Granična vrijednost funkcije

$f: X \to Y$, $X, Y \subseteq R$

Definicija: kažemo da je A granična vrijednost funkcije f u tački x_0 i pišemo $\lim\limits_{x \to x_0} f(x) = A$ ako je funkcija f definisana u nekoj okolini tačke x_0, sa isključenjem možda tačke x_0 i važi:

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon))(\forall x \in X)(0 < |x - x_0| < \delta \Rightarrow |f(x) - A| < \varepsilon)$$

Osnovne granične vrijednosti:

1° $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

2° $\lim\limits_{x \to 0} \frac{\ln(1+x)}{x} = 1$

3° $\lim\limits_{x \to 0} (1+x)^{\frac{1}{x}} = e$

4° $\lim\limits_{x \to \infty} (1+\frac{1}{x})^{x} = e$

5° $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$

6° $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \ln a$

7° $\lim\limits_{x \to 0} \frac{\log_a(1+x)}{x} = \ln a$

1° $$\lim_{x\to 2}\frac{e^{x-2}-1}{x-2}=\left[\begin{array}{l}\lim_{x\to 0}\frac{e^x-1}{x}=1\\ x-2=t\\ x\to 2,\ t\to 0\end{array}\right]=\lim_{t\to 0}\frac{e^t-1}{t}=1$$

2° $$\lim_{x\to 3}\frac{x^3-5x^2+6x}{x^2-9}=\lim_{x\to 3}\frac{x^3-5x^2+6x}{(x-3)^2}=\lim_{x\to 3}\frac{x^3-5x^2+6x}{(x-3)(x+3)}=$$ izvlači se x

$$=\lim_{x\to 3}\frac{x(x^2-5x+6)}{(x-3)(x+3)}=\lim_{x\to 3}\frac{x(x-2)\cancel{(x-3)}}{\cancel{(x-3)}(x+3)}=\lim_{x\to 3}\frac{x(x-2)}{(x+3)}=$$

$$\boxed{(x^2-5x+6)=x^2-(3x+2x)+6=(x-2)(x-3)}$$

$$=\frac{3(3-2)}{3+3}=\frac{9-6}{6}=\frac{3}{6}=\frac{1}{2}$$

3° $$\lim_{x\to +\infty}\frac{2x^4+x^2+2x+3}{3x^4+x^2+x+3}=\lim_{x\to +\infty}\frac{\cancel{x^4}\left(2+\frac{1}{x^2}+\frac{2}{x^3}+\frac{3}{x^4}\right)}{\cancel{x^4}\left(3+\frac{1}{x^2}+\frac{1}{x^3}+\frac{3}{x^4}\right)}=\frac{2}{3}$$

veći stepen polinoma brže raste

4° $$\lim_{x\to 0}\frac{x^7+x^6+3x^4}{x^6+x^5+x^4}=\lim_{x\to 0}\frac{\cancel{x^4}(x^3+x^2+3)}{\cancel{x^4}(x^2+x+1)}=\frac{3}{1}=3$$

5° $$\lim_{x\to -\infty}\frac{\sqrt{5x^2-3}}{x}=\lim_{x\to -\infty}\frac{\sqrt{x^2\left(5-\frac{3}{x^2}\right)}}{x}=\lim_{x\to -\infty}\frac{|x|\cdot\sqrt{5-\frac{3}{x^2}}}{x}=\lim_{x\to -\infty}\frac{-x\cdot\sqrt{5-\frac{3}{x^2}}}{x}$$

$$=\lim_{x\to -\infty}-\sqrt{5-\frac{3}{x^2}}=-\sqrt{5}$$

$$\sqrt[2]{x^2}=|x| \qquad |x|=\begin{cases}x, & x\geq 0\\ -x, & x<0\end{cases}$$

$$\sqrt[3]{x^3}=x$$

$$\sqrt{(-5)^2}=|-5|$$

$$\sqrt[3]{-8}=-2$$

$$\sqrt[3]{8}=2$$

$$\lim_{x\to +\infty}\frac{x^5+3x^2+2x+5}{4x^4+x^2+x+3}=\lim_{x\to +\infty}\frac{x^5\left(1+\frac{3}{x^3}+\frac{2}{x^4}+\frac{5}{x^5}\right)}{x^4\left(4+\frac{1}{x^2}+\frac{1}{x^3}+\frac{3}{x^4}\right)}=\frac{x}{4}=\frac{\infty}{4}=\infty$$

$$\lim_{x\to +\infty}\frac{\sqrt[3]{x^3+2}}{x}=\lim_{x\to +\infty}\frac{\sqrt[3]{x^3\left(1+\frac{2}{x^3}\right)}}{x}=\lim_{x\to +\infty}\frac{\cancel{x}\cdot\sqrt[3]{1+\frac{2}{x^3}}}{\cancel{x}}=\sqrt[3]{1}=1$$

