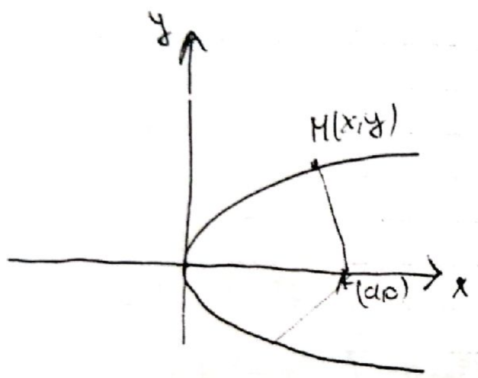
Eksteremne vrijednosti

1. Na paraboli $y^2 = 2px$ naći tačku najbližu tački A ~~sa~~ $(a,0)$



$$d(M,A) = \sqrt{(x-a)^2 + y^2}$$

$$f(x) = \sqrt{(x-a)^2 + y^2} = \sqrt{(x-a)^2 + 2px}$$

Tražimo minimum fu-je f.

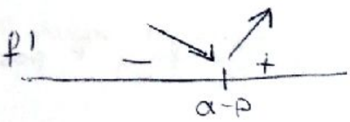
$$f'(x) = \frac{1}{2\sqrt{(x-a)^2 + \cancel{y^2}\, 2px}} \cdot \left(2\cdot(x-a) + 2p\right)$$

$$f'(x) = \frac{x-a+p}{\sqrt{(x-a)^2+2px}}$$

$$f'(x) = 0 \iff x-a+p = 0$$
$$x = a-p$$

$$f'(x) > 0 \iff x-a+p > 0$$
$$\iff x > a-p \quad \nearrow$$

$$f(x) < 0 \iff x-a+p < 0$$
$$\iff x < a-p \quad \searrow$$



U tački $x = a-p$ funkcija f postiže minimum.

$y^2 = 2px$

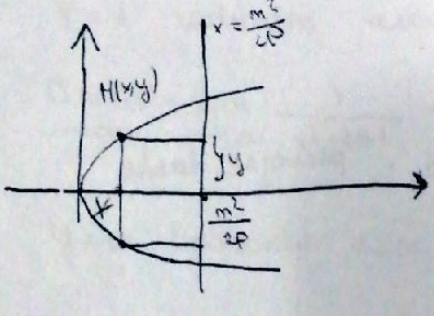
$y^2 = 2p(a-p)$

$y = \pm\sqrt{2p(a-p)}$

$M_1\left(a-p,\ \sqrt{2p(a-p)}\right)$

$M_2\left(a-p,\ -\sqrt{2p(a-p)}\right)$

2. Naći pravougaonik maksimalne površine ograničen parabolom $y^2 = 2px$ i pravom $x = \frac{m^2}{2p}$ ako jedna stranica tog pravougaonika leži na datoj pravoj.



$P = a\cdot b$

$b = 2y$

$a = \frac{m^2}{2p} - x$

$P = \left(\frac{m^2}{2p} - x\right)\cdot 2y$

$P(y) = \left(\frac{m^2}{2p} - \frac{y^2}{2p}\right)\cdot 2y$

$P(y) = \frac{1}{p}\left(m^2 y - y^3\right)$

$$P'(y) = \frac{1}{p}(m^2 - 3y^2)$$

$$P'(y) = 0 \iff m^2 - 3y^2 = 0$$
$$\iff y^2 = \frac{m^2}{3}$$
$$\iff y = \pm \frac{m}{\sqrt{3}}$$

Uzećemo $y = \frac{m}{\sqrt{3}}$ (jer M pripada I kvadrantu)

Tražim drugi izvod

$$P''(y) = \frac{1}{p}(-6y)$$

$$P''\left(\frac{m}{\sqrt{3}}\right) = \frac{1}{p}\left(-6 \cdot \frac{m}{\sqrt{3}}\right) < 0 \iff$$ u tački $y = \frac{m}{\sqrt{3}}$ f-ja P postiže maksimum.

$$P_{max} = P\left(\frac{m}{\sqrt{3}}\right) = \frac{1}{p}\left(m^2 \cdot \frac{m}{\sqrt{3}} - \frac{m^3}{3\sqrt{3}}\right) = \frac{1}{p} m^3 \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}\, m^2}{9p}$$

$$x = \frac{y^2}{2p}$$

$$x = \frac{\frac{m^2}{3}}{2p} = \frac{m^2}{6p} \qquad M\left(\frac{m^2}{6p}, \frac{m\sqrt{3}}{3}\right)$$

