Određeni integrali

Njutn-Lajbnicova formula

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a),$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \text{ i}$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy$$

tj. varijablu po kojoj integriramo možemo označiti kako želimo.

1. $$\int_a^b c \cdot f(x)dx = c\int_a^b f(x)dx \,,\ c \in \mathbb{R},$$

2. $$\int_a^b \big(f(x) \pm g(x)\big)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx \,,$$

3. $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \,,\ a < c < b.$$

4. Ako je $f(x) \leq g(x)$, za svaki $x \in [a, b]$, onda je i

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx \,.$$

Primjer 1. Primjenom Leibniz–Newtonova formule nađimo sljedeće određene integrale elementarnih funkcija:

a) $$\int_0^1 x^3 dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4} - 0 = \frac{1}{4};$$

b) $$\int_{-1}^1 x^2 dx = \frac{x^3}{3}\Big|_{-1}^1 = \frac{1}{3} - \frac{(-1)^3}{3} = \frac{2}{3};$$

c) $$\int_0^\pi \sin x dx = -\cos x\Big|_0^\pi = -\cos\pi + \cos 0 = -(-1) + 1 = 2;$$

d) $$\int_0^1 \frac{dx}{1+x^2} = arctgx\Big|_0^1 = arctg1 - arctg0 = \frac{\pi}{4} - 0 = \frac{\pi}{4};$$

e) $$\int_0^2 e^x dx = e^x\Big|_0^2 = e^2 - e^0 = e^2 - 1;$$

f) $$\int_1^e \frac{dx}{x} = \ln x\Big|_1^e = \ln e - \ln 1 = 1 - 0 = 1;$$

g) $$\int_0^\pi \sin^2 x dx = \int_0^\pi \frac{1-\cos 2x}{2} dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]\Bigg|_0^\pi = \frac{1}{2}\left[\pi - \frac{\sin 2\pi}{2} - 0 + \frac{\sin 0}{2}\right] = \frac{\pi}{2};$$

Primjer 2. Izračunajmo sljedeće određene integrale:

a) $$\int_0^8 (1+\sqrt{2x}+\sqrt[3]{x})dx = \left(x+\sqrt{2}\frac{2}{3}x^{\frac{3}{2}}+\frac{3}{4}x^{\frac{4}{3}}\right)\Bigg|_0^8 =$$

$$=\left(8+\frac{2\sqrt{2}}{3}16\sqrt{2}+\frac{3}{4}\cdot 16\right)=8+16\left(\frac{4}{3}+\frac{3}{4}\right)=8+16\cdot\frac{25}{12}=\frac{124}{3};$$

b) $$\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} = \arcsin\frac{\sqrt{2}}{2}-\arcsin(-\frac{\sqrt{2}}{2}) = \frac{\pi}{4}-(-\frac{\pi}{4}) = \frac{\pi}{2};$$

c) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{1+x}{1-x}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2}\left[\ln\frac{\frac{3}{2}}{\frac{1}{2}}-\ln\frac{\frac{1}{2}}{\frac{3}{2}}\right] = \frac{1}{2}\ln 3^2 = \ln 3;$$

d) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\cos^2 x} = tgx\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = tg\frac{\pi}{4}-tg(-\frac{\pi}{4}) = 1+1=2;$$

e) $$\int_2^3 \frac{x+1}{x}dx = \int_2^3 (1+\frac{1}{x})dx = x\Big|_2^3 + \ln x\Big|_2^3 = 3-2+\ln 3-\ln 2 = 1+\ln\frac{3}{2}.$$

2.3 Metoda supstitucije kod određenog integrala

U određenom integralu $\int_a^b f(x)dx$, kod zamjene varijabli naročito treba paziti da se i granice a i b, koje se odnose na varijablu x, promjene i nađu odgovarajuće granice novo uvedene varijable.