8) $$\lim_{x\to 3}\frac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}=\lim_{x\to 3}\frac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}\cdot\frac{\sqrt{x+13}+2\sqrt{x+1}}{\sqrt{x+13}+2\sqrt{x+1}}=$$

$$=\lim_{x\to 3}\frac{x+13-4(x+1)}{(x-3)^2\cdot(\sqrt{x+13}+2\sqrt{x+1})}=\lim_{x\to 3}\frac{x+13-4x+4}{(x-3)(x+3)\cdot(\sqrt{x+13}+2\sqrt{x+1})}=$$

$$=\lim_{x\to 3}\frac{3x-9}{(x-3)(x+3)(\sqrt{x+13}+2\sqrt{x+1})}=\lim_{x\to 3}\frac{3\cancel{(x-3)}}{\cancel{(x-3)}(x+3)(\sqrt{x+13}+2\sqrt{x+1})}$$

$$=\frac{3}{(3+3)(\sqrt{3+13}+2\cdot\sqrt{3+1})}=\frac{3}{6\cdot(\sqrt{16}+2\cdot\sqrt{4})}=\frac{\cancel{3}}{\cancel{6}\cdot(4+4)}=\frac{1}{2\cdot 8}=$$

Razlika kvadrata: $a^2-b^2=(a-b)(a+b)$

Razlika kubova: $a^3-b^3=(a-b)(a^2+ab+b^2)$

9) $$\lim_{x\to 0}\frac{x}{\sqrt[3]{8+x}-2}=\lim_{x\to 0}\frac{x}{\sqrt[3]{8+x}-2}\cdot\frac{\sqrt[3]{(8+x)^2}+2\sqrt[3]{8+x}+2^2}{\sqrt[3]{(8+x)^2}+2\sqrt[3]{8+x}+2^2}$$

$$=\lim_{x\to 0}\frac{x(\sqrt[3]{(8+x)^2}+2\sqrt[3]{8+x}+4)}{\cancel{8}+x-\cancel{8}}=\lim_{x\to 0}\frac{\cancel{x}(\sqrt[3]{(8+x)^2}+2\sqrt[3]{8+x}+4)}{\cancel{x}}=$$

$$=\lim_{x\to 0}\sqrt[3]{(8+x)^2}+2\sqrt[3]{8+x}+4=\sqrt[3]{8^2}+2\sqrt[3]{8}+4=\sqrt[3]{64}+2\cdot 2+$$

$$=4+4+4=12\qquad\left(\sqrt[3]{(2^3)^2}=\sqrt[3]{2^6}=2^2=4\right)$$

$$\lim_{x\to+\infty}\left(\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}\right)=\lim_{x\to+\infty}\left(\underbrace{\sqrt[3]{x^3+3x^2}-x}_{a^3-b^3}+\underbrace{x-\sqrt{x^2-2x}}_{a^2-b^2}\right)=$$

$$=\lim_{x\to+\infty}\sqrt[3]{x^3+3x^2}-x\cdot\frac{\sqrt[3]{(x^3+3x^2)^2}+x\sqrt[3]{x^3+3x^2}+x^2}{\sqrt[3]{(x^3+3x^2)^2}+x\sqrt[3]{x^3+3x^2}+x^2}+x-\sqrt{x^2-2x}\cdot\frac{x+\sqrt{x^2-}}{x+\sqrt{x^2-}}$$

$$=\lim_{x\to+\infty}\frac{\cancel{x^3}+3x^2-\cancel{x^3}}{\sqrt[3]{(x^3+3x^2)^2}+x\sqrt[3]{x^3+3x^2}+x^2}+\frac{\cancel{x^2}-\cancel{x^2}-2x}{x+\sqrt{x^2-2x}}=$$

$$=\lim_{x\to+\infty}\frac{3x^2}{\sqrt[3]{\left(x^3\left(1+\frac{3}{x}\right)\right)^2}+x\cdot\sqrt[3]{x^3\left(1+\frac{3}{x}\right)}+x^2}+\frac{-2x}{x+\sqrt{x^2\left(1-\frac{2}{x}\right)}}=$$

$$=\lim_{x\to+\infty}\frac{3x^2}{x^2\sqrt[3]{1+\frac{3}{x}}+x^2\sqrt[3]{\left(1+\frac{3}{x}\right)}+x^2}+\frac{-2x}{2x+\left(\sqrt{1-\frac{2}{x}}\right)}$$

