1 Odrediti prvi i drugi izvod za funkciju

$x^3-y^3=2$ gdje je $y=f(x)$. Naći y' i y'' u tački $x=1$

$x^3-y^3=2$

$3x^2-3y^2\cdot y'=0$

$x^2-y^2\cdot y'=0$

$y'=\frac{x^2}{y^2}$

$$y''=(y')'=\left(\frac{x^2}{y^2}\right)'=\frac{2x\cdot y^2-x^2\,2y\cdot y'}{y^4}=\frac{2xy-2x^2\cdot y'}{y^3}=\frac{2xy-2x^2\cdot\frac{x^2}{y^2}}{y^3}=$$

$$=\frac{\frac{2xy^3-2x^4}{y^2}}{y^3}=\frac{2xy^3-2x^4}{y^5}$$

$x=1 \quad 1^3-y^3=2$

$y^3=-1$

$y=-1$

$y'(1)=\frac{1^2}{(-1)^2}=\frac{1}{1}=1$

$y''(1)=\frac{2\cdot1(-1)^3-2\cdot1^4}{(-1)^5}=\frac{-2-2}{-1}=4$

$dy=y'\cdot dx$

$d^{(n)}y=y^{(n)}\cdot dx^n$

$y^{(n)}=\frac{d^n\cdot y}{dx^n}$

2. Naći $\frac{d^2y}{dx^2}$ za funkciju zadatu imp. jednačinom $\operatorname{arctg} y = y - x$

$d^2y = y'' \cdot dx^2$

$y'' = \frac{d^2y}{dx^2}$

$\operatorname{arctg} y = y - x$

$\frac{1}{1+y^2} \cdot y' = y' - 1$

$\left(1 - \frac{1}{1+y^2}\right) \cdot y' = 1$

$\frac{y^2}{1+y^2} \cdot y' = 1$

$y' = \frac{1+y^2}{y^2}$

$y' = 1 + \frac{1}{y^2}$

$y'' = \left(1 + \frac{1}{y^2}\right)' = -2y^{-3} \cdot y' = \frac{-2}{y^3} \cdot \frac{1+y^2}{y^2} =$

$= -2 \cdot \frac{1+y^2}{y^5}$

$\frac{d^2y}{dx^2}$

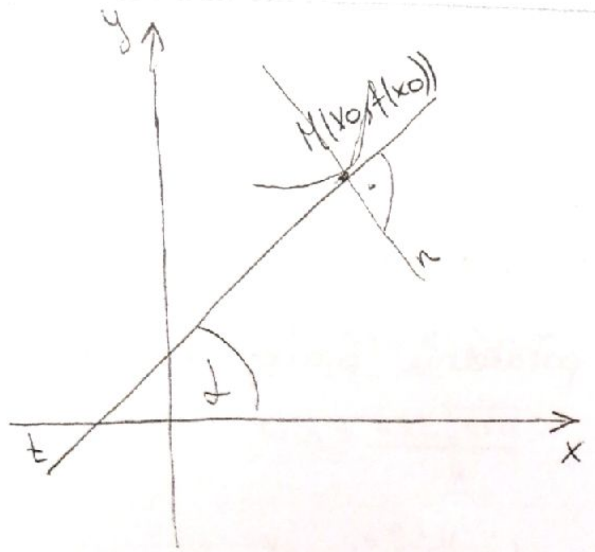
③ Tangente i normale

- Neka funkcija f ima izvod u tački x_0

Ako je $0 \leq \alpha \leq \pi$ ugao između tangente u tački $M(x_0, f(x_0))$ i pozitivnog dijela Ox-ose, tada važi:

$tg\alpha = f'(x_0)$

$k = f'(x_0)$ koeficijent pravca tangente na grafiku u tački $M(x_0, f(x_0))$



Jednačina tangente u tački $M(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)\,(x - x_0)$$

koef. pravca tange(nte)
$k_1 = f'(x_0)$

Jednačina normale u tački $M(x_0, f(x_0))$

$$y - f(x_0) = -\frac{1}{f'(x_0)} \cdot (x - x_0)$$

koef. pravca normale
$k_2 = \frac{-1}{f'(x_0)}$

- Napomena:

