$$I=\int \frac{x-3}{\sqrt{x^2-6x+1}}\,dx$$

REŠENJE:

$$I=\int \frac{x-3}{\sqrt{x^2-6x+1}}\,dx=\frac{1}{2}\int\frac{2x-6}{\sqrt{x^2-6x+1}}\,dx$$

$$=\frac{1}{2}\int\frac{d(x^2-6x+1)}{\sqrt{x^2-6x+1}}=\sqrt{x^2-6x+1}+C$$

$$I=\int \frac{x+4}{\sqrt{2-x-x^2}}\,dx$$

REŠENJE:

$$I=-\frac{1}{2}\int\frac{-2x-8}{\sqrt{2-x-x^2}}\,dx$$

$$=-\frac{1}{2}\int\frac{-2x-1-7}{\sqrt{2-x-x^2}}\,dx$$

$$=-\frac{1}{2}\left[\int\frac{-2x-1}{\sqrt{2-x-x^2}}\,dx-7\int\frac{dx}{\sqrt{2-x-x^2}}\right]$$

$$=-\frac{1}{2}\left[\int\frac{d(2-x-x^2)}{\sqrt{2-x-x^2}}-7\int\frac{dx}{\sqrt{\frac{9}{4}-(\frac{1}{2}+x)^2}}\right]$$

$$=-\frac{2}{2}\sqrt{2-x-x^2}+\frac{7}{2}\arcsin\frac{\frac{1}{2}+x}{\frac{3}{2}}+C$$

$$=-\frac{2}{2}\sqrt{2-x-x^2}+\frac{7}{2}\arcsin\frac{2x+1}{3}+C$$

$$I = \int \frac{dx}{(x+2)^2 \sqrt{x^2+5}}$$

RJEŠENJE:

INT. OBLIKA $\int \frac{dx}{(mx+n)^2 \sqrt{ax^2+bx+c}}$, SMJENA $mx+n = \frac{1}{t}$

$$x+2 = \frac{1}{t}; \quad dx = -\frac{dt}{t^2}; \quad x = \frac{1}{t} - 2; \quad t = \frac{1}{x+2}$$

$$\sqrt{x^2+5} = \sqrt{\frac{1}{t^2} - \frac{4}{t} + 4 + 5} = \frac{\sqrt{1-4t+9t^2}}{t}$$

$$I = \int \frac{-\frac{dt}{t^2}}{\frac{1}{t^2}\frac{\sqrt{1-4t+9t^2}}{t}} = -\int \frac{t\,dt}{\sqrt{1-4t+9t^2}}$$

$$= -\frac{1}{18}\int \frac{18t\,dt}{\sqrt{1-4t+9t^2}} = -\frac{1}{18}\int \frac{18t-4+4}{\sqrt{1-4t+9t^2}}\,dt$$

$$= -\frac{1}{18}\left[\int \frac{18t-4}{\sqrt{1-4t+9t^2}}\,dt + 4\int \frac{dt}{\sqrt{\frac{5}{9} + \left(3t-\frac{2}{3}\right)^2}}\right]$$

$$= -\frac{1}{18}\int \frac{d(1-4t+9t^2)}{\sqrt{1-4t+9t^2}} - \frac{4}{18}\int \frac{dt}{\sqrt{\frac{5}{9} + \left(3t-\frac{2}{3}\right)^2}}$$

$$= -\frac{1}{9}\sqrt{1-4t+9t^2} - \frac{4}{18}\ln\left(\left(3t-\frac{2}{3}\right) + \sqrt{\frac{5}{9}+\left(3t-\frac{2}{3}\right)^2}\right)$$

$$= -\frac{1}{9}\sqrt{1-\frac{4}{(x+2)}+9\frac{1}{(x+2)^2}} - \frac{2}{9}\ln\left(\frac{3}{x+2}-\frac{2}{3}+\sqrt{1-\frac{4}{x+2}+\frac{9}{(x+2)^2}}\right)$$

$$= -\frac{1}{9}\sqrt{\frac{x^2+4x+4-\cancel{4x}-8+9}{(x+2)^2}} - \frac{2}{9}\ln\left(\frac{9-2x-4}{3(x+2)} + \frac{3\sqrt{x^2+4x+4-4x-8+9}}{3(x+2)}\right)$$

$$= -\frac{1}{9}\frac{\sqrt{x^2+5}}{x+2} - \frac{2}{9}\ln\frac{5-2x+3\sqrt{x^2+5}}{3(x+2)} + C$$

$$I=\int \ln(x^2+1)\,dx$$

REŠENJE

$$u=\ln(x^2+1) \qquad dx=dv$$
$$du=\frac{2x}{x^2+1}\,dx \qquad v=x$$

$$I= x\ln(x^2+1)-\int x\cdot\frac{2x}{x^2+1}\,dx$$
$$= x\ln(x^2+1)-2\int\frac{x^2}{x^2+1}\,dx$$
$$= x\ln(x^2+1)-2\int\frac{x^2+1-1}{x^2+1}\,dx$$
$$= x\ln(x^2+1)-2\cdot\int\left(1-\frac{1}{x^2+1}\right)dx$$
$$= x\ln(x^2+1)-2(x-\operatorname{arctg}x)+C$$

IZRAČUNATI $I=\int(-x^6+30x^4-x^2+2)e^{-x}dx$

REŠENJE:

$$\int(-x^6+30x^4-x^2+2)e^{-x}dx= P_6(x)e^{-x}+C$$
$$=(Ax^6+Bx^5+Cx^4+Dx^3+Ex^2+Fx+G)e^{-x} \;/'$$

$$(-x^6+30x^4-x^2+2)e^{-x}= (P_6(x))'\cdot e^{-x}+P_6(x)(e^{-x})'$$
$$= e^{-x}(P_6'-P_6)$$
$$=e^{-x}(6Ax^5+5Bx^4+4Cx^3+3Dx^2+2Ex+F$$
$$-Ax^6-Bx^5-Cx^4-Dx^3-Ex^2-Fx-G) \;/ e^{x}$$

$$-x^6+30x^4-x^2+2=-Ax^6+(6A-B)x^5+(5B-C)x^4+(4C-D)x^3+$$
$$+(3D-E)x^2+(2E-F)x+(F-G)$$

x^6 : $-1=-A$; $A=1$

x^5 : $0=6A-B$; $B=6$

x^4 : $30=5B-C$; $C=0$

x^3 : $0=4C-D$; $D=0$

x^2 : $-1=3D-E$; $E=1$ $\Rightarrow$ $\underline{I=(x^6+6x^5+x^2+2x)e^{-x}+C}$

x : $0=2E-F$; $F=2$

x^0 : $2=F-G$; $G=0$

IZRAČUNATI REKURENTNU FORMULU ZA

$I_n = \int x^n e^{ax} dx; \quad a \neq 0$

RJEŠENJE:

$u = x^n \qquad dv = e^{ax} dx$

$du = n x^{n-1} dx \qquad v = \frac{1}{a} e^{ax}$

$I_n = x^n \frac{1}{a} e^{ax} - \int \frac{1}{a} e^{ax} n x^{n-1} dx$

$= \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx$

$I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$

IZRAČUNATI: $I = \int e^{ax} \sin bx \, dx$

RJEŠENJE

$u = e^{ax} \qquad dv = \sin bx \, dx$

$du = a e^{ax} dx \qquad v = -\frac{1}{b} \cos bx$

$I = -\frac{1}{b} e^{ax} \cos bx + \int \frac{1}{b} \cos bx \, a e^{ax} dx$

$I = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx$

$u = e^{ax} \qquad dv = \cos bx \, dx$

$du = a e^{ax} dx \qquad v = \frac{1}{b} \sin bx$

$I = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \left[\frac{1}{b} e^{ax} \sin bx - \int \frac{1}{b} \sin bx \, a e^{ax} dx \right]$

$I = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} I$

$I + \frac{a^2}{b^2} I = e^{ax} \left(\frac{-\cos bx}{b} + \frac{a \sin bx}{b^2} \right)$

$I \left(\frac{a^2 + b^2}{b^2} \right) = e^{ax} \frac{-b \cos bx + a \sin bx}{b^2}$

$I = e^{ax} \frac{a \sin bx - b \cos bx}{a^2 + b^2}$

$$\int \frac{x^4+1}{x^5+x^4-x^3-x^2}\,dx$$

РЕШЕНИЕ:

$$x^5+x^4-x^3-x^2 = x^2(x^3+x^2-x-1) =$$
$$= x^2[x^2(x+1)-(x+1)]$$
$$= x^2(x+1)(x^2-1)$$
$$= x^2(x+1)^2(x-1)$$

$$\frac{x^4+1}{x^5+x^4-x^3-x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{x-1}$$

$$x^4+1 = A\,x(x+1)^2(x-1) + B(x+1)^2(x-1) + C\,x^2(x+1)(x-1) +$$
$$+ D\,x^2(x-1) + E\,x^2(x+1)^2$$

ЗА $x=0$ $\quad 1=-B \quad ; B=-1$

$x=1 \quad 2=4E ; \quad E=\frac{1}{2}$

$x=-1 \quad 2=-2D ; \quad D=-1$

КЕ $x^4 \quad 1=A+C+E \quad ; A+C=\frac{1}{2} \quad ; C=\frac{1}{2}-A$

ЗА $x=2 \quad 17=18A+9B+12C+4D+36E$

$17=18A-9-6-6A-4+18$

$12A=12 ; \quad A=1 ; \quad C=-\frac{1}{2}$

$$\int \frac{x^4+1}{x^5+x^4-x^3-x^2}\,dx =$$

$$= \int \left[\frac{1}{x} - \frac{1}{x^2} - \frac{1}{2}\frac{1}{x+1} - \frac{1}{(x+1)^2} + \frac{1}{2}\frac{1}{x-1}\right] dx$$

$$= \ln|x| + \frac{1}{x} - \frac{1}{2}\ln|x+1| + \frac{1}{x+1} + \frac{1}{2}\ln|x-1| + C$$

$$\int \frac{2x^4+5x^2-2}{2x^3-x-1}\,dx$$

REŠENJE:

$$2x^4+5x^2-2 : 2x^3-x-1 = x$$
$$\underline{2x^4-x^2-x}$$
$$6x^2+x-2$$

$$\int \frac{2x^4+5x^2-2}{2x^3-x-1}\,dx$$
$$=\int\left[x+\frac{6x^2+x-2}{2x^3-x-1}\right]dx$$

$$P(x)=2x^3-x-1;\quad P(x_0)=0 \Rightarrow P(x)=(x-x_0)\,P_1(x)$$
$$P(1)=0 \Rightarrow 2x^3-x-1=(x-1)\,P_1(x)$$

$$2x^3-x-1 : x-1 = 2x^2+2x+1$$
$$\underline{2x^3-2x^2}$$
$$2x^2-x-1$$
$$\underline{2x^2-2x}$$
$$x-1$$

$$\Rightarrow 2x^3-x-1=(x-1)(2x^2+2x+1)$$

$$\frac{6x^2+x-2}{2x^3-x-1}=\frac{A}{x-1}+\frac{Bx+C}{2x^2+2x+1}$$
$$6x^2+x-2=A(2x^2+2x+1)+(Bx+C)(x-1)$$

$x=1$ $\quad 5=5A;\quad A=1$

x^2 $\quad 6=2A+B;\quad B=4$

$x=0$ $\quad -2=A-C;\quad C=A+2;\ C=3$

$$\int \frac{2x^4+5x^2-2}{2x^3-x-1}\,dx=\int\left[x+\frac{1}{x-1}+\frac{4x+3}{2x^2+2x+1}\right]dx$$
$$=\frac{x^2}{2}+\ln|x-1|+\int\left[\frac{4x+2}{2x^2+2x+1}+\frac{1}{2x^2+2x+1}\right]dx$$
$$=\frac{x^2}{2}+\ln|x-1|+\int\frac{d(2x^2+2x+1)}{2x^2+2x+1}+\int\frac{d(2x+1)}{(2x+1)^2+1}$$
$$=\frac{x^2}{2}+\ln|x-1|+\ln(2x^2+2x+1)+\operatorname{arctg}(2x+1)+C$$

$$\int \frac{dx}{3\sin x + 4\cos x + 5}$$

REŠENJE:

SMJENA $t = tg\frac{x}{2}$;

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \frac{\frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1 + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \frac{1 - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{1 + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{1-t^2}{1+t^2}$$

$t = tg\frac{x}{2}$; $\frac{x}{2} = arctg\, t$; $x = 2\, arctg\, t$; $dx = \frac{2dt}{1+t^2}$

$$\int \frac{dx}{3\sin x + 4\cos x + 5} = \int \frac{\frac{2dt}{1+t^2}}{3\frac{2t}{1+t^2} + 4\frac{1-t^2}{1+t^2} + 5}$$

$$= 2\int \frac{dt}{6t + 4 - 4t^2 + 5 + 5t^2} = 2\int \frac{dt}{t^2 + 6t + 9}$$

$$= 2\int \frac{dt}{(t+3)^2} = 2 \cdot \frac{-1}{t+3} + C$$

$$= \frac{-2}{tg\frac{x}{2} + 3} + C$$

$$\int \frac{2\sin x + 3\cos x}{\sin^2 x \cos x + 9\cos^3 x}\,dx$$

REŠENJE: $\int R(\sin x; \cos x)\,dx; \quad R(-\sin x; -\cos x) = R(\sin x; \cos x)$