$$=\lim_{x\to+\infty}\frac{\cancel{3x^2}}{\cancel{3x^2}\left(\sqrt[3]{1+\cancel{\tfrac{3}{x}}_{0}}+\sqrt[3]{1+\cancel{\tfrac{3}{x}}_{0}}\right)}+\frac{\cancel{-2x}}{\cancel{2x}+\sqrt{1-\cancel{\tfrac{2}{x}}_{0}}}=\frac{1}{\sqrt[3]{1}+\sqrt[3]{1}}+\frac{-1}{\sqrt[2]{1}}$$

$$=\frac{1}{2}+(-1)=-\frac{3}{2}$$

(11) $\lim\limits_{x\to 0}\frac{\sin 5x}{x}=\lim\limits_{x\to 0}\frac{\sin 5x}{5x}\cdot 5=5\cdot\lim\limits_{x\to 0}\frac{\sin 5x}{5x}=\left\lceil\begin{matrix}5x=t\\ x\to 0,\ t\to 0\end{matrix}\right\rfloor=$

$5\cdot\lim\limits_{t\to 0}\frac{\sin t}{t}=5\cdot 1=5$

(12) $\lim\limits_{x\to 0}\frac{e^{2x}-e^{3x}}{x}=\lim\limits_{x\to 0}\frac{e^{2x}-1+1-e^{3x}}{x}=$

$\lim\limits_{x\to 0}\frac{e^{2x}-1-(e^{3x}-1)}{x}=l_1-l_2=2-3=-1$

$\lim\limits_{x\to 0}\frac{e^{x}-1}{x}=1$!!!

$l_1=\lim\limits_{x\to 0}\frac{e^{2x}-1}{x}=\lim\limits_{x\to 0}\frac{e^{2x}-1}{2x}\cdot 2=2\cdot\lim\limits_{x\to 0}\frac{e^{2x}-1}{2x}=2\cdot 1=2$

$l_2=\lim\limits_{x\to 0}\frac{e^{3x}-1}{x}=\lim\limits_{x\to 0}\frac{e^{3x}-1}{3x}\cdot 3=3\cdot 1=3$

(13) $\lim\limits_{x\to 0}\frac{\ln(1+4x)}{\ln(1+5x)}=\lim\limits_{x\to 0}\frac{\frac{\ln(1+4x)}{4x}\cdot 4x}{\frac{\ln(1+5x)}{5x}\cdot 5x}=\frac{4}{5}$

$\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1$

(14) $\lim\limits_{x\to 0}\frac{\operatorname{tg}x}{x}=\lim\limits_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x}=\lim\limits_{x\to 0}\frac{\sin x}{x\cdot\cos x}=\lim\limits_{x\to 0}\frac{\sin x}{x\,\cancel{\cos x}}\cdot\lim\limits_{x\to 0}\frac{1}{\cos x}=1\cdot 1=1$

(15) $\lim\limits_{x\to 0}\frac{\arcsin x}{x}=\left\lceil\begin{matrix}\arcsin x=t\\ x=\sin t\\ x\to 0\Rightarrow t\to 0\end{matrix}\right\rfloor=\lim\limits_{t\to 0}\frac{t}{\sin t}=\lim\limits_{t\to 0}\frac{1}{\underbrace{\frac{\sin t}{t}}_{1}}=\frac{1}{1}=1$

(16) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=$

$\cos 2t=\cos^2 t-\sin^2 t$

$\cos x=\cos 2\cdot\frac{x}{2}=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$

$\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{1-(\cos^2\frac{x}{2}-\sin^2\frac{x}{2})}{x^2}=\lim\limits_{x\to 0}\frac{1-\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{x^2}=\lim\limits_{x\to 0}\frac{\sin^2\frac{x}{2}+\sin^2\frac{x}{2}}{x^2}=$

$\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=2\cdot\lim\limits_{x\to 0}\left(\frac{\sin\frac{x}{2}}{x}\right)^2=2\lim\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}\cdot 2}\right)^2=2\cdot\frac{1}{2^2}\cdot\lim\limits_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}$