$y = k_1 x + n_1$

$y = k_2 x + n_2$

$k_1 k_2 = -1$ prave su ortogenalne

$k_1 = k_2$ prave su paralelne

(1.) Naći jednačinu tangente i normale na grafik funkcije $f(x) = \sqrt{x}$ u tački za koju je $x_0 = 4$

$M(x_0, f(x_0))$

$M(4, f(2))$

$f(x) = \sqrt{4} = 2$

$M(4, 2)$

$t: \; y - 2 = f(4) \cdot (x - 4)$

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$

$y - 2 = \frac{1}{4}(x - 4)$

$t: \; y = \frac{1}{4}x + 1$

$k_t = \frac{1}{4}$

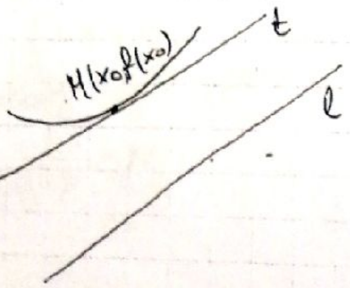
$n: \; y - 2 = \frac{-1}{f'(4)} \cdot (x - 4)$

$y - 2 = -4 \cdot (x - 4)$

$y = -4x + 18$

$k_n = -4 \qquad k_t \cdot k_n = -1$

(2.) Na krivoj $y = x^2 + 3x - 4$ naći tangentu paralelnu pravoj $l: y = 3x - 3$.



$t \parallel l \Rightarrow k_t = k_l$

$k_t = 3$

$f'(x_0) = 3$

$y' = 2x + 3$

$y'(x_0) = 2x_0 + 3$

$y'(x_0) = 3 \Rightarrow 2x_0 + 3 = 3$

$2x_0 = 0$

$x_0 = 0$

$f(x_0) = f(0) = -4$

$M(0, f(0))$

$M(0, -4)$

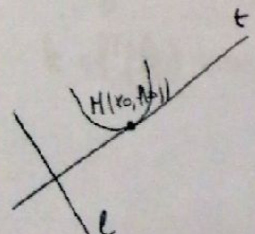
$t: y - (-4) = 3(x - 0)$

$t: y + 4 = 3x$

$t: y = 3x - 4$

Iz prave koeficijent!

(3.) Napisati jednačinu tangente krive $y = x^3 + 3x^2 - 5$ koja je ortogonalna na pravu $l: y = -\frac{1}{9}x + 2$



$t \perp l \Rightarrow k_t \cdot k_l = -1$

$k_t = -\frac{1}{k_l} \qquad k_l = -\frac{1}{9}$

$k_t = -(-9) = 9$

$k_t = f'(x_0)$

$f'(x_0) = 9$

$y' = 3x^2 + 6x$

$y'(x_0) = 3x_0^2 + 6x_0$

$f'(x_0) = 9 \Rightarrow \quad 3x_0^2 + 6x_0 = 9$

$x_0^2 + 2x_0 - 3 = 0$

$x_{0_{1,2}} = \dfrac{-2 \pm \sqrt{4+12}}{2}$

$x_{0_1} = \dfrac{-2+4}{2} = 1$

$x_{0_2} = -3$

1. $x_0 = 1$

$f(1) = 1^3 + 3 \cdot 1^2 - 5 = -1$

$M(1, -1)$

$t: y - (-1) = 9 \cdot (x - 1)$

$t: y = 9x - 10$

2. $x_0 = -3$

$f(-3) = (-3)^3 + 3 \cdot (-3)^2 - 5$

$f(-3) = -27 + 27 - 5 = 5$

$M(-3, f(-3))$

$M(-3, -5)$

$t: y - (-5) = 9\,(x - (-3))$

$t: y = 9x + 22$

4. Odrediti ugao pod kojim se sijeku krive $y = (x-2)^2$ i $y = -x^2 + 4x + 4$

- Nađemo presječne tačke krivih

$$\begin{cases} y = (x-2)^2 \\ y = x^2 + 4x + 4 \end{cases}$$

$x^2 - 4x + 4 = -x^2 + 4x + 4$

$2x^2 - 8x = 0$

$x^2 - 4x = 0$

$x(x-4) = 0 \Leftrightarrow x = 0 \lor x = 4$

prvi slučaj
ako je
1° $x=0$

$$y=f(0)=(0-2)^2=4$$

$$M(0,4)$$

* Ugao između krivih je ugao između tangenti na te krive postavljenih na presječnoj tački.

