Nizovi

$f: N \to R$

$f(n) = a_n$

Niz $(a_n)_{n\in N}$ konvergira ka a ako važi:

$(\forall \varepsilon > 0)\ (\exists N(\varepsilon))\ (\forall n \geq N \mid a_n - a \mid < \varepsilon)$

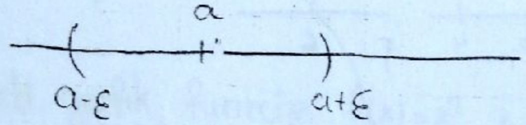
$-\varepsilon < a_n - a < \varepsilon$

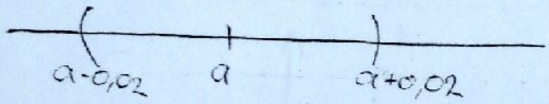
$a - \varepsilon < a_n < a + \varepsilon$

$a_n \in (a-\varepsilon,\ a+\varepsilon)$

$|x| < 5$

$-5 < x < 5$





$\lim\limits_{n\to\infty} a_n = a$

1. $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1,\ a > 0$

4. $\lim\limits_{n\to\infty} \frac{n^k}{a^n} = 0,\ a > 1$

5. $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$, $n! = 1 \cdot 2 \cdot \ldots \cdot n$

6. $\lim\limits_{n\to\infty} q^n = \begin{cases} 0 & |q| < 1 \\ 1 & q = 1 \\ -\infty & q > 1 \end{cases}$

a) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

b) $\lim\limits_{n\to\infty} (a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

c) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim a_n}{\lim\limits_{n\to\infty} b_n}$

ako oba postoje

1. $\lim\limits_{n\to\infty} \frac{5n^2+2n}{n^3+1} = \lim\limits_{n\to\infty} \frac{n^2(5+\frac{2}{n})}{n^3(1+\frac{1}{n^3})} = \lim\limits_{n\to\infty} \frac{1}{n} \cdot \frac{5+\frac{2}{n}}{1+\frac{1}{n^3}} = 0 \cdot 5 = 0$

2. $\lim\limits_{n\to\infty} \frac{(2n+1)(n+1)n}{n^3+3n^2+3n+1} = \lim\limits_{n\to\infty} \frac{2n^3+3n^2+n}{n^3+3n^2+3n+1} = \lim\limits_{n\to\infty} \frac{n^3(2+\frac{3}{n}+\frac{1}{n^2})}{n^3(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3})} = 2$

3 $\lim\limits_{n\to\infty} (\underbrace{n-2}_{a} - \underbrace{\sqrt{n^2-n+1}}_{b}) = \lim\limits_{n\to\infty} (n-2-\sqrt{n^2-n+1}) \cdot \frac{n-2+\sqrt{n^2-n+1}}{n-2+\sqrt{n^2-n+1}} =$

$= \lim\limits_{n\to\infty} \frac{n(\frac{3}{n}-1)}{n(1-\frac{2}{n}+\sqrt{1-\frac{1}{n}+\frac{1}{n^2}})} = -\frac{1}{2}$

4. $\lim\limits_{n\to\infty} \frac{(-2)^n+5^n}{(-2)^{n+1}+5^{n+1}} = \lim\limits_{n\to\infty} \frac{5^n((-\frac{2}{5})^n+1)}{5^{n+1}((-\frac{2}{5})^{n+1}+1)} = \lim\limits_{n\to\infty} \frac{1}{5} \cdot \frac{(-\frac{2}{5})^n+1}{(-\frac{2}{5})^{n+1}+1} = \frac{1}{5}$

geometrijski niz

5. $\lim\limits_{n\to\infty} \frac{1+2+2^2+\dots 2^n}{1+3+3^2+\dots 3^n} = \lim\limits_{n\to\infty} \frac{1 \cdot \frac{1-2^{n+1}}{1-2}}{1 \cdot \frac{1-3^{n+1}}{1-3}} = \lim\limits_{n\to\infty} 2 \cdot \frac{1-2^{n+1}}{1-3^{n+1}} =$