(17.) $$\lim_{x\to 0}\frac{x^2}{\sqrt{1+x\cdot\sin x}-\sqrt{\cos x}}=\lim_{x\to 0}\frac{x^2}{\sqrt{1+x\cdot\sin x}-\sqrt{\cos x}}\cdot\frac{\sqrt{1+x\sin x}+\sqrt{\cos x}}{\sqrt{1+x\sin x}+\sqrt{\cos x}}=$$

$$\lim_{x\to 0}\frac{x^2\left(\sqrt{1+x\sin x}+\sqrt{\cos x}\right)}{1+x\cdot\sin x-\cos x}=\lim_{x\to 0}\frac{\sqrt{1+x\cdot\sin x}+\sqrt{\cos x}}{\frac{1-\cos x+x\cdot\sin x}{x^2}}=$$

$$\frac{1-\cos x}{x^2}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x\sin x}+\sqrt{\cos x}}{\underbrace{\frac{1-\cos x}{x^2}}_{\frac{1}{2}}+\frac{\sin x}{x}}=\frac{1+1}{\frac{1}{2}+1}=\frac{4}{3}$$

(18.) $$\lim_{x\to 1}\left(\frac{3}{1-\sqrt{x}}-\frac{2}{1-\sqrt[3]{x}}\right)=\ulcorner x^{\frac{1}{2}},\ x^{\frac{1}{3}}\quad=$$

$x=t^6$, $6=NZS(2,3)$

$\sqrt{x}=t^3$ $\quad\sqrt[3]{x}=t^2$

$t=\sqrt[6]{x}$, $x\to 1$, $t\to 1$

$$\lim_{t\to 1}\left(\frac{3}{1-t^3}-\frac{2}{1-t^2}\right)=\lim_{t\to 1}\left(\frac{3}{(1-t)(1+t+t^2)}-\frac{2}{(1-t)(1+t)}\right)=$$

$$\lim_{t\to 1}\frac{3(1+t)-2(1+t+t^2)}{(1-t)(1+t)(1+t+t^2)}=\lim_{t\to 1}\frac{-2t^2+t+1}{(1-t)(1+t)(1+t+t^2)}=$$

$$\lim_{t\to 1}\frac{(-2)(t+\frac{1}{2})(t-1)}{(1-t)(1+t)(1+t+t^2)}=\frac{2\cdot\frac{3}{2}}{2\cdot(1+1+1)}=\frac{1}{2}$$

(19.) $$\lim_{x\to+\infty}\frac{\ln(2+e^{3x})}{\ln(3+e^{2x})}=\lim_{x\to+\infty}\frac{\ln e^{3x}\left(\frac{2}{e^{3x}}+1\right)}{\ln e^{2x}\left(\frac{3}{e^{2x}}+1\right)}=\lim_{x\to+\infty}\frac{\ln e^{3x}+\ln\left(\frac{2}{e^{3x}}+1\right)}{\ln e^{2x}+\ln\left(\frac{3}{e^{2x}}+1\right)}=$$

$$\lim_{x\to+\infty}\frac{3x+\ln\left(\frac{2}{e^{3x}}+1\right)}{2x+\ln\left(\frac{3}{e^{2x}}+1\right)}=\lim_{x\to+\infty}\frac{x\left(3+\frac{1}{x}\cdot\ln\left(\frac{2}{e^{3x}}+1\right)\right)}{x\cdot\left(2+\frac{1}{x}\cdot\ln\left(\frac{3}{e^{2x}}+1\right)\right)}=\frac{3}{2}$$

(20.) $$\lim_{x\to 0}(1+2x)^{\frac{5}{x}}=\lim_{x\to 0}(1+2x)^{\frac{1}{2x}\cdot 2x\cdot\frac{5}{x}}\ \underset{(*)}{=}\ e^{10}$$

$$(*)\ \lim_{x\to 0}(1+2x)^{\frac{1}{2x}}=\ulcorner 2x=z,\ x\to 0,\ z\to 0\quad=$$

$$=\lim_{z\to 0}(1+z)^{\frac{1}{z}}=e$$

21. $\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x = \lim_{x\to+\infty}\left(1+\frac{x+1}{x-1}-1\right)^x = \lim_{x\to+\infty}\left(1+\frac{2}{x-1}\right)^x$

$\lim_{x\to+\infty}\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}\cdot\frac{2}{x-1}\cdot x} = \lim_{x\to+\infty}\left(\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}}\right)^{\frac{2x}{x-1}} =$

$e^{\lim_{x\to+\infty}\frac{2x}{x-1}} = e^{\lim_{x\to+\infty}\frac{2x}{x(1-\frac{1}{x})}} = e^2$

22. $\lim_{x\to 0}\left(\frac{2^x+5^x}{2}\right)^{\frac{1}{x}} = \lim_{x\to 0}\left(1+\frac{2^x+5^x}{2}-1\right)^{\frac{1}{x}} =$

$\lim_{x\to 0}\left(1+\frac{2^x+5^x-2}{2}\right)^{\frac{1}{x}} = \lim_{x\to 0}\left(1+\frac{2^x+5^x-2}{2}\right)^{\frac{2}{2^x+5^x-2}\cdot\frac{2^x+5^x-2}{2}\cdot\frac{1}{x}} =$