Grafik funkcije

1. Oblast def.
2. Parnost
3. Nule funkcije i znak (i presjek sa Oy-osom)
4. ponašanje na krajevima oblasti definisanosti
5. Prvi izvod i tok funkcije, ekstremne vrijednosti
6. Drugi izvod, intervali konveksnosti i konkavnosti, prevojne tačke

Grafik funkcije

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Funkcije

(1.) $y = \dfrac{x}{(x-1)^2}$

Razlomak imenioc $>0 \to \neq$ različit od 0
paran | podkorijena veličina veća od 0
ln od nekog izraza, izraz >0

1. $D_f: (x-1)^2 \neq 0$

$x - 1 \neq 0$

$x \neq 1$

$D_f: (-\infty, 1) \cup (1, +\infty)$

2. $f(-x) = \dfrac{-x}{(-x-1)^2} = \dfrac{-x}{(x+1)^2} \neq f(x)$ $\neq f(-x)$

ni parna ni neparna

3. $y = 0 \iff \dfrac{x}{(x-1)^2} = 0$

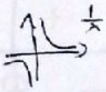
$\iff x = 0 \quad (0,0)$

$\dfrac{x}{(x-1)^2}$

$y > 0 \iff x > 0 \quad x \in (0, +\infty)$

$y < 0 \iff x < 0 \quad x \in (-\infty, 0)$

4. $\lim\limits_{x \to 1+} f(x) = \lim\limits_{x \to 1+} \dfrac{x}{(x-1)^2} = +\infty$



$\lim\limits_{x \to 1-} f(x) = \lim\limits_{x \to 1-} \dfrac{x}{(x-1)^2} = +\infty$

$x = 1$ vertikalna asimptota

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{x}{x^2 - 2x + 1} = \lim\limits_{x \to +\infty} \dfrac{x}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)} = \lim\limits_{x \to +\infty} \dfrac{1}{x} \cdot \dfrac{1}{1 - \frac{2}{x} + \frac{1}{x^2}} = 0$

$y = 0$ horizontal. asim. $(x \to +\infty)$

$$\lim_{x \to -\infty} f(x) = 0 \qquad y = 0 \quad H.A. \quad (x \to -\infty)$$

Napomena: $y=0$ jednačina x-ose

Nema kose asimp. (jer ima H)

5. $y = \dfrac{x}{(x-1)^2}$

$$y' = \frac{1 \cdot (x-1)^2 - x \cdot 2(x-1) \cdot 1}{(x-1)^4}$$

$$y' = \frac{(x-1)[(x-1) - 2x]}{(x-1)^4}$$

$$y' = \frac{-1 - x}{(x-1)^3}$$

$$y' = -\frac{(1+x)}{(x-1)^3} \quad , \quad y' = 0 \iff x = -1$$

y' nije def. u $x = 1$, ali $x = 1 \notin D_f$.

	$-\infty$	-1	1	$+\infty$
$1+x$	$-$	0	$+$	$+$
$(x-1)^3$	$-$		$-$	0 $+$
"$-$"	$-$		$-$	$-$
y'	$-$		$+$	nedef. $-$
y	$\searrow$		$\nearrow$	nedef. $\searrow$

U $x = -1$ f postiže minimum $\quad y_{min} = f(-1) = \dfrac{-1}{(-1-1)^2} = \dfrac{-1}{4}$

$M\left(-1, -\frac{1}{4}\right)$

6. $y' = -\frac{1+x}{(x-1)^3}$

$$y'' = -\frac{1(x-1)^3 - (1+x)\cdot 3(x-1)^2\cdot 1}{(x-1)^6}$$

$$y'' = -\frac{(x-1)^2(x-1-3(1+x))}{\cancel{x-1}\,(x-1)^6}$$

$$y'' = -\frac{-2x-4}{(x-1)^4}$$

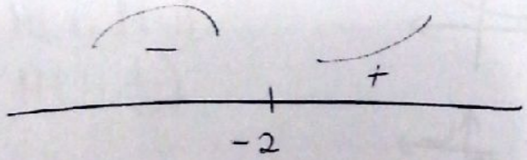
$$y'' = 2\cdot\frac{x+2}{(x-1)^4}$$

$y'' = 0 \iff x = -2$

y'' nije def u $x = 1$, $x = 1 \notin D_f$.