$2 \cdot \lim\limits_{n\to\infty} \frac{2^{n+1}(\frac{1}{2^{n+1}}-1)}{3^{n+1}(\frac{1}{3^{n+1}}-1)} = 2 \cdot \lim\limits_{n\to\infty} \left(\frac{2}{3}\right)^{n+1} \cdot \frac{\frac{1}{2^{n+1}}-1}{\frac{1}{3^{n+1}}-1} = 2$

3 $\lim\limits_{n\to\infty} \left(\frac{(n-2)^2-(\sqrt{n^2-n+1})^2}{n-2+\sqrt{n^2-n+1}} = \lim\limits_{n\to\infty} \frac{(n-2)-(n^2-n+1)}{n-2+\sqrt{n^2-n+1}} = \lim\limits_{n\to\infty} \frac{n^2-2n+4-n^2+n-1}{n-2+\sqrt{n^2(1-\frac{1}{n}+\frac{1}{n^2})}}\right. =$

$\lim\limits_{n\to\infty} \frac{-n+3}{n-2+n\sqrt{1-\frac{1}{n}+\frac{1}{n^2}}} = \lim\limits_{n\to\infty} \frac{-n+3}{n(1-\frac{2}{n}+\sqrt{1-\frac{1}{n}+\frac{1}{n^2}})}$

Vježba 4. 15.10.2019. IV sedmica

1. Naći graničnu vrijednost čiji je opšti član x_n

$x_n = \sqrt{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{2} \dots \sqrt[2n]{2}$

$x_n = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \dots 2^{\frac{1}{2n}}$

$x_n = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}}$ $= \frac{1}{2} \cdot \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1-(\frac{1}{2})^n$

$x_n = 2^{1-(\frac{1}{2})^n}$

$\lim\limits_{n\to\infty} x_n = \lim\limits_{x\to\infty} 2^{1-(\frac{1}{2})^n} = 2^1 = 2$

(2) $$\lim_{n\to\infty} \sqrt[3]{n^2}\cdot(\underbrace{\sqrt[3]{n-1}}_{a} - \underbrace{\sqrt[3]{n+1}}_{b}) = \lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3]{n-1} - \sqrt[3]{n+1}\right)\cdot\left(\frac{\sqrt[3]{(n-1)}^2 + \sqrt[3]{n^2-1} + \sqrt[3]{(n+1)^2}}{\sqrt[3]{(n-1)^2} + \sqrt[3]{n-1} + \sqrt[3]{n+1}^2}\right)$$

$$= \lim_{n\to\infty} \sqrt[3]{n^2}\cdot\frac{n-1-(n+1)}{\sqrt[3]{(n-1)^2} + \sqrt[3]{n^2-1} + \sqrt[3]{(n+1)^2}} = \lim_{n\to\infty}$$

$$= \frac{\sqrt[3]{n^2}\cdot -2}{\sqrt{n^2\left(1-\frac{2}{n}+\frac{1}{n^2}\right)} + \sqrt[3]{n^2\left(1-\frac{1}{n^2}\right)} + \sqrt[3]{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}} =$$

$$= \lim_{n\to\infty} \sqrt[3]{n^2}\cdot\frac{-2}{\sqrt[3]{n^2}\cdot\left(\sqrt[3]{1-\frac{2}{n}+\frac{1}{n^2}} + \sqrt[3]{1-\frac{1}{n^2}} + \sqrt[3]{1+\frac{2}{n}+\frac{1}{n^2}}\right)} = -\frac{2}{3}$$

(3) Nađi graničnu vrijednost niza čiji je opšti član a_n

$$a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} \cdots + \frac{1}{n(n+1)}$$

$$a_1 = \frac{1}{1\cdot 2}$$

$$a_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}$$

$\vdots$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$\frac{1}{n(n+1)} = \frac{A(n+1)+Bn}{n(n+1)}$$

$$1 = An + A + Bn$$

$$1 = (A+B)n + A \implies A+B=0$$

$$A = 1 \qquad B = -1$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$n=1 \quad \frac{1}{1\cdot 2} = \frac{1}{1} - \frac{1}{2}$