Primjer 3. Metodom supstitucije rješimo sljedeće određene integrale:

a) $$\int_0^4 x^3\sqrt{x^2+9}\,dx = \int_0^4 x^2\sqrt{x^2+9}\cdot xdx = \left| \begin{array}{l} x^2+9=t^2, 2xdx=2tdt \\ x=0 \Rightarrow t=3 \\ x=4 \Rightarrow t=5 \end{array} \right| = \int_3^5 (t^2-9)\cdot t\cdot tdt =$$

$$= \left(\frac{t^5}{5} - 9\frac{t^3}{3}\right)\Big|_3^5 = 625 - \frac{225}{2} - \frac{243}{5} + \frac{81}{2} = \frac{1412}{5};$$

b) $$\int_1^e \frac{\sin \ln x}{x}dx = \left| \begin{array}{l} \ln x = t, \frac{1}{x}dx = dt \\ x=1 \Rightarrow t=\ln 1 = 0 \\ x=e \Rightarrow t = \ln e = 1 \end{array} \right| = \int_0^1 \sin tdt = -\cos t\Big|_0^1 = -\cos 1 + \cos 0 = 1 - \cos 1;$$

c) $$\int_0^{\frac{\pi}{2}} \cos x \sin^2 xdx = \left| \begin{array}{l} \sin x = t, \cos xdx = dt \\ x=0 \Rightarrow t = \sin 0 = 0 \\ x=\frac{\pi}{2} \Rightarrow t = \sin\frac{\pi}{2} = 1 \end{array} \right| = \int_0^1 t^2dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3};$$

d) $$\int_0^{\ln 2} \frac{e^x-1}{\sqrt{e^x-1}}dx = \int_0^{\pi}(e^x-1)dx = \left| \begin{array}{l} e^x-1=t^2, e^xdx = 2tdt, dx = \frac{2tdt}{t^2+1} \\ e^x = t^2+1 \\ x=0 \Rightarrow t^2 = e^0-1 = 0 \Rightarrow t=0 \\ x=\ln 2 \Rightarrow t^2 = e^{\ln 2}-1 = 2-1 = 1 \Rightarrow t=1 \end{array} \right| =$$

$$= 2\int_0^1 t\frac{tdt}{t^2+1} = 2\int_0^1 \frac{t^2+1-1}{t^2+1}dt = 2\int_0^1 \left(1-\frac{1}{t^2+1}\right)dt = 2[t - arctgt]\Big|_0^1 =$$

$$= 2[1 - arctg1 - 0 + arctg0] = 2\left(1-\frac{\pi}{4}\right) = 2 - \frac{\pi}{2};$$

e) $$\int_{e}^{e^2}\frac{dx}{x\ln^3 x}=\left|\begin{array}{l}\ln x=t,\frac{dx}{x}=dt\\ x=0\Rightarrow t=\ln e=1\\ x=e^2\Rightarrow t=\ln e^2=2\end{array}\right|=\int_1^2\frac{dt}{t^3}=-\frac{1}{2t^2}\Big|_1^2=\frac{3}{8}.$$

Parcijalna integracija kod određenog integrala

$$\int_a^b u\,dv=(u\cdot v)\Big|_a^b-\int_a^b v\,du.$$

Primjer 4. Primjenom parcijalne integracije izračunajmo sljedeće određene integrale:

a) $$\int_1^{e^2}\frac{\ln x}{x^2}dx=\left|\begin{array}{l}u=\ln x,du=\frac{dx}{x}\\ dv=\frac{dx}{x^2},v=\frac{-1}{x}\end{array}\right|=-\frac{\ln x}{x}\Big|_1^{e^2}-\int_1^{e^2}(-\frac{1}{x})\frac{dx}{x}=-\frac{\ln x}{x}\Big|_1^{e^2}+\int_1^{e^2}\frac{dx}{x^2}=$$

$$=-\frac{\ln e^2}{e^2}+\frac{\ln 1}{1}-\frac{1}{x}\Big|_1^{e^2}=-\frac{2}{e^2}-\frac{1}{e^2}+1=1-\frac{3}{e^2};$$

b) $$\int_0^1\arcsin x\,dx=\left|\begin{array}{l}u=\arcsin x\,dx,du=\frac{dx}{\sqrt{1-x^2}}\\ dv=dx,v=x\end{array}\right|=x\arcsin x\Big|_0^1-\int_0^1\frac{xdx}{\sqrt{1-x^2}}=$$

$$=\left|\begin{array}{l}1-x^2=t^2,-2xdx=2tdt\\ x=0\Rightarrow t=1\\ x=1\Rightarrow t=0\end{array}\right|=\arcsin 1+\int_0^1 dt=\frac{\pi}{2}+1;$$