SMJENA $t = tg\,x$

$$\int \frac{2\sin x + 3\cos x}{\sin^2 x \cos x + 9\cos^3 x}\,dx = \int \frac{\frac{2\sin x}{\cos^3 x} + \frac{3\cos x}{\cos^3 x}}{\frac{\sin^2 x\cos x}{\cos^3 x} + \frac{9\cos^3 x}{\cos^3 x}}\,dx$$

$$= \int \frac{2\,tg\,x \cdot \frac{1}{\cos^2 x} + 3 \cdot \frac{1}{\cos^2 x}}{tg^2 x + 9}\,dx$$

$$= \int \frac{2\,tg\,x + 3}{tg^2 x + 9}\,\frac{dx}{\cos^2 x}$$

$$= \int \frac{2t+3}{t^2+9}\,dt = \int \frac{2t\,dt}{t^2+9} + 3\int \frac{dt}{t^2+9}$$

$$= \int \frac{d(t^2+9)}{t^2+9} + 3 \cdot \frac{1}{9} \int \frac{3\,d(\frac{t}{3})}{(\frac{t}{3})^2+1}$$

$$= \int \frac{d(t^2+9)}{t^2+9} + \int \frac{d(\frac{t}{3})}{(\frac{t}{3})^2+1}$$

$$= \ln(t^2+9) + arc\,tg\,\frac{t}{3} + C$$

$$= \ln(tg^2 x + 9) + arc\,tg\,\frac{tg\,x}{3} + C$$

③ Izračunati površinu figure ograničene sa $7x^2-9y+9=0$ i

$5x^2-9y+27=0$

$-9y=-7x^2-9 \qquad 9y=5x^2+27$

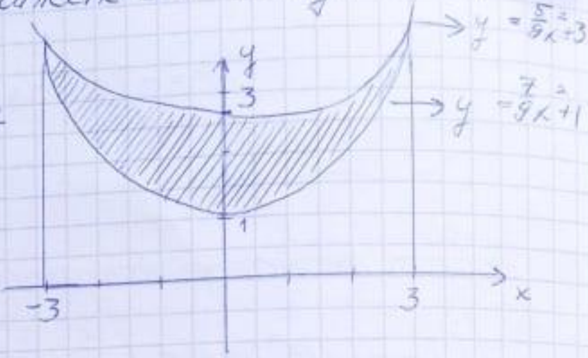
$y=\frac{7}{9}x^2+1 \qquad y=\frac{5}{9}x^2+3$

$\frac{7}{9}x^2+1=\frac{5}{9}x^2+3 \ /\cdot 9$

$7x^2-5x^2=18$

$2x^2=18$

$x=\pm 3$



$P=\int_{-3}^{3}\left(\frac{5}{9}x^2+3-\frac{7}{9}x^2-1\right)dx=\dots=8$

④ Izračunati P figure ograničene sa $y=-x^2+10x-16$ i $y=x+2$

$y=-x^2+10x-16=0$

$x_{1/2}=\frac{-10\pm\sqrt{100-64}}{-2} \Rightarrow x_{1/2}=\frac{-10\pm 6}{-2} \Rightarrow x_1=2$

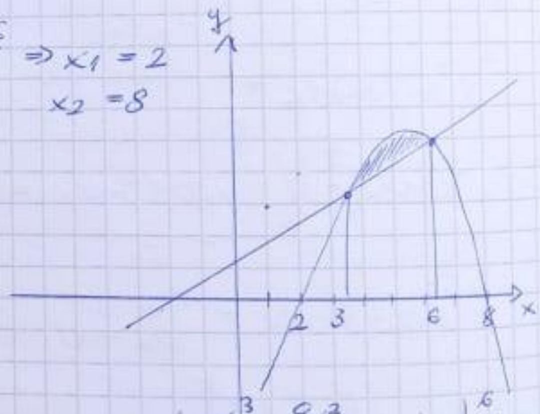
$x_2=8$

$-x^2+10x-16=x+2$

$-x^2+9x-18=0 \Rightarrow x^2-9x+18=0$

$x_{1/2}=\frac{+9\pm\sqrt{81-72}}{2} \Rightarrow x_{1/2}=\frac{9\pm 3}{2}$

$x_1=6 \quad x_2=3$



$P=\int_{3}^{6}(-x^2+10x-16-x-2)dx=\int_{3}^{6}(-x^2+9x-18)dx=\left(\frac{-x^3}{3}+\frac{9x^2}{2}-18x\right)\Big|_{3}^{6}=$

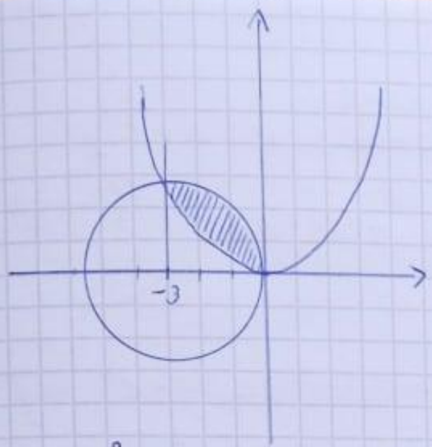
$=\left(\frac{-6^3}{3}+9\frac{6^2}{2}-18\cdot 6\right)-\left(\frac{-3^3}{3}+9\frac{3^2}{2}-18\cdot 3\right)$

⑤ Izračunati P figure ograničene sa $3y=x^2$ i $x^2+y^2+6x=0$

$y=\frac{x^2}{3}$; $(x+3)^2-9+y^2=0$

$(x+3)^2+y^2=9$

$C(-3,0) \quad r=3$



Presjek: $x^2 + \left(\frac{x^2}{3}\right)^2 + 6x = 0$

$$x^2 + \frac{x^4}{9} + 6x = 0$$

$$x\left(\frac{x^3}{9} + x + 6\right) = 0$$

$$\underline{x = 0} \ \vee \ \frac{x^3}{9} + x + 6 = 0 \ / \cdot 9$$

$$x^3 + 9x + 54 = 0$$

$$\underline{x = -3}$$

~~parabola~~ ? kružnica: $y^2 = 9 - (x+3)^2$

$$y = \pm\sqrt{9-(x+3)^2}$$

$$P = \int_{-3}^{0} \left(\sqrt{9-(x+3)^2} - \frac{x^2}{3}\right) dx = \underbrace{\int_{-3}^{0} \sqrt{9-(x+3)^2}\, dx}_{I_1} - \frac{1}{3}\frac{x^3}{3}\Big|_{-3}^{0} = I_1 - \frac{1}{9}(0+27)$$

I_1: $x + 3 = 3\sin t$

$dx = 3\cos t\, dt$

x	-3	0
t	0	$\frac{\pi}{2}$

$$I_1 = 3\int_0^{\frac{\pi}{2}} \sqrt{9 - 9\sin^2 t} \cdot \cos t\, dt = 3 \cdot 3 \int_0^{\frac{\pi}{2}} \cos^2 t\, dt =$$

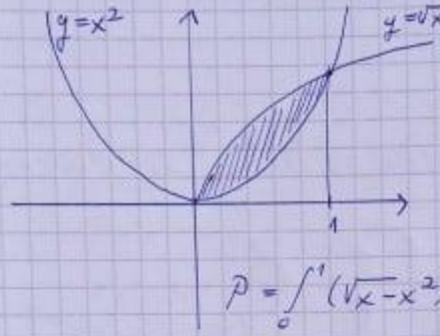
$1 - \cos t = 2\sin^2\frac{t}{2}$

$1 + \cos t = 2\cos^2\frac{t}{2}$

$$= 9\int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\, dt = 9\left(\frac{1}{2}t\Big|_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2t\, dt\right) =$$

$$= 9\left(\frac{\pi}{4} - \frac{1}{2}\cdot\frac{1}{2}\sin 2t\Big|_0^{\frac{\pi}{2}}\right) = \frac{9\pi}{4} - \frac{9}{4}(0) \qquad \underline{P = \frac{9\pi}{4} - 3}$$

