dt}{t+1}=\int dt-\int\frac{dt}{t+1}=t-\ln|t+1|+C=e^x-\ln\left(e^x+1\right)+C$$

9. $$\int\frac{dx}{x\sqrt{4-x^2}}=\left\langle\begin{matrix}x=\frac{2}{t}\\dx=-\frac{2}{t^2}dt\end{matrix}\right\rangle=-\int\frac{1}{\frac{2}{t}\sqrt{4-\frac{4}{t^2}}}\frac{2}{t^2}dt=-\int\frac{1}{2\sqrt{t^2-1}}dt=-\frac{1}{2}\ln\left|t+\sqrt{t^2-1}\right|+C$$

$$=-\frac{1}{2}\ln\left|\frac{2}{x}+\sqrt{\left(\frac{2}{x}\right)^2-1}\right|+C=-\frac{1}{2}\ln\left|\frac{2}{x}+\frac{\sqrt{4-x^2}}{x}\right|+C$$

Метод парцијалне интеграције

$$\int udv=uv-\int vdu$$

1. $$\int\ln xdx=\left\langle u=\ln x;\quad dv=dx;\quad du=\frac{dx}{x};\quad v=x;\right\rangle=x\ln x-\int dx=x\ln x-x+C$$

2. $$\int x^2\cos xdx=\left\langle u=x^2;\quad dv=\cos dx;\quad du=2xdx;\quad v=\sin x\right\rangle=x^2\sin x-\int 2x\sin xdx=$$

$$\left\langle u=x;\quad dv=\sin dx;\quad du=dx;\quad v=-\cos x\right\rangle=x^2\sin x-2\left(-x\cos x-\int-\cos dx\right)=$$

$$=x^2\sin x+2x\cos x-2\sin x+C$$

3. $$\int e^{ax}\sin bxdx=\left\langle u=e^{ax};\quad dv=\sin bxdx;\quad du=ae^{ax}dx;\quad v=\frac{-\cos bx}{b}\right\rangle=$$

$$=\frac{-e^{ax}\cos bx}{b}-\int\frac{-ae^{ax}\cos bx}{b}dx=\frac{-e^{ax}\cos bx}{b}+\frac{a}{b}\int e^{ax}\cos bxdx=$$

$$\left\langle u=e^{ax};\quad dv=\cos bxdx;\quad du=ae^{ax}dx;\quad v=\frac{\sin bx}{b}\right\rangle=\frac{-e^{ax}\cos bx}{b}+\frac{a}{b}\left(\frac{e^{ax}\sin bx}{b}-\int\frac{ae^{ax}\sin bxdx}{b}\right)=$$

$$=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx-\frac{a^2}{b^2}\int e^{ax}\sin bxdx$$

Како је $\int e^{ax}\sin bxdx=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx-\frac{a^2}{b^2}\int e^{ax}\sin bxdx$ то је

$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bxdx=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx$ тј.

$$\int e^{ax}\sin bxdx=e^{ax}\frac{-b\cos bx+a\sin bx}{a^2+b^2}+C$$

4. $I_n = \int \frac{dx}{\left(x^2+a^2\right)^n} = \left\langle u = \frac{1}{\left(x^2+a^2\right)^n}; \quad dv = dx; \quad du = \frac{-n2x}{\left(x^2+a^2\right)^{n+1}}dx; \quad v = x \right\rangle =$

$$I_n = \frac{x}{\left(x^2+a^2\right)^n} + 2n\int \frac{x^2}{\left(x^2+a^2\right)^{n+1}}dx = \frac{x}{\left(x^2+a^2\right)^n} + 2n\int \frac{x^2+a^2-a^2}{\left(x^2+a^2\right)^{n+1}}dx$$

$$I_n = \frac{x}{\left(x^2+a^2\right)^n} + 2n\left(\int \frac{dx}{\left(x^2+a^2\right)^n} - a^2\int \frac{dx}{\left(x^2+a^2\right)^{n+1}}\right)$$

$$I_n = \frac{x}{\left(x^2+a^2\right)^n} + 2n\left(I_n - a^2 I_{n+1}\right)$$

$$2na^2 I_{n+1} = \frac{x}{\left(x^2+a^2\right)^n} + (2n-1)I_n$$

$$I_{n+1} = \frac{1}{2na^2}\frac{x}{\left(x^2+a^2\right)^n} + \frac{2n-1}{2na^2}I_n \quad n = 1,2,3,4,.... \text{ или}$$

$$I_n = \frac{1}{2(n-1)a^2}\frac{x}{\left(x^2+a^2\right)^{n-1}} + \frac{2n-3}{2(n-1)a^2}I_{n-1} \quad n = 2,3,4,....$$

$$I_1 = \frac{1}{a}arctg\,\frac{x}{a} + C\,,$$

$$I_2 = \frac{1}{2a^2}\frac{x}{x^2+a^2} + \frac{1}{2a^3}arctg\,\frac{x}{a} + C\,,$$

$$I_3 = \frac{1}{4a^2}\frac{x}{\left(x^2+a^2\right)^2} + \frac{3}{4a^2}\left(\frac{1}{2a^2}\frac{x}{x^2+a^2} + \frac{1}{2a^3}arctg\,\frac{x}{a}\right) + C$$

$$I_3 = \frac{1}{4a^2}\frac{x}{\left(x^2+a^2\right)^2} + \frac{3}{8a^4}\frac{x}{x^2+a^2} + \frac{3}{8a^5}arctg\,\frac{x}{a} + C$$

5. $\int (2x^5+5x^4-8x^3-10x^2+2x-2)e^{2x}dx$

$$\int (2x^5+5x^4-8x^3-10x^2+2x-2)e^{2x}dx = \left(Ax^5+Bx^4+Cx^3+Dx^2+Ex+F\right)e^{2x} + C$$

$$(2x^5+5x^4-8x^3-10x^2+2x-2)e^{2x} = \left(Ax^5+Bx^4+Cx^3+Dx^2+Ex+F\right)' e^{2x} + \left(Ax^5+Bx^4+Cx^3+Dx^2+Ex+F\right)2e^{2x}$$

$$(2x^5+5x^4-8x^3-10x^2+2x-2)e^{2x} = [\ 2Ax^5+\left(5A+2B\right)x^4+\left(4B+2C\right)x^3+\left(3C+2D\right)x^2+\left(2D+2E\right)x+\left(E+2F\right)\ ]2e^{2x}$$

Изједначавањем коефицијената уз одговарајуће степене добијамо

$2 = 2A$		$A = 1$
$5 = 5A+2B$		$B = 0$
$-8 = 4B+2C$	тј.	$C = -4$ па је
$-10 = 3C+2D$		$D = 1$
$2 = 2D+2E$		$E = 0$
$-2 = E+2F$		$F = -1$

$$\int (2x^5+5x^4-8x^3-10x^2+2x-2)e^{2x}dx = \left(x^5-4x^3+x^2+-1\right)e^{2x} + C$$