$\sqrt{1-x^2}=t$

c) $$\int_0^1 x\ln(x+3)dx = \left|\begin{matrix} u=\ln(x+3)\Rightarrow du=\dfrac{1}{x+3}dx \\ dv=xdx\Rightarrow v=\dfrac{x^2}{2}\end{matrix}\right| = \left[\ln(x+3)\cdot\frac{x^2}{2}\right]\Bigg|_0^1 - \int_0^1 \frac{x^2}{2}\frac{dx}{x+3} =$$

$$=\frac{1}{2}\ln 4-\frac{1}{2}\int_0^1\frac{x^2}{x+3}dx=\frac{1}{2}\ln 4-\frac{1}{2}\int_0^1(x-3+\frac{9}{x+3})dx=\frac{1}{2}\ln 4-\frac{1}{2}\left[\frac{x^2}{2}-3x+9\ln(x+3)\right]\Bigg|_0^1=$$

$$=\frac{1}{2}\ln 4-\frac{1}{2}\left[\frac{1}{2}-3+9\ln 4-9\ln 3\right]=\frac{1}{2}\ln 4+\frac{5}{4}-\frac{9}{2}\ln 4+\frac{9}{2}\ln 3=\frac{5}{4}-4\ln 4+\frac{9}{2}\ln 3\,;$$

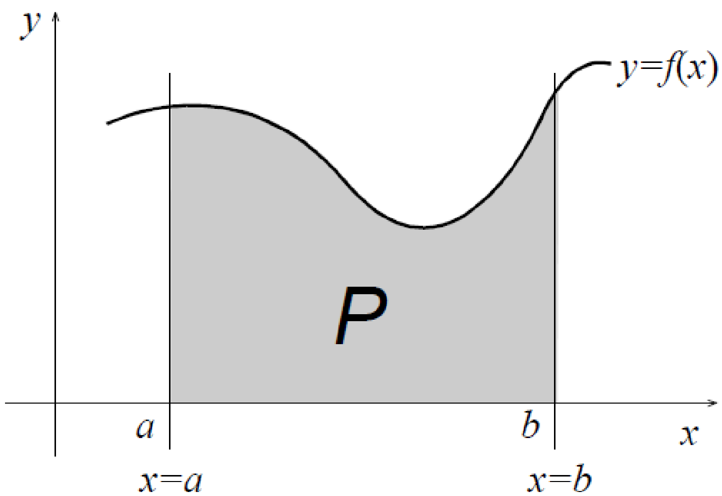
d) $$\int_0^{\ln 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}dx=\left|\begin{matrix} e^x-1=t^2,\ e^x dx=2tdt,\ dx=\dfrac{2tdt}{t^2+1} \\ x=0\Rightarrow t=0,\ x=\ln 5\Rightarrow t=2\end{matrix}\right|=\int_0^2\frac{t\cdot 2tdt}{t^2+1+3}=$$

$$=2\int_0^2\frac{t^2+4-4}{t^2+4}dt=2\int_0^2\frac{t^2+4}{t^2+4}dt+2\int_0^2\frac{-4}{t^2+4}dt=2t\Big|_0^2-8\cdot\frac{1}{2}arctg\frac{t}{2}\Big|_0^2=$$

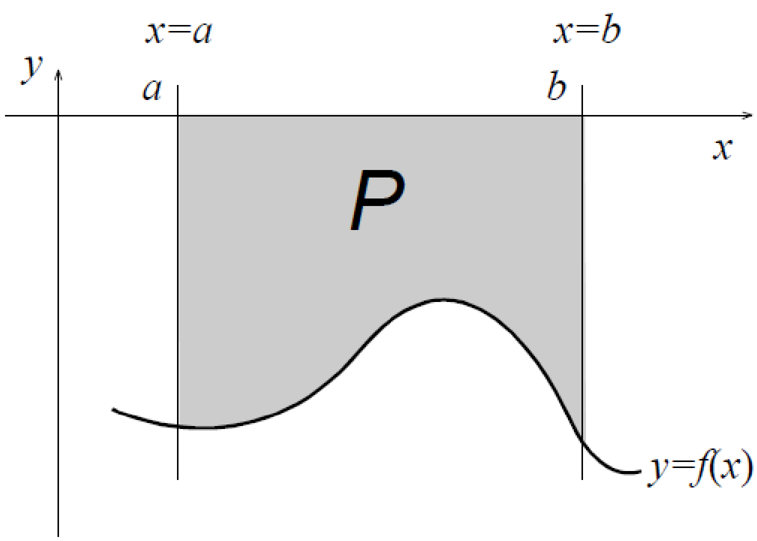
$$=4-4arctg1=4-4\cdot\frac{\pi}{4}=4-\pi\,.$$