⑦ Izračunati površinu figure ograničene linijama $y=x^2$ i $y=\sqrt{x}$



$y=x^2 \qquad x^2=\sqrt{x} \;/^2$

$y=\sqrt{x} \qquad x^4-x=0$

$x(x^3-1)=0$

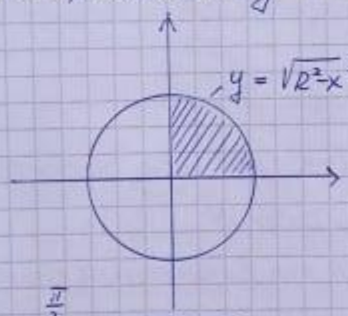
$\underline{x=0} \lor \underline{x=1}$

$$P=\int_0^1(\sqrt{x}-x^2)dx=\int_0^1\sqrt{x}dx-\int_0^1x^2dx=\frac{x^{\frac{3}{2}}\cdot 2}{3}\Big|_0^1-\frac{x^3}{3}\Big|_0^1=$$

$$=\frac{2}{3}(1-0)-\frac{1}{3}(1-0)=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$$

⑧ Naći površinu kruga $x^2+y^2=R^2$

$y=\pm\sqrt{R^2-x^2}$



$$P=4\int_0^R\sqrt{R^2-x^2}dx=$$

$$=\left[\begin{array}{l} x=R\sin t \Rightarrow dx=R\cos t \\ \begin{array}{c|c|c} x & 0 & R \\ \hline t & 0 & \frac{\pi}{2} \end{array}\end{array}\right]=$$

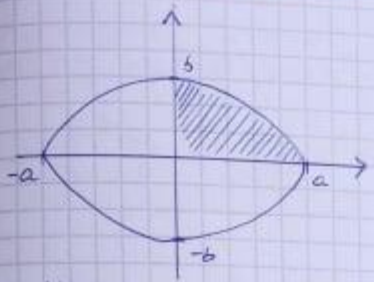
$$=4\int_0^{\frac{\pi}{2}}\sqrt{R^2-R^2\sin^2 t}\cdot R\cos t\,dt=4\int_0^{\frac{\pi}{2}}R^2\cos^2 t\,dt=4R^2\int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}dt=$$

$$=2R^2\,t\Big|_0^{\frac{\pi}{2}}+2R^2\cdot\frac{1}{2}\sin 2t\Big|_0^{\frac{\pi}{2}}=2R^2\left(\frac{\pi}{2}-0\right)+R^2(\sin\pi-\sin 0)=$$

$$=\pi R^2+R^2(0-0)=\underline{R^2\pi}$$

⑨ Izračunati P elipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

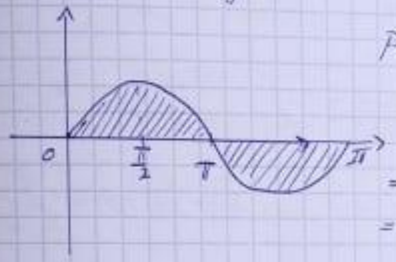
$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}\Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right)\Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$



$P = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx =$

$= \lceil x = a\sin t \rfloor = \underline{ab\pi}$

⑩ Naći površinu ograničenu krivom $y=\sin x$, $x=0$, $x=2\pi$



$P = \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi}\sin x\,dx =$

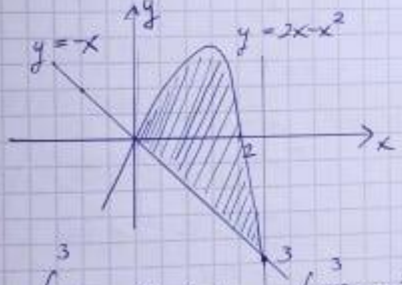
$= -\cos x\,|_0^{\pi} + \cos x\,|_{\pi}^{2\pi} =$

$= -(\cos\pi - \cos 0) + (\cos 2\pi - \cos\pi) =$

$= -(-1-1)+1+1 = \underline{4}$

⑪ Naći površinu ograničenu parabolom $y=2x-x^2$ i pravom $y=-x$

$y=2x-x^2 = x(2-x) \Rightarrow x=0 \vee x=2$



presjek:

$2x-x^2=-x$

$3x-x^2=0$

$x(3-x)=0 \Rightarrow \underline{x=0} \vee \underline{x=3}$

$P=\int_0^3(2x-x^2+x)\,dx = \int_0^3(3x-x^2)\,dx = 3\frac{x^2}{2}\Big|_0^3 - \frac{x^3}{3}\Big|_0^3 =$

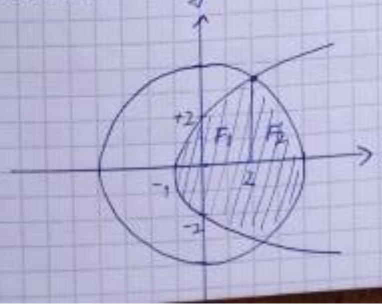
$= 3\cdot\frac{9}{2} - 3\cdot\frac{0^2}{2} - \left(\frac{3^3}{3}-\frac{0^3}{3}\right) = \frac{27}{2}-\frac{27}{3} = \underline{\frac{27}{6}}$

⑫ Izračunati P ograničeno krivom: $x^2+y^2=16$

$y^2=4(x+1) \Rightarrow y=\pm 2\sqrt{x+1}$

$x=0,\ y=\pm 2$ } crtanje parabole

$y=0,\ x=-1$



presjek: $x^2+4x+4=16$

$x^2+4x-12=0$

$x_1=2 \quad x_2=-6$

$P = P(F_1) + P(F_2)$

$P(F_1) = \int_{-1}^{2} 2\sqrt{x+1}\,dx - \int_{-1}^{2} -2\sqrt{x+1}\,dx = 2\int_{-1}^{2}\sqrt{x+1}\,dx + 2\int_{-1}^{2}\sqrt{x+1}\,dx = 4\int_{-1}^{2}\sqrt{x+1}\,dx$

$= \begin{bmatrix} x+1=t \\ dx=dt \end{bmatrix} \begin{array}{c|c|c} x & -1 & 2 \\ y & 0 & 3 \end{array} = 4\int_{0}^{3}\sqrt{t}\,dt = 4\int_{0}^{3} t^{\frac{1}{2}}\,dt = 4\cdot\frac{2t^{\frac{3}{2}}}{3}\Big|_0^3 =$

$= \frac{8}{3}\sqrt{27} = 8\sqrt{3}$

$P(F_2) = \int_{2}^{4}\sqrt{16-x^2}\,dx = \begin{bmatrix} x = 4\sin t \\ dx = 4\cos t\,dt \\ t = \arcsin\frac{x}{4} \end{bmatrix} \begin{array}{c|c|c} x & 2 & 4 \\ t & \frac{\pi}{6} & \frac{\pi}{2} \end{array} =$

$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{16-16\sin^2 t}\,\cos t\,dt = 8\cdot 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 t\,dt = 32\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt =$