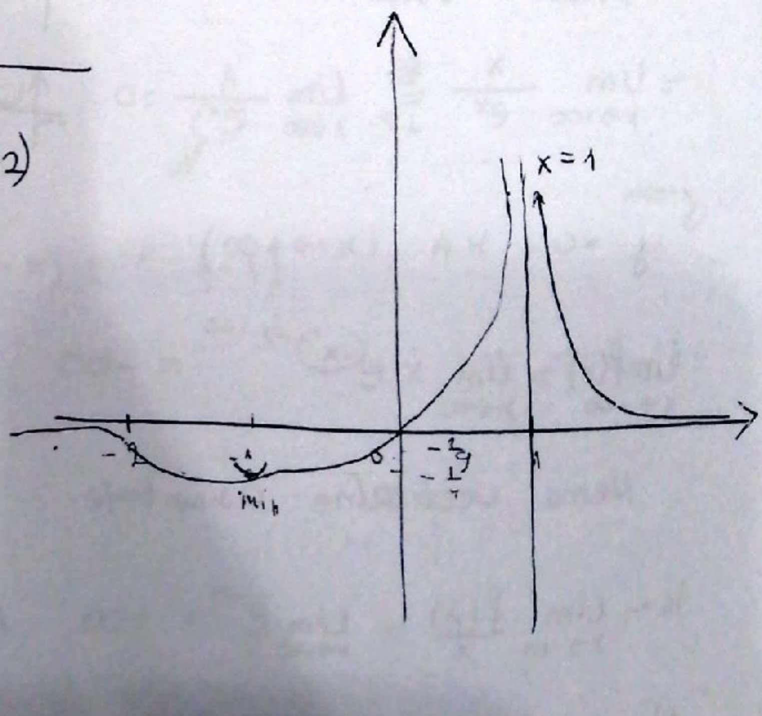
$y'' > 0 \iff x+2 > 0$
$\iff x > -2$

$y'' < 0 \iff x+2 < 0$
$\iff x < -2$



Prevojna tačka $P(-2, f(-2))$

$$f(-2) = \frac{-2}{(-2-1)^2} = -\frac{2}{9}$$



① $y = x \cdot e^{-x}$

1. $Df = \mathbb{R}$

2. $y(-x) = -x \cdot e^{-(-x)} = -x \cdot e^{x} \neq y(x)$
$\neq -y(x)$

y nije parna

y nije neparna

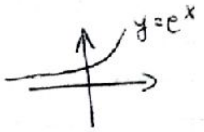
3. $y = 0 \iff x \cdot \frac{1}{\underset{>0}{e^x}} = 0$

$\iff x = 0$

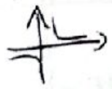
$y > 0 \iff x > 0$

$y < 0 \iff x < 0$

4. $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} x \cdot e^{-x} \overset{\infty \cdot 0}{=}$



$= \lim\limits_{x \to +\infty} \frac{x}{e^x} \underset{L.P.}{\overset{\frac{\infty}{\infty}}{=}} \lim\limits_{x \to +\infty} \frac{1}{\underset{\infty}{e^x}} = 0$



x-osa

$y = 0$ H.A. $(x \to +\infty)$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} x \cdot e^{\overset{+\infty}{-x}} = -\infty$

Nema vertikalne asimptote

$k = \lim\limits_{x \to -\infty} \frac{f(x)}{x} = \lim\limits_{x \to -\infty} e^{-x} = +\infty$ Nema kose asimptote za $x \to -\infty$

Ako nema hor. tražimo kosu!