$e^{\lim_{x\to 0}\frac{2^x+5^x-2}{2x}} \overset{(*)}{=} e^{\ln\sqrt{10}} = \sqrt{10}$

$\overset{(*)}{=} \lim_{x\to 0}\frac{2^x+5^x-2}{2x} = \lim_{x\to 0}\frac{2^x-1+5^x-1}{2x} =$

$\frac{1}{2}\left(\lim_{x\to 0}\frac{2^x-1}{x} + \lim_{x\to 0}\frac{5^x-1}{x}\right) = \frac{1}{2}(\ln 2+\ln 5) = \frac{1}{2}\ln 10 =$

$\lim_{x\to 0}\frac{a^x-1}{x} = \ln a \quad = \lim_{x\to 0}\sqrt{10}$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$$

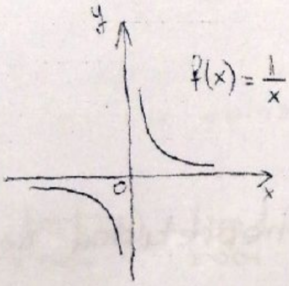
① $$\lim_{x\to 0}\frac{\ln(\cos 5x)}{\ln(\cos 7x)}=\lim_{x\to 0}\frac{\frac{\ln(1+\cos 5x-1)}{\cos 5x-1}\cdot(\cos 5x-1)}{\frac{\ln(1+\cos 7x-1)}{\cos 7x-1}\cdot(\cos 7x-1)}=$$

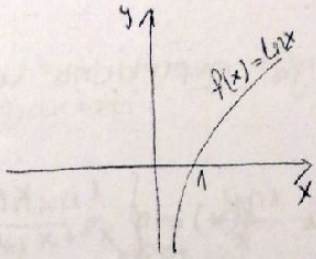
$$\lim_{x\to 0}\frac{\frac{\ln(1+\cos 5x-1)}{\cos 5x-1}}{\frac{\ln(1+\cos 7x-1)}{\cos 7x-1}}=\lim_{x\to 0}\frac{\cos 5x-1}{\cos 7x-1}=1\lim_{x\to 0}\frac{\overbrace{\frac{1-\cos 5x}{25x^2}}^{\to\frac{1}{2}}\cdot 25x^2}{\underbrace{\frac{1-\cos 7x}{49x^2}}_{\frac{1}{2}}\cdot 49x^2}=$$

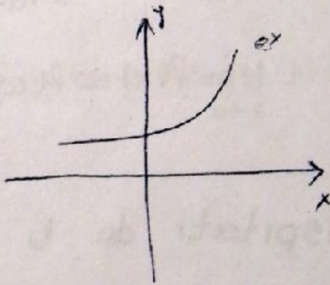
$$\lim_{x\to 0}\frac{\frac{1}{2}}{\frac{1}{2}}\cdot\frac{25}{49}=\frac{25}{49}$$

② Nači lijevu i desnu graničnu vrijednost funkcije $f(x)=\frac{2}{x-3}$ u točki $x=3$

$$\lim_{x\to 3+}f(x)=\lim_{x\to 3+}\frac{2}{x-3}=\lim_{\varepsilon\to 0+}\frac{2}{3+\varepsilon-3}=2\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}=+\infty$$







$D_f=\mathbb{R}\setminus\{0\}$, stalno opada

$\lim_{x\to 0+}\frac{1}{x}=+\infty$

$\lim_{x\to 0-}\frac{1}{x}=-\infty$

$\lim_{x\to +\infty}\frac{1}{x}=0$

$\lim_{x\to -\infty}\frac{1}{x}=0$

$D_f=\mathbb{R}^+$ rastuća

$\lim_{x\to 0+}\ln x=-\infty$

$\lim_{x\to +\infty}\ln x=+\infty$

$\ln x>0,\ x>1$

$\ln x<0,\ 0<x<1$

$D_f=\mathbb{R}$ rastuća

$\lim_{x\to +\infty}e^x=+\infty$

$\lim_{x\to -\infty}e^x=0$

$e^x>0,\ \forall x\in\mathbb{R}$

Drugi način: $\lim_{x\to 3+}\frac{2}{x-3}=2\cdot\lim_{x\to 3+}\frac{1}{x-3}=+\infty$

$$\lim_{x\to 3-}f(x)=\lim_{x\to 3-}\frac{2}{x-3}=\lim_{\varepsilon\to 0+}\frac{2}{3-\varepsilon+3}=-2\cdot\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}=-\infty$$