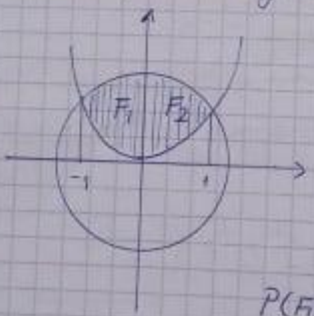
$= 16t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} + 16\cdot\frac{1}{2}\sin 2t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 16\left(\frac{\pi}{2}-\frac{\pi}{6}\right) + 8\left(\sin\pi - \sin\frac{\pi}{3}\right) = \frac{16\pi}{3} + 8\left(0-\frac{\sqrt{3}}{2}\right) =$

$= \frac{16\pi}{3} - 4\sqrt{3}$

$P = 8\sqrt{3} + \frac{16\pi}{3} - 4\sqrt{3} \Rightarrow \underline{P = \frac{16\pi}{3} + 4\sqrt{3}}$

(13) Izračunati površinu ograničenu sa: $x^2+y^2=2$

$y = x^2$



presjek: $x^2 + x^4 = 2$

$\left[x^2 = t\right] \Rightarrow t + t^2 - 2 = 0$

$t_1 = 1 \quad t_2 = -2\ (\perp)$

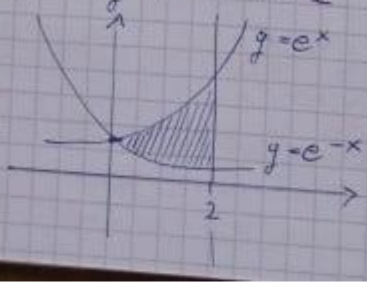
$\Downarrow$

$\underline{x = \pm 1}$

$P(F_1) = \int_{-1}^{0}\left(\sqrt{2-x^2} - x^2\right)dx \ldots$

$P(F_2) = \int_{0}^{1}\left(\sqrt{2-x^2} - x^2\right)dx \ldots$

(14) $y = e^x \quad y = e^{-x} \quad x = 2$



$P = \int_{0}^{2}\left(e^x - e^{-x}\right)dx = e^x\Big|_0^2 + \frac{1}{e^x}\Big|_0^2 =$

$= e^2 - 1 + \frac{1}{e^2} - 1 = \frac{e^4+1}{e^2} - 2$

Одредити екстремуме функције $z(x;y)=x^2+y^2-2x+6y+10$ под условом $x^2+y^2-6x+2y+9=0$.

$$F(x;y;\lambda)=x^2+y^2-2x+6y+10+\lambda(x^2+y^2-6x+2y+9)$$

$$\frac{\partial F}{\partial x}=2x-2+\lambda(2x-6)=0$$

$$\frac{\partial F}{\partial y}=2y+6+\lambda(2y+2)=0$$

$$\frac{\partial F}{\partial \lambda}=x^2+y^2-6x+2y+9=0$$

$$2x-2=\lambda(6-2x)\;;\quad \lambda=\frac{2x-2}{6-2x}$$

$$2y+6=\lambda(-2y-2)\;;\quad \lambda=\cancel{\frac{-2y+x}{2y+6}}\;\frac{2y+6}{-2y-2}$$

$$\frac{2x-2}{6-2x}=\cancel{\frac{-2y+x}{2y+6}}\;\frac{2y+6}{-2y-2}$$

$$\cancel{(2x-2)(2y+6)=(6-2x)(-2y-2)}$$

$$\cancel{4xy-4y+12x-12=-12y+4xy-12+4x}$$

$$\cancel{8y=-8x}$$

$$\cancel{y=-x}$$

$$\cancel{x^2+y^2-6x+2y+9=0\;;\;y=-x}$$

$$\cancel{x^2+x^2-6x-2x+9=0}$$

$$\cancel{2x^2-8x+9=0}$$

$$\cancel{x_{1/2}=8\pm\sqrt{64-}}$$

$$(2x-2)(-2y-2)=(6-2x)(2y+6)$$

$$-4xy+4y-4x+4=12y-4xy+36-12x$$

$$8x-8y=32$$

$$x-y=4\;;\quad y=x-4$$

$$x^2+y^2-6x+2y+9=0$$

$$x^2+x^2-8x+16-6x+2x-8+9=0$$

$$2x^2-12x+17=0\;;\quad x_{1/2}=\frac{12\pm\sqrt{144-136}}{4}=3\pm\frac{2\sqrt2}{4}$$

$$x_1=3-\frac{\sqrt2}{2}\qquad x_2=3+\frac{\sqrt2}{2}\qquad A\left(3-\frac{\sqrt2}{2};-1-\frac{\sqrt2}{2}\right)$$

$$y_1=-1-\frac{\sqrt2}{2}\qquad y_2=-1+\frac{\sqrt2}{2}\qquad B\left(3+\frac{\sqrt2}{2};-1+\frac{\sqrt2}{2}\right)\;(*)$$

$$(*)\;\lambda=\frac{2x-2}{6-2x}$$

$$\lambda_A=\frac{6-\sqrt2-2}{2-6+\sqrt2}=\frac{4-\sqrt2}{\sqrt2}$$

$$\lambda_B=\frac{6+\sqrt2-2}{6-6-\sqrt2}=\frac{4+\sqrt2}{-\sqrt2}$$

$$\frac{\partial^2F}{\partial x^2}=2+2\lambda$$

$$\frac{\partial^2F}{\partial x\partial y}=0$$

$$\frac{\partial^2F}{\partial y^2}=2+2\lambda$$

$$d^2F=\frac{\partial^2F}{\partial x^2}dx^2+\frac{\partial^2F}{\partial x\partial y}dx\,dy+\frac{\partial^2F}{\partial y^2}dy^2$$

$$d^2F=(2+2\lambda)(dx^2+dy^2)$$

$$d^2F_A=4\sqrt2(dx^2+dy^2)>0$$

$$d^2F_B=-4\sqrt2(dx^2+dy^2)<0$$

$$z(x;y)=(x-1)^2+(y+3)^2$$

$$z(A)=\left(2-\frac{\sqrt2}{2}\right)^2+\left(2-\frac{\sqrt2}{2}\right)^2=2\cdot\left(4-2\sqrt2+\frac12\right)=9-4\sqrt2$$

$$z(B)=\left(2+\frac{\sqrt2}{2}\right)^2+\left(2+\frac{\sqrt2}{2}\right)^2=2\cdot\left(4+2\sqrt2+\frac12\right)=9+4\sqrt2$$

$$z_{min}=z(A)=z\left(3-\frac{\sqrt2}{2};-1-\frac{\sqrt2}{2}\right)=9-4\sqrt2$$

$$z_{max}=z(B)=z\left(3+\frac{\sqrt2}{2};-1+\frac{\sqrt2}{2}\right)=9+4\sqrt2$$

NAĆI EKSTREME FUNKCIJE

$z = xy$; PRI USLOVU: $x^2+y^2=3$

$$F(x;y;\lambda) = xy + \lambda(x^2+y^2-3)$$

$\frac{\partial F}{\partial x} = y + 2x\lambda = 0 \quad \Rightarrow y = -2x\lambda$

$\frac{\partial F}{\partial y} = x + 2y\lambda = 0 \qquad x + 2(-2x\lambda)\lambda = 0$