$\sqrt{e^x-1}=t$

PRIMJENE ODREĐENOG INTEGRALA



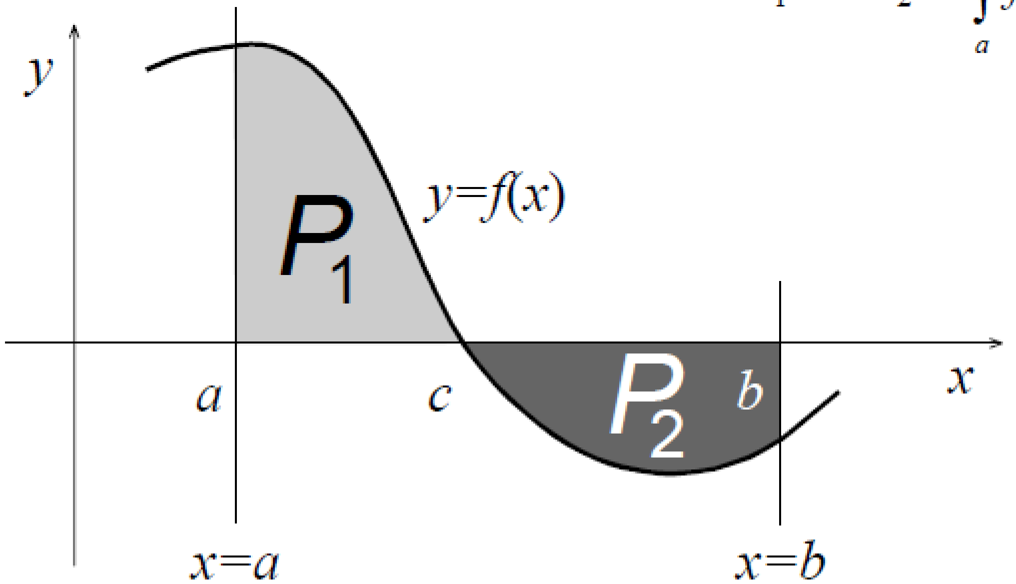
$$P = \int_a^b f(x)dx$$

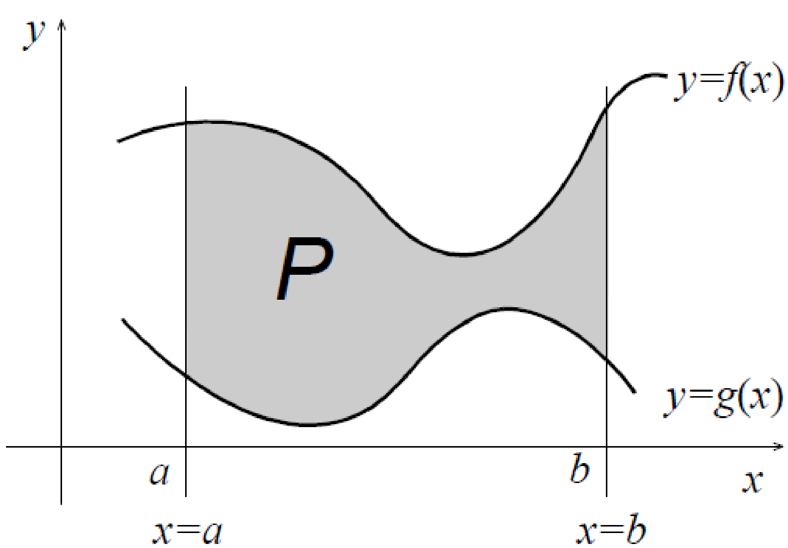


$$P = -\int_a^b f(x)dx\,.$$

$$P = \int_a^b |f(x)|dx = \int_a^c |f(x)|dx + \int_c^b |f(x)|dx$$

$$P = P_1 + P_2 = \int_a^c f(x)dx - \int_c^b f(x)dx .$$

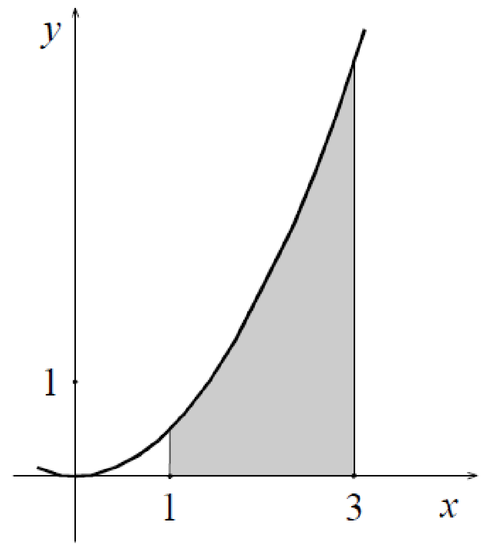




$$P = \int_a^b [f(x) - g(x)]dx .$$