Napomena: Kada imamo $y=k_1x+n_1$, $y=k_2x+n$ $\quad tg\alpha=\dfrac{k_2-k_1}{1+k_1k_2}$

koeficijent pravca tangente u tački $M(0,4)$ na krivu $y=(x-2)^2$ je

$$k_1=y'(0)$$

$$y'=2(x-2)\cdot 1$$

$$y'(0)=2\cdot(-2)=-4$$

$$k_1=-4$$

koeficijent pravca tangente u tački $M(0,4)$ na krivu $y=-x^2+4x+4$

$$k_2=y'(0)$$

$$y'=-2x+4$$

$$y'(0)=4$$

$$k_2=4$$

$$tg\alpha=\frac{k_2-k_1}{1+k_1\cdot k_2}$$

$$tg\alpha=\frac{4-(-4)}{1+4(-4)}=\frac{8}{-15}$$

Drugi slučaj
2° $x_0=4$

$k_1=y'(4)=4$

$k_2=y'(4)=-4$

$$tg\alpha=\frac{k_2-k_1}{1+k_1k_2}=\frac{-4-4}{1+4\cdot(-4)}=\frac{-8}{-15}=\frac{8}{15}$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

se svode na $\frac{0}{0}$ ili $\frac{\infty}{\infty}$

Lopitalovo pravilo

Neodređeni izrazi: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$, 1^{∞}, 0^0, ∞^0

1. a) $$\lim_{x\to 0} \frac{x\cdot ctgx - 1}{x^2} = \lim_{x\to 0} \frac{x\cdot\frac{\cos x}{\sin x} - 1}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{x\cdot\cos x - \sin x}{\sin x}}{x^2} = \lim_{x\to 0} \frac{x\cdot\cos x - \sin x}{x^2 \cdot \sin x} \overset{\frac{0}{0}}{\underset{L.P}{=}}$$

$$\lim_{x\to 0} \frac{1\cdot\cos + x(-\sin) - \cos x}{2x\sin x + x^2\cdot\cos x} = \lim_{x\to 0} \frac{-x\cdot\sin x}{x(2\sin x + x\cdot\cos x)}$$

$$\lim_{x\to 0} \frac{-\sin x}{2\cdot\sin x + x\cdot\cos x} \overset{\frac{0}{0}}{\underset{L.P.}{=}} \lim_{x\to\infty} \frac{-\cos x}{2\cos x + 1\cdot\cos x + x\cdot(-\sin x)} =$$

$$\lim_{x\to 0} \frac{-\cos x}{3\cos x - x\cdot\sin x} = \frac{-1}{3}$$

2. $$\lim_{x\to+\infty} \frac{\frac{\pi}{2} - arctgx}{\ln\left(1+\frac{1}{x^2}\right)} \overset{\frac{0}{0}}{\underset{L.P}{=}} \lim_{x\to+\infty} \frac{-\frac{1}{1+x^2}}{\frac{1}{1+\frac{1}{x^2}}\cdot\frac{-2}{x^3}} = \lim_{x\to+\infty} \frac{-\frac{1}{1+x^2}}{\frac{1}{\frac{x^2+1}{x^2}}\cdot\frac{-2}{x^3}} =$$

$$\lim_{x\to+\infty} \frac{-\frac{1}{1+x^2}}{\frac{x^2}{x^2+1}\cdot\frac{-2}{x^3}} = \frac{1}{2} \lim_{x\to+\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x(x^2+1)}} = \frac{1}{2} \lim_{x\to+\infty} x = +\infty$$