$x - 4x\lambda^2 = 0$

$\frac{\partial F}{\partial \lambda} = x^2+y^2-3 = 0 \qquad x(1-4\lambda^2)=0$

$x=0 \lor 1-4\lambda^2=0$

$x=0$ NE MOŽE ZBOG TOGA ŠTO IZ $y=-2x\lambda \Rightarrow y=0$

$\begin{cases} x=0 \\ y=0 \\ x^2+y^2=3 \end{cases}$ NEMOGUĆE

$\Rightarrow \quad 1-4\lambda^2=0;\ 1=4\lambda^2;\ \lambda^2=\frac{1}{4};\ \lambda=\pm\frac{1}{2}$

$\lambda=\frac{1}{2} \quad \begin{rcases} y+2x\lambda=0 \\ x+2y\lambda=0 \\ x^2+y^2=3 \end{rcases} \Rightarrow \begin{rcases} y+x=0 \\ x+y=0 \\ x^2+y^2=3 \end{rcases} \begin{rcases} y=-x \\ x^2+x^2=3 \end{rcases} \begin{cases} 2x^2=3 \\ x^2=3/2 \\ x=\pm\sqrt{\frac{3}{2}} \\ y=\mp\sqrt{\frac{3}{2}} \end{cases} \quad A\left(\sqrt{\frac{3}{2}}; -\sqrt{\frac{3}{2}}\right) \quad B\left(-\sqrt{\frac{3}{2}}; \sqrt{\frac{3}{2}}\right)$

$\lambda=-\frac{1}{2} \quad \begin{rcases} y+2x\lambda=0 \\ x+2y\lambda=0 \\ x^2+y^2=3 \end{rcases} \begin{rcases} y-x=0 \\ x-y=0 \\ x^2+y^2=3 \end{rcases} \Rightarrow \begin{rcases} y=x \\ x^2+x^2=3 \end{rcases} \Rightarrow \quad C\left(\sqrt{\frac{3}{2}}; \sqrt{\frac{3}{2}}\right) \quad D\left(-\sqrt{\frac{3}{2}}; -\sqrt{\frac{3}{2}}\right)$

A, B, C, D SU STACIONARNE TAČKE

$\frac{\partial^2 F}{\partial x^2} = 2\lambda;\quad \frac{\partial^2 F}{\partial y^2} = 2\lambda \cdot \frac{\partial^2 F}{\partial x \partial y} = 1$

$d^2F = \frac{\partial^2 F}{\partial x^2} \cdot dx^2 + 2\frac{\partial^2 F}{\partial x \partial y} dx dy + \frac{\partial^2 F}{\partial y^2} dy^2$

$A\left(\sqrt{\frac{3}{2}}; -\sqrt{\frac{3}{2}}\right)$, $\lambda=\frac{1}{2}$ $\quad d^2F(A) = 2\lambda dx^2 + 2dxdy + 2\lambda dy^2 = dx^2 + 2dxdy + dy^2$

$= (dx+dy)^2 > 0$ A JE LOKALNI MIN

$d^2F(B) = \ldots =$ ISTO $\Rightarrow$ B JE LOKALNI MIN

$C\left(\sqrt{\frac{3}{2}}; \sqrt{\frac{3}{2}}\right)$, $\lambda=-\frac{1}{2}$ $\quad d^2F(C) = 2\lambda dx^2 + 2dxdy + 2\lambda dy^2 = -dx^2 + 2dxdy - dy^2$

$= -(dx-dy)^2 < 0 \Rightarrow$ C JE LOKALNI MAX.

$d^2F(D) = \ldots =$ ISTO $\Rightarrow$ D JE LOKALNI MAX.

Ријешити диференцијалну једначину: $(y^2+2)dx = 2y(2x+y^2-4)dy$.

Решење:

$$\frac{dx}{dy} = \frac{2y}{y^2+2}\cdot 2x + \frac{2y(y^2-4)}{y^2+2}$$

$$\frac{dx}{dy} - \frac{4y}{y^2+2}x - \frac{2y(y^2-4)}{y^2+2} = 0;$$

$$x = e^{-\int -\frac{4y}{y^2+2}dy}\left[C + \int \frac{2y(y^2-4)}{y^2+2} e^{\int -\frac{4y}{y^2+2}dy} dy\right]$$

$$x = e^{2\ln(y^2+2)}\left[C + \int \frac{2y(y^2-4)}{y^2+2} e^{-2\ln(y^2+2)} dy\right]$$

$$x = (y^2+2)^2\left[C + \int \frac{2y(y^2-4)}{(y^2+2)^3} dy\right]$$

$$y^2+2 = t;\quad 2y\,dy = dt;\quad y^2 = t-2$$

$$\int \frac{2y(y^2-4)}{(y^2+2)^3}dy = \int \frac{t-6}{t^3}dt = -\frac{1}{t} + 3\frac{1}{t^2} = \frac{-t+3}{t^2} = \frac{-y^2+1}{(y^2+2)^2}$$

$$x = (y^2+2)^2\left[C + \frac{1-y^2}{(y^2+2)^2}\right]$$

$$\boxed{x = C(y^2+2)^2 + 1 - y^2}$$

II начин $\quad \lambda(y) = \frac{1}{(y^2+2)^3}$

Ријешити диференцијалну једначину: $(x^2 - \sin^2 y)dx + x\sin 2y\,dy = 0$.

$\frac{\partial P}{\partial y} = -2\sin y\cos y$; $\frac{\partial Q}{\partial x} = \sin 2y$

$\lambda = \lambda(x)$ $\quad \frac{\partial(\lambda P)}{\partial y} = \frac{\partial(\lambda Q)}{\partial x}$; $\lambda\frac{\partial P}{\partial y} = \lambda\frac{\partial Q}{\partial x} + Q\frac{d\lambda}{dx}$

$\frac{d\lambda}{\lambda} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}dx$; $\frac{d\lambda}{\lambda} = \frac{-\sin 2y - \sin 2y}{x\sin 2y}dx$

$\frac{d\lambda}{\lambda} = -\frac{2}{x}dx$; $\ln\lambda = -2\ln x$; $\lambda = \frac{1}{x^2}$

$\left(1 - \frac{\sin^2 y}{x^2}\right)dx + \frac{1}{x}\sin 2y\,dy = 0$

$\frac{\partial P}{\partial y} = -\frac{2\sin y\cos y}{x^2}$; $\frac{\partial Q}{\partial x} = -\frac{1}{x^2}\sin 2y$

$\frac{\partial u}{\partial x} = \left(1 - \frac{\sin^2 y}{x^2}\right) / \int dx \Rightarrow u = x + \frac{\sin^2 y}{x} + \varphi(y)$

$\frac{\partial u}{\partial y} = \frac{2\sin y\cos y}{x} + \varphi'(y)$

$\frac{\partial u}{\partial y} = \frac{1}{x}\sin 2y$

$\varphi'(y) = 0$, $\varphi(y) = C$

$$\boxed{x + \frac{\sin^2 y}{x} = C}$$

$$\boxed{y = \arcsin\sqrt{Cx - x^2}}$$

II način $\quad \sin y = z(x)$; $\cos y\, y' = z'$ $\quad$ [može i $\sin^2 y = z(x)$]

$x^2 - \sin^2 y + x\,2\sin y\cos y\cdot y' = 0$

$x^2 - z^2 + x\cdot 2zz' = 0$ $\quad z^2 = u(x)$; $2zz' = u'$

$x^2 - u + xu' = 0$;

$u' - \frac{1}{x}u + x = 0$; $u = e^{-\int -\frac{1}{x}dx}\left[C - \int x e^{\int -\frac{1}{x}dx}dx\right]$

$u = x[C - x]$; $\quad z = \sqrt{Cx - x^2}$;

$$\boxed{y = \arcsin\sqrt{Cx - x^2}}$$

Риjешити диференциjалну jедначину: $(2x^2y+2y+5)dx+(2x^3+2x)dy=0$.