$n=2 \quad \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$

$\vdots$

$$a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

$$a_n \le x_n \le b_n \quad , \forall n \in N$$

$$\left.\begin{array}{l}\lim\limits_{n\to\infty} a_n = l \\ \lim\limits_{n\to\infty} b_n = l\end{array}\right\} \Longrightarrow \lim_{n\to\infty} x_n = l$$

④ Primjenom teoreme o ukljeÅ¡tenju dokazati konvergenciju i naći graničnu vrijednost niza čiji je opÅ¡ti član. x_n

$$x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

$$* \; x_n \le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}} = n \cdot \frac{1}{\sqrt{n^2+1}}$$

$$* \; x_n \ge \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}} = n \cdot \frac{1}{\sqrt{n^2+n}}$$

$$a_n = \frac{n}{\sqrt{n^2+n}} \quad , \quad b_n = \frac{n}{\sqrt{n^2+1}}$$

1. $a_n \le x_n \le b_n , \quad \forall n \in N$

2. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2(1+\frac{1}{n})}} = \lim\limits_{n\to\infty} \frac{n}{n\sqrt{1+\frac{1}{n}}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\underbrace{\frac{1}{n}}_{0}}} = 1$

3. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \frac{n}{n \cdot \sqrt{1+\frac{1}{n}}} = 1$

Iz 1,2,3 na osnovu teoreme o ukljeÅ¡tenju, niz x_n konvergira i tada je $\lim\limits_{n\to\infty} x_n = 1$

⑤ $x_n = \frac{1}{\sqrt{n^2+\ln 1}} + \frac{1}{\sqrt{n^2+\ln 2}} + \dots + \frac{1}{\sqrt{n^2+\ln n}}$

$x_n \leq \frac{1}{\sqrt{n^2+\ln 1}} + \frac{1}{\sqrt{n^2+\ln 1}} + \dots + \frac{1}{\sqrt{n^2+\ln 1}} =$

$x_n \leq n \cdot \frac{1}{\sqrt{n^2+\ln 1}}$

$x_n \geq \frac{1}{\sqrt{n^2+\ln n}} + \frac{1}{\sqrt{n^2+\ln n}} + \dots + \frac{1}{\sqrt{n^2+\ln n}}$

$x_n \geq n \cdot \frac{1}{\sqrt{n^2 \cdot \ln n}}$

$\frac{n}{\sqrt{n^2+\ln n}} \leq x_n \leq \frac{n}{\sqrt{n^2+\ln 1}}$

$a_n = \frac{n}{\sqrt{n^2+\ln n}}$, $b_n = \frac{n}{\sqrt{n^2+\ln 1}}$

1. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{\sqrt{n \cdot \ln n}} = \lim\limits_{n\to\infty} \frac{n}{n\sqrt{1+\frac{\ln n}{n}}} = 1$

2. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{n}{n\sqrt{1+\frac{\ln 1}{n}}} = 1$

3. $a_n \leq x_n \leq b_n$, $\forall n \in \mathbb{N}$

$\left|_2\right.$ 1, 2, 3 $\Longrightarrow \lim\limits_{n\to\infty} x_n = 1$

⑥ Ispitati konvergenciju i naći graničnu vrijednost niza x_n

$$x_n = \frac{\cos n}{n}$$

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

$$a_n = -\frac{1}{n}, \quad b_n = \frac{1}{n}$$

1. $a_n \le x_n \le b_n$

2. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(-\frac{1}{n}\right) = 0$

3. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ Iz 1, 2, 3 slijedi da je $\lim_{n\to\infty} x_n = 0$

$$\left[\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \right] \qquad \lim_{n\to\infty} \left(1 + \frac{1}{\text{izraz}}\right)^{\text{izraz}}$$

⑦ $\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{2n \cdot \frac{1}{2}}$ $\underline{\underline{(*)}}$