3. a) $$\lim_{x\to+\infty} \frac{3x^2+2x-2}{x^2-1} \overset{\frac{\infty}{\infty}}{\underset{L.P}{=}} \lim_{x\to+\infty} \frac{6x+2}{2x} = \lim_{x\to+\infty} \frac{3x+1}{x} \overset{\frac{\infty}{\infty}}{\underset{L.P}{=}}$$

$$\lim_{x\to+\infty} \frac{3}{1} = 3$$

b) $$\lim_{x\to+\infty} \frac{2\cdot\ln x}{x^b}, \quad b>0$$

$$\lim_{x\to+\infty} \frac{2\ln x}{x^b} \overset{\frac{\infty}{\infty}}{\underset{L.P.}{=}} \lim_{x\to+\infty} \frac{2\cdot\frac{1}{x}}{b\cdot x^{b-1}} = \frac{2}{b}\cdot\lim_{x\to+\infty} \frac{1}{x\cdot x^{b-1}} = \frac{2}{b}\lim_{x\to+\infty} \frac{1}{x^b} = 0$$

∞^0

4. $$\lim_{x\to+\infty}(3x^2+3^x)^{\frac{1}{x}} = e^{\lim\limits_{x\to+\infty}\ln(3x^2+3^x)^{\frac{1}{x}}} = e^{\lim\limits_{x\to+\infty}\frac{1}{x}\cdot\ln(3x^2+3^x)} \overset{(*)}{=} e^{\ln 3} = 3$$

inverzne funk.

$$(*) = \lim_{x\to+\infty}\frac{1}{x}\ln(3x^2+3^x) = \lim_{x\to+\infty}\frac{\ln(3x^2+3^x)}{x} \underset{L.P.}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to+\infty}\frac{\frac{1}{3x^2+3^x}\cdot(6x+3^x\ln 3)}{1}$$

$$= \lim_{x\to+\infty}\frac{6x+3^x\cdot\ln 3}{3x^2+3^x} \underset{L.P.}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to+\infty}\frac{6+3^x\cdot\ln 3\cdot\ln 3}{6x+3^x\cdot\ln 3} \underset{L.P.}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to+\infty}\frac{3^x\cdot\ln 3\cdot(\ln 3)^2}{6+3^x\cdot(\ln 3)^2} \underset{L.P.}{\overset{\frac{\infty}{\infty}}{=}}$$

$$\lim_{x\to+\infty}\frac{3^x\cdot\ln 3\cdot(\ln 3)^3}{3^x\cdot\ln 3\cdot(\ln 3)^2} = \ln 3$$

0^0

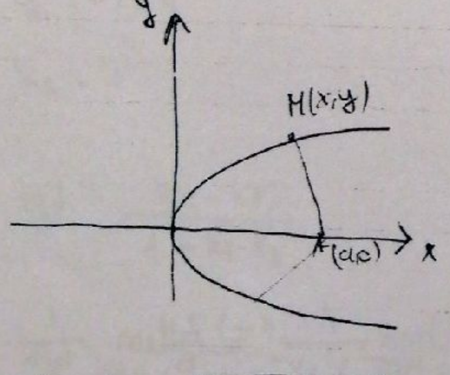
5. $$\lim_{x\to 0+}(e^x-1)^x = e^{\lim\limits_{x\to 0+}\ln(e^x-1)} = e^{\lim\limits_{x\to 0+}x\cdot\ln(e^x-1)} \overset{(*)}{=} e^0 = 1$$

$$(*) = \lim_{x\to 0+}x\cdot\ln(e^x-1) \overset{0\cdot(-\infty)}{=} \lim_{x\to 0+}\frac{\ln(e^x-1)}{\frac{1}{x}} \underset{L.P.}{\overset{\frac{\infty}{+\infty}}{=}} \lim_{x\to 0+}\frac{\frac{1}{e^x-1}\cdot e^x}{-\frac{1}{x^2}} =$$

$$= -\lim_{x\to 0+}\frac{x^2\cdot e^x}{e^x-1} \underset{L.P.}{\overset{\frac{0}{0}}{=}} -\lim_{x\to 0+}\frac{2xe^x\cdot x^2\cdot e^x}{e^x} = -\lim_{x\to 0+}\frac{2x+x^2}{1} = 0$$

Ekstremne vrijednosti

1. Na paraboli $y^2 = 2px$ naći tačku najbližu tački A ~~se~~ $(a,0)$



$$d(M,A) = \sqrt{(x-a)^2+y^2}$$

$$f(x) = \sqrt{(x-a)^2+y^2} = \sqrt{(x-a)^2+2px}$$

Tražimo minimum fu-je f.