5. $y = x \cdot e^{-x}$

$y = e^{-x} + x \cdot e^{-x} \cdot (-1)$

$y' = (1-x) \cdot e^{-x}$ ili

$y' = \dfrac{1-x}{e^{x}}$

$y' = 0 \iff 1 - x = 0$

$\iff x = 1$

y' je definisano za $\forall x \in R$

$y' > 0 \iff 1 - x > 0$

$\iff x < 1$

$y' < 0 \iff 1 - x < 0$

$\iff x > 1$

y'	$+$ $\nearrow$	1	$-$ $\searrow$

U tački $x = 1$ f-ja postiže maksimum. To je $f_{max} = y_{max} = f(1)=$

$1 \cdot e^{-1} = \dfrac{1}{e}$

$M(1, f(1))$

$M\left(1, \dfrac{1}{e}\right)$

⑥ $y' = (1-x) \cdot e^{-x}$

$y'' = (y')' = (-1) \cdot e^{-x} + (1 - x) \cdot e^{-x} \cdot (-1)$

$y'' = (-1 - 1 \cdot (1-x)) \cdot e^{-x}$

$y'' = (x-2) \cdot e^{-x}$

$y'' = \dfrac{x-2}{e^{x}}$

$y'' = 0 \iff x-2=0$
$\iff x=2$

y'' je definisano za $\forall x \in R$

$y'' > 0 \iff x-2>0$
$\iff x>2$

$y''<0 \iff x-2<0$
$x<2$

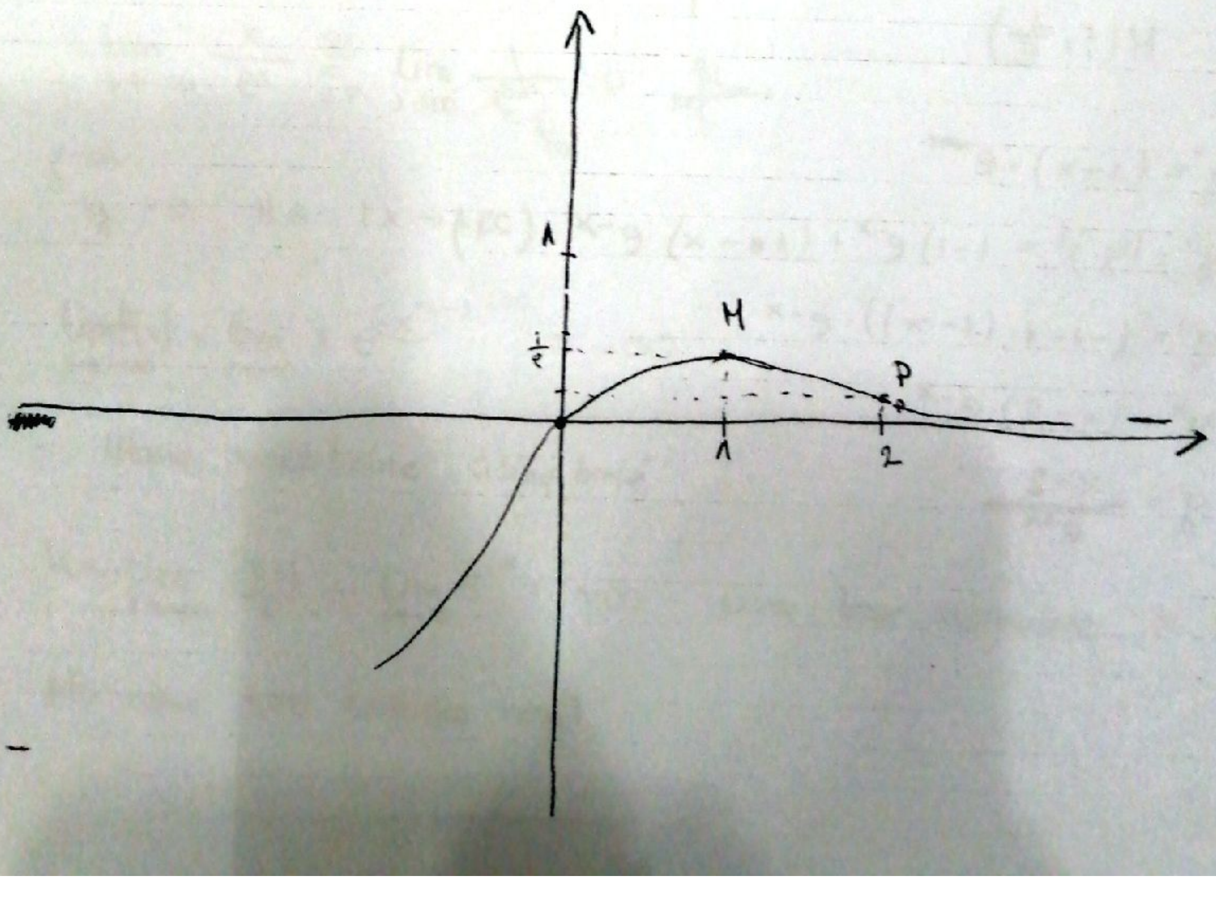
konkavna konveksna

y'' — 2 +

$P(2, f(2))$

$f(2) = 2 \cdot e^{-2} = \frac{2}{e^2}$

$P\left(2, \frac{2}{e^2}\right)$



②

$$y = \frac{6x - x^2 - 9}{x-2}$$

$$y = \frac{-x^2 + 6x - 9}{x-2}$$

$$y = \frac{-(x-3)^2}{x-2} = \frac{(x-3)^2}{2-x}$$

1. $D_f \cdot \; x - 2 \neq 0$

$x \neq 2$

$D_f \; (-\infty, 2) \cup (2, +\infty)$

2. $y(-x) = \frac{-(-x-3)^2}{-x-2} = \frac{(x+3)^2}{x+2} \neq y(x)$

$\neq -y(x)$

nit je parna, nit neparna

3. $y = 0 \iff -\frac{(x-3)^2}{x-2} = 0$

$\iff x = 3 \qquad N(3,0)$

$y > 0 \iff 2 - x > 0$

$\iff x < 2$

$y < 0 \iff 2 - x < 0$

$x > 2$

4. $\lim\limits_{x \to 2+} f(x) = \lim\limits_{x \to 2+} \frac{(x-3)^2}{\underbrace{2 - x}_{0-}} = -\infty$

$x = 2 \quad$ V. A.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{(x-3)^2}{2-x} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(x-3)^2}{2-x} = \lim_{x\to+\infty} \frac{x^2\left(1-\frac{6}{x}+\frac{9}{x^2}\right)}{x\left(\frac{2}{x}-1\right)} = \lim_{x\to+\infty} \frac{x\cdot\left(1-\frac{6}{x}+\frac{9}{x^2}\right)}{\frac{2}{x}-1} = -\infty$$