$(*) = \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{2n}$ $[2n = m] \Rightarrow n \to \infty \Rightarrow m \to \infty$

$$\lim_{n\to\infty} \left(1 + \frac{1}{m}\right)^m = e$$

$$= e^{\frac{1}{2}} = \sqrt{e}$$

(8.) $\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = \lim_{n\to\infty}\left(1+\frac{n+1}{n-1}-1\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^n =$

$= \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^{\frac{n-1}{2}\cdot\frac{2}{n-1}\cdot n} \underline{\underline{(*)}}\ e^{\lim_{n\to\infty}\frac{2n}{n-1}} = e^{\lim_{n\to\infty}\frac{2n}{n(n-\frac{1}{n})}} = e^2$

$(*): \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^{\frac{n-1}{2}} = \left\lceil \frac{n-1}{2} = m \right.,\ n\to\infty,\ m\to\infty =$

$\left. \frac{2}{n-1} = \frac{1}{m} \right\rfloor$

$\lim_{n\to\infty}\left(1+\frac{1}{m}\right)^m = e$

ili $\lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n-1}{2}}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n-1}{2}}\right)^{\frac{n-1}{2}\cdot\frac{2}{n-1}\cdot n} =$

$= e^{\lim_{n\to\infty}\frac{2n}{n-1}} = e^2$

(9.) $\lim_{n\to\infty}\left(\frac{2n^2+3n+1}{2n^2-n-2}\right)^n = \lim_{n\to\infty}\left(1+\frac{2n^2+3n+1}{2n^2-n-2}-1\right)^n =$

$\lim_{n\to\infty}\left(1+\frac{4n+3}{2n^2-n-2}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{2n^2-n-2}{4n+3}}\right)^{\frac{2n^2-n-2}{4n+3}\cdot\frac{4n+3}{2n^2-n-2}\cdot n} =$

$e^{\lim_{n\to\infty}\frac{4n^2+3n}{2n^2-n-2}} = e^{\lim_{n\to\infty}\frac{n^2(4+\frac{3}{n}}{n^2(2-\frac{1}{2}-\frac{2}{n^2})}} = e^2$

Monotoni nizovi

$a_n < a_{n+1}$, $\forall n \in N$, a_n je monotono rastući niz

$a_n > a_{n+1}$, $\forall n \in N$, a_n je monotono opadajući niz

$a_n \leq a_{n+1}$, $\forall n \in N$, a_n je monotono neopadajući niz

$a_n \geq a_{n+1}$, $\forall n \in N$, a_n je monotono nerastući niz

Ako $\exists M \in N$, tako da $|a_n| \leq M$, (a_n) ograničen niz

Ako je niz monoton i ograničen onda je on konvergentan

(10.) Dokazati da niz konvergira ako je $a_n = \frac{1}{3+1} + \frac{1}{3^2+1} + \dots + \frac{1}{3^n+1}$

$$a_{n+1} - a_n = \frac{1}{3+1} + \frac{1}{3^2+1} \dots + \frac{1}{3^n+1} + \frac{1}{3^{n+1}+1} - \frac{1}{3+1} - \frac{1}{3^2+1} \dots - \frac{1}{3^n+1} - \frac{1}{3^{n+1}+1} =$$

$$= \frac{1}{3^{n+1}+1} > 0, \ \forall n \in N$$

$a_1 = \frac{1}{3+1}$

$a_2 = \frac{1}{3+1} + \frac{1}{3^2+1}$

$a_{n+1} > a_n, \ \forall n \in N \Rightarrow$ monotono rastući .... (1)

$a_1 \ a_2 \ a_3 \ a_4 \dots$

Kako je niz rastući, to je $a_1 \leq a_n$, $\forall n \in N$

$\frac{1}{4} \leq a_n, \ \forall n \in N$

$$a_n = \frac{1}{3+1} + \frac{1}{3^2+1} + \dots \frac{1}{3^n+1} \leq \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n} =$$

$$= \frac{1}{3} \cdot \frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}} = \frac{1}{3} \cdot \frac{3}{2} \left(\left(1 - \left(\frac{1}{3}\right)^n\right) = \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^n\right) < \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$\frac{1}{4} \leq a_n < \frac{1}{2}$, $\forall n \in N \Rightarrow a_n$ - ograničen niz .... (2)

Iz 1,2 slijedi da je niz konvergentan!