Primjer 1. Izračunajmo površine područja omeđena grafovima funkcija:

a) $f(x) = \frac{x^2}{2}$ i pravcima $x = 1$, $x = 3$, $y = 0$. Skicirajmo površinu.

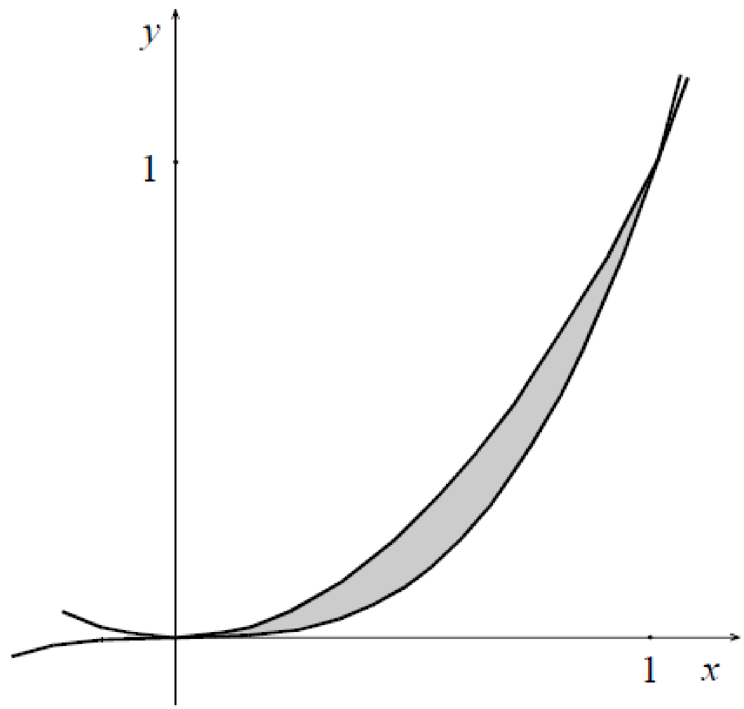


Slika 5.

Iz slike se vidi da je površina područja smještena nad intervalom [1, 3], pa je prema (1)

$$P = \int_1^3 \frac{x^2}{2}dx = \frac{1}{2}\frac{x^3}{3}\Big|_1^3 = \frac{1}{2}\left(9 - \frac{1}{3}\right) = \frac{13}{3};$$

b) $f(x) = x^2$ i $g(x) = x^3$.