Решење:

$$(2x^3+2x)dy=-((2x^2+2)y+5)dx$$

$$y'=-\frac{2x^2+2}{2x^3+2x}y-\frac{5}{2x^3+2x}$$

$$y'+\frac{2(x^2+1)}{2x(x^2+1)}y+\frac{5}{2x^3+2x}=0$$

$$y'+\frac{1}{x}y+\frac{5}{2x^3+2x}=0$$

$$y=e^{-\int\frac{1}{x}dx}\left[C-\int\frac{5}{2x^3+2x}e^{\int\frac{1}{x}dx}dx\right]$$

$$y=\frac{1}{x}\left[\frac{C}{2}-\int\frac{5}{2(x^2+1)}dx\right]$$

$$y=\frac{1}{x}\left[\frac{C}{2}-\frac{5}{2}\operatorname{arctg}x\right]$$

$$\boxed{y=\frac{C-5\operatorname{arctg}x}{2x}}$$

II način množi $\lambda(x)=\frac{1}{x^2+1}$

Ријешити диференцијалну једначину: $x^2(3y+2x)y' + 3x(y+x)^2 = 0$.

$$y' = -\frac{3x(y+x)^2}{x^2(3y+2x)} \; ; \; y' = -\frac{3(y+x)^2}{3xy+2x^2} \; ; \; y' = -\frac{3\left(\frac{y}{x}+1\right)^2}{3\frac{y}{x}+2}$$

$$\frac{y}{x} = z(x); \quad y = xz(x); \quad y' = z + xz'$$

$$z + xz' = -\frac{3(z+1)^2}{3z+2} \; ; \; xz' = -\frac{3z^2+2z}{3z+2} - \frac{3z^2+6z+3}{3z+2}$$

$$xz' = -\frac{6z^2+8z+3}{3z+2}$$

$$\frac{3z+2}{6z^2+8z+3}\,dz = -\frac{dx}{x} \quad /\cdot 4$$

$$\frac{12z+8}{6z^2+8z+3}\,dz = -\frac{4\,dx}{x}$$

$$\frac{d(6z^2+8z+3)}{6z^2+8z+3} = -4\frac{dx}{x}$$

$$\ln(6z^2+8z+3) = -4\ln x + \ln C$$

$$6z^2+8z+3 = \frac{C}{x^4}$$

$$6\frac{y^2}{x^2} + 8\frac{y}{x} + 3 = \frac{C}{x^4} \quad /\cdot x^4$$

$$\boxed{6x^2y^2 + 8x^3y + 3x^4 = C}$$

II način $\quad 3x(y^2+2xy+x^2)\,dx + x^2(3y+2x)\,dy = 0$

$$\frac{\partial P}{\partial y} = 6xy + 6x^2 \; ; \; \frac{\partial Q}{\partial x} = 6xy + 6x^2$$

Ријешити диференцијалну једначину: $\cos x(3y^2\cos x-\sin x)dy-y\,dx=0$.

$$y\,dx+\cos x(\sin x-3y^2\cos x)\,dy=0$$

$$\frac{\partial P}{\partial y}=1;\quad \frac{\partial Q}{\partial x}=-\sin x(\sin x-3y^2\cos x)+\cos x(\cos x+3y^2\sin x)$$

$$=-\sin^2 x+3y^2\sin x\cos x+\cos^2 x+3y^2\sin x\cos x$$

$$=-\sin^2 x+6y^2\sin x\cos x+1-\sin^2 x$$

$$=1+2\sin x(3y^2\cos x-\sin x)$$

$\lambda=\lambda(x)$

$$\frac{d\lambda}{\lambda}=\frac{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}{Q}dx;\quad \frac{d\lambda}{\lambda}=\frac{-2\sin x(3y^2\cos x-\sin x)}{\cos x(\sin x-3y^2\cos x)}dx$$

$$\frac{d\lambda}{\lambda}=\frac{2\sin x}{\cos x}dx;\quad \ln\lambda=-2\ln\cos x;\quad \lambda=\frac{1}{\cos^2 x}$$

$$\frac{y}{\cos^2 x}dx+(\operatorname{tg}x-3y^2)dy=0$$

$$\frac{\partial P}{\partial y}=\frac{1}{\cos^2 x};\quad \frac{\partial Q}{\partial x}=\frac{1}{\cos^2 x}$$

$$\frac{\partial u}{\partial x}=\frac{y}{\cos^2 x}\quad /\int dx\Rightarrow u(x,y)=y\operatorname{tg}x+\varphi(y)\quad /\frac{\partial}{\partial y}$$

$$\frac{\partial u}{\partial y}=\operatorname{tg}x-3y^2\qquad \frac{\partial u}{\partial y}=\operatorname{tg}x+\varphi'(y)$$

$$\operatorname{tg}x+\varphi'(y)=\operatorname{tg}x-3y^2$$

$$\varphi'(y)=-3y^2;\quad \varphi(y)=-y^3$$

$$y\operatorname{tg}x-y^3=C$$

II način

$$\frac{dx}{dy}=3y\cos^2 x-\frac{1}{y}\sin x\cos x\quad /:\cos^2 x$$

$$\frac{x'}{\cos^2 x}=3y-\frac{1}{y}\operatorname{tg}x;\quad \operatorname{tg}x=z(y);\quad \frac{x'}{\cos^2 x}=z'$$

$$z'+\frac{1}{y}z-3y=0;$$

$$z=e^{-\int\frac{1}{y}dy}\left[C+3\int y\,e^{\int\frac{1}{y}dy}dy\right]$$

$$z=\frac{1}{y}[C+y^3];\quad \operatorname{tg}x=\frac{1}{y}[C+y^3]$$

$$y\operatorname{tg}x-y^3=C$$

$(y\cos x + y^3)dx - \sin x\, dy = 0$

rešenje:

$$\frac{dy}{dx} = \frac{y\cos x + y^3}{\sin x}$$

$$y' - \operatorname{ctg} x\, y - \frac{1}{\sin x} y^3 = 0 \quad / \cdot y^{-3}, \; y \neq 0 \qquad y = 0 \text{ je rešenje}$$

$$y^{-3} y' - \operatorname{ctg} x\, y^{-2} - \frac{1}{\sin x} = 0$$

$$y^{-2} = z(x) \qquad y = z^{-\frac{1}{2}}$$

$$-\frac{z'}{2} - \operatorname{ctg} x\, z - \frac{1}{\sin x} = 0 \qquad -2y^{-3}y' = z'$$

$$z' + 2\operatorname{ctg} x\, z + \frac{2}{\sin x} = 0$$

$$z = e^{-\int 2\operatorname{ctg} x\, dx}\left[c - \int \frac{2}{\sin x} e^{\int 2\operatorname{ctg} x\, dx} dx\right]$$

$$z = e^{-2\ln \sin x}\left[c - \int \frac{2}{\sin x} \sin^2 x\, dx\right]$$

$$z = \frac{1}{\sin^2 x}\left[c + 2\cos x\right]$$

$$z = \frac{c + 2\cos x}{\sin^2 x}, \; \frac{1}{y^2} = z; \; y^2 = \frac{1}{z}$$

$$y = \frac{\pm \sin x}{\sqrt{c + 2\cos x}}$$

$$y = 0$$

$(x^2+xy)y' = x\sqrt{x^2-y^2}+xy+y^2$

Решење:

$(1+\frac{y}{x})y' = \sqrt{1-(\frac{y}{x})^2}+\frac{y}{x}+(\frac{y}{x})^2$ хомогена

$\frac{y}{x}=z(x);\ y=xz;\ y'=z+xz'$

$(1+z)(z+xz') = \sqrt{1-z^2}+z+z^2$

$z+xz' = \frac{\sqrt{1-z^2}}{1+z}+\frac{z(1+z)}{1+z}$

$xz' = \frac{\sqrt{1-z^2}}{1+z}$

$\frac{1+z}{\sqrt{1-z^2}}dz = \frac{dx}{x}\ /\int$

$\int\frac{dz}{\sqrt{1-z^2}} + \frac{1}{2}\int\frac{2z\,dz}{\sqrt{1-z^2}} = \ln x+\ln C$

$\arcsin z - \frac{1}{2}\ln\sqrt{1-z^2} = \ln x+\ln C$

$\arcsin\frac{y}{x} = \ln\frac{Cx\sqrt{1-z^2}}{2}$

$\arcsin\frac{y}{x} = \ln\frac{C\sqrt{x^2-y^2}}{2}$

Ријешити диференцијалну једначину: $xyy' - x^2\sqrt{y^2+1} = (x+1)(y^2+1)$.

Решење:

смјена $y^2+1 = z(x)$; $2yy' = z'$

$2xyy' - 2x^2\sqrt{y^2+1} = 2(x+1)(y^2+1)$

$xz' - 2x^2\sqrt{z} = 2(x+1)z$

$z' - 2x\sqrt{z} - 2\frac{x+1}{x}z = 0 \;/\cdot\frac{1}{\sqrt{z}}$, $\sqrt{z} \neq 0$ Бернулијева

$z^{-\frac{1}{2}}z' - 2x - \frac{2(x+1)}{x}z^{\frac{1}{2}} = 0$

$z^{\frac{1}{2}} = u(x)$; $\frac{1}{2}z^{-\frac{1}{2}}z' = u'$

$2u' - \frac{2x+2}{x}u - 2x = 0$

$u' - \frac{x+1}{x}u - x = 0$

$u = e^{-\int -\frac{x+1}{x}dx}\left[C + \int x e^{\int -\frac{x+1}{x}dx}dx\right]$

$u = e^{x+\ln x}\left[C + \int x e^{-x-\ln x}dx\right]$

$u = xe^{x}\left[C + \int x\frac{e^{-x}}{x}dx\right]$; $u = xe^{x}(C - e^{-x})$

$u = Cxe^{x} - x$; $z = u^2$; $y^2+1 = z$

$$\boxed{y^2+1 = (Cxe^{x} - x)^2}$$

$$\boxed{y = \pm\sqrt{(Cxe^{x} - x)^2 - 1}}$$

Ријешити диференцијалну једначину: $xy' + (\sin y - 3x^2 \cos y)\cos y = 0$.

Решење:

$$(\sin y - 3x^2 \cos y)\cos y \, dx + x\, dy = 0$$

$$\frac{\partial P}{\partial y} = \cos^2 y - \sin^2 y + 3x^2 \cdot 2\cos y \sin y \; ; \; \frac{\partial Q}{\partial x} = 1$$

$$\lambda = \lambda(y) \; ; \; \frac{d\lambda}{\lambda} = \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} dy \; ; \; \frac{d\lambda}{\lambda} = \frac{1 - \cos^2 y + \sin^2 y - 6x^2 \cos y \sin y}{(\sin y - 3x^2 \cos y)\cos y} dy$$

$$\frac{d\lambda}{\lambda} = \frac{2\sin^2 y - 6x^2 \cos y \sin y}{(\sin y - 3x^2 \cos y)\cos y} dy \; ; \; \frac{d\lambda}{\lambda} = \frac{2\sin y(\sin y - 3x^2 \cos y)}{(\sin y - 3x^2 \cos y)\cos y} dy$$

$$\frac{d\lambda}{\lambda} = 2\frac{\sin y}{\cos y} dy \; ; \; \ln\lambda = -2\ln\cos y \; ; \; \lambda = \frac{1}{\cos^2 y}$$

$$\cancel{\sin y} \; (\sin y - 3x^2 \cos y)\cos y \, dx + x\, dy = 0 \quad / \cdot \frac{1}{\cos^2 y} \neq 0$$

$$(\operatorname{tg} y - 3x^2)\, dx + \frac{x}{\cos^2 y} dy = 0$$

$$\frac{\partial P}{\partial y} = \frac{1}{\cos^2 y} \; ; \; \frac{\partial Q}{\partial x} = \frac{1}{\cos^2 y} \; ; \; \exists\, U = U(x;y) \; ; \; U(x;y) = C$$

$$\frac{\partial U}{\partial x} = \operatorname{tg} y - 3x^2 \; / \int dx \Rightarrow U(x;y) = x \operatorname{tg} y - x^3 + \varphi(y) \quad / \frac{\partial}{\partial y}$$

$$\frac{\partial U}{\partial y} = \frac{x}{\cos^2 y} \qquad\qquad \frac{\partial U}{\partial y} = \frac{x}{\cos^2 y} + \varphi'(y)$$

$$\varphi'(y) = 0, \; \varphi(y) = C$$

$$x \operatorname{tg} y - x^3 = C$$

$$\boxed{x \operatorname{tg} y - x^3 = C}$$

$$x \operatorname{tg} y = C + x^3$$

$$\operatorname{tg} y = \frac{C + x^3}{x}$$

$$\boxed{y = \operatorname{arc}\operatorname{tg} \frac{C + x^3}{x}}$$

$$\boxed{y = \frac{\pi}{2} + k\pi}$$

II načim

$$xy' + \sin y \cos y - 3x^2 \cos^2 y = 0$$

$$x\frac{y'}{\cos^2 y} + \frac{\sin y}{\cos y} - 3x^2 = 0$$

smjena $\operatorname{tg} y = z(x)$

$$\frac{y'}{\cos^2 y} = z'$$

$$x z' + z - 3x^2 = 0$$

$$\cancel{z = e^{\int \operatorname{tg} y}}$$

$$z' + \frac{1}{x} z - 3x = 0$$

$$z = e^{-\int \frac{1}{x} dx}\left[C + 3\int x e^{\int \frac{1}{x} dx}\right]$$

$$z = \frac{1}{x}\left[C + x^3\right]$$

$$x \operatorname{tg} y = C + x^3$$