(11.) Dokazati da niz (a_n) konvergira ako je $a_n = (1-\frac{1}{2})(1-\frac{1}{4})\cdots(1-\frac{1}{2^{n+1}})$

$a_1 = (1-\frac{1}{2})(1-\frac{1}{4})$

$a_2 = (1-\frac{1}{2})(1-\frac{1}{4})(1-\frac{1}{2^3})$

$$\frac{a_{n+1}}{a_n} = \frac{(1-\frac{1}{2})(1-\frac{1}{4})\cdots(1-\frac{1}{2^{n+1}})\cdot(1-\frac{1}{2^{n+2}})}{(1-\frac{1}{2})(1-\frac{1}{4})\cdots(1-\frac{1}{2^{n+1}})} =$$

$$= 1 - \frac{1}{2^{n+2}} < 1$$

$$\left.\begin{array}{l} \frac{a_{n+1}}{a_n} < 1 \\ a_n > 0,\ \forall n \in N \end{array}\right\} a_{n+1} < a_n,\ \forall n \in N \Rightarrow \text{niz monotono opadajući} \ldots (1)$$

a_1
a_2
a_3
$\vdots$

Kada je niz opadajući onda važi $a_n \leq a_1$, $\forall n \in N$. Već smo uočili da je $a_n > 0$

$0 < a_n < \frac{3}{8}$, $\forall n \in N$ (a_n) je ograničen ... (2)

Iz 1,2 slijedi da je niz konvergentan

(12) Dokazati da niz konvergira (a_n) ako je $a_n = \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} \cdots + \frac{1}{(3n+1)(3n+4)}$

$$a_{n+1} - a_n = \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{(3n+1)(3n+4)} - \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots - \frac{1}{(3n+1)(3n+4)} =$$

$$= \frac{1}{(3n+4)(3n+7)} > 0$$

$a_{n+1} \cdot a_n > 0,\ \forall n \in N$

$a_{n+1} > a_n,\ \forall n \in N \Rightarrow (a_n)$ je monotono rastući ... (1)

Slijedi $a_1 < a_n,\ \forall n \in N$

$\frac{1}{28} \leq a_n,\ \forall n \in N$

$$\frac{1}{(3n+1)(3n+4)} = \frac{A}{3n+1} + \frac{B}{(3n+4)}$$

$$\frac{1}{(3n+1)(3n+4)} = \frac{A(3n+4)+B(3n+1)}{(3n+1)(3n+4)}$$

$1 = (3A+3B)\cdot n + 4A + B \Rightarrow A+B=0 \Rightarrow B = -A$

$4A + B = 1 \qquad 3A = 1$

$A = \frac{1}{3}\ ;\ B = -\frac{1}{3}$

$$\frac{1}{(3n+1)(3n+4)} = \frac{1}{3}\cdot\frac{1}{3n+1} - \frac{1}{3}\cdot\frac{1}{3n+4} = \frac{1}{3}\left(\frac{1}{3n+1} - \frac{1}{3n+4}\right)$$

$n=1 \qquad \frac{1}{4\cdot 7} = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right)$

$n=2 \qquad \frac{1}{4\cdot 7} = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{10}\right)$

$\vdots$

$$a_n = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \cdots + \frac{1}{3n+1} - \frac{1}{3n+4}\right)$$

$$a_n = \frac{1}{3}\left(\frac{1}{4} \cdot \frac{1}{3n+4} < \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}\right.$$

$\frac{1}{28} \leq a_n < \frac{1}{12},\ \forall n \in N$ (a_n) je ograničen niz ... (2)

Iz 1, 2 slijedi da je niz konvergentan!