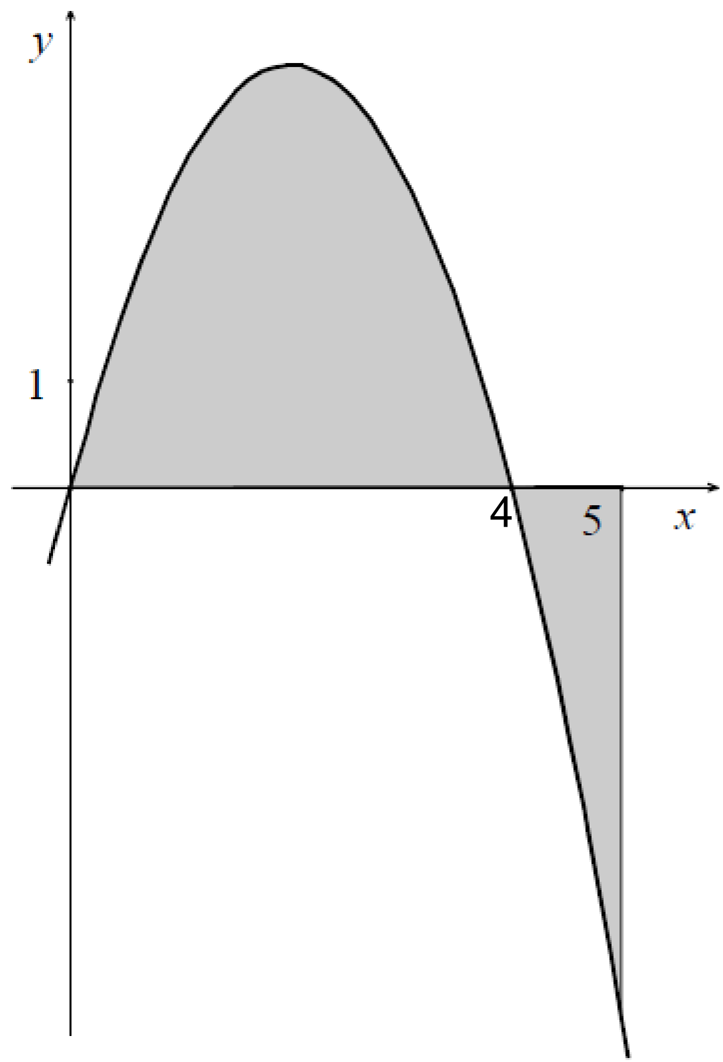


Slika 6.

Površina područja, koje je smješteno nad intervalom [0, 1] između grafova funkcija f i g prema (4) je

$$P = \int_0^1 x^2 dx - \int_0^1 x^3 dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12};$$

c) $f(x) = 4x - x^2$ i pravcima $x = 5$, $y = 0$.



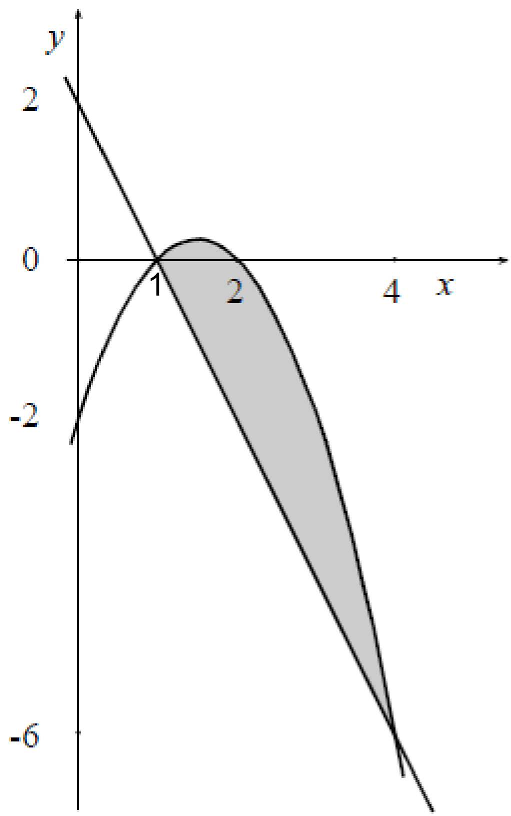
Slika 7.

Nultočke funkcije f su $x_1 = 0$ i $x_2 = 4$, pa je prema (3)

$$P = P_1 + P_2 = \int_0^4 \left(4x - x^2\right)dx + \int_4^5 \left(x^2 - 4x\right)dx = \left(2x^2 - \frac{x^3}{3}\right)\Bigg|_0^4 + \left(\frac{x^3}{3} - 2x^2\right)\Bigg|_4^5 =$$

$$= 2\cdot 16 - \frac{64}{3} + \frac{125}{3} - 2\cdot 25 - \frac{64}{3} + 2\cdot 16 = 13;$$

d) $f(x) = -x^2 + 3x - 2$ i pravcem $y = -2x + 2$.



Slika 8.

Grafovi tih funkcija se sjeku u apscisama u kojima je

$$-x^2 + 3x - 2 = -2x + 2,$$

Odakle je

$$x^2 - 5x + 4 = 0,$$

pa je $x_1 = 1$ i $x_2 = 4$.

$$P = \int_1^4 (-x^2 + 3x - 2 + 2x - 2)dx = \int_1^4 (-x^2 + 5x - 4)dx = \left(-\frac{x^3}{3} + 5\frac{x^2}{2} - 4x\right)\Bigg|_1^4 =$$

$$= -\frac{64}{3} + \frac{80}{2} - 8 + \frac{1}{3} - \frac{5}{2} + 4 = \frac{9}{2};$$