nema konačnog br. nema H. A.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x\cdot\left(1-\frac{6}{x}+\frac{9}{x^2}\right)}{\frac{2}{x}-1} = +\infty$$

Nema H. A.

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{(x-3)^2}{x\cdot(2-x)} = \lim_{x\to\pm\infty} \frac{x^2\left(1-\frac{6}{x}+\frac{9}{x^2}\right)}{x^2\left(\frac{2}{x}-1\right)} = \frac{1}{-1} = -1$$

$$n = \lim_{x\to+\infty} \left(f(x)-kx\right)$$

$$n = \lim_{x\to\pm\infty} \left(\frac{(x-3)^2}{2-x} - (-1)\cdot x\right) = \lim_{x\to\pm\infty} \frac{x^2-6x+9+2x-x^2}{2-x} =$$

$$= \lim_{x\to\pm\infty} \frac{9-4x}{2-x} = \lim_{x\to\pm\infty} \frac{x\left(\frac{9}{x}-4\right)}{x\left(\frac{2}{x}-1\right)} = 4$$

$y = kx + n$

$y = -x + 4$ K. A.

(5.) $y = \dfrac{(x-3)^2}{2-x}$

$$y' = \frac{2\cdot(x-3)(2-x) - (x-3)^2(-1)}{(2-x)^2}$$

$$y' = \frac{(x-3)[2\cdot(2-x)+(x-3)]}{(2-x)^2}$$

$$y' = \frac{(x-3)\cdot(1-x)}{(2-x)^2}$$

$$y'=0 \iff (x-3)(1-x)=0$$
$$\iff x=3 \lor x=1$$

y' nije definisano za $x=2$ ali $x=2 \notin D_f$.

	$-\infty$	1		3	$+\infty$
$x-3$	$-$		$-$	0	$+$
$1-x$	$+$	0	$-$		$-$
y'	$-$	0	$+$	0	$-$
y	$\searrow$		$\nearrow$		$\searrow$

U $x=1$ f postiže minimum

$$y_{min} = f(1) = \frac{(1-3)^2}{2-1} = 4$$

$M_1(1,4)$

U $x=3$ f postiže maksimum

$$y_{max} = f(3) = \frac{(3-3)^2}{2-3} = 0$$

$M_2(3,0)$

6. $y' = \dfrac{-x^2+4x-3}{(2-x)^2}$

$$y'' = \frac{(-2x+4)(2-x)^2 - (-x^2+4x-3)\cdot 2(2-x)\cdot(-1)}{(2-x)^4}$$

$$y'' = \frac{2(2-x)\left[(2-x)^2 + (-x^2+4x-3)\right]}{(2-x)^4}$$

$$y'' = 2 \cdot \frac{4-4x+x^2-x^2+4x-3}{(2-x)^3} = \frac{2}{(2-x)^3}$$

$y'' \neq 0 \quad , \forall x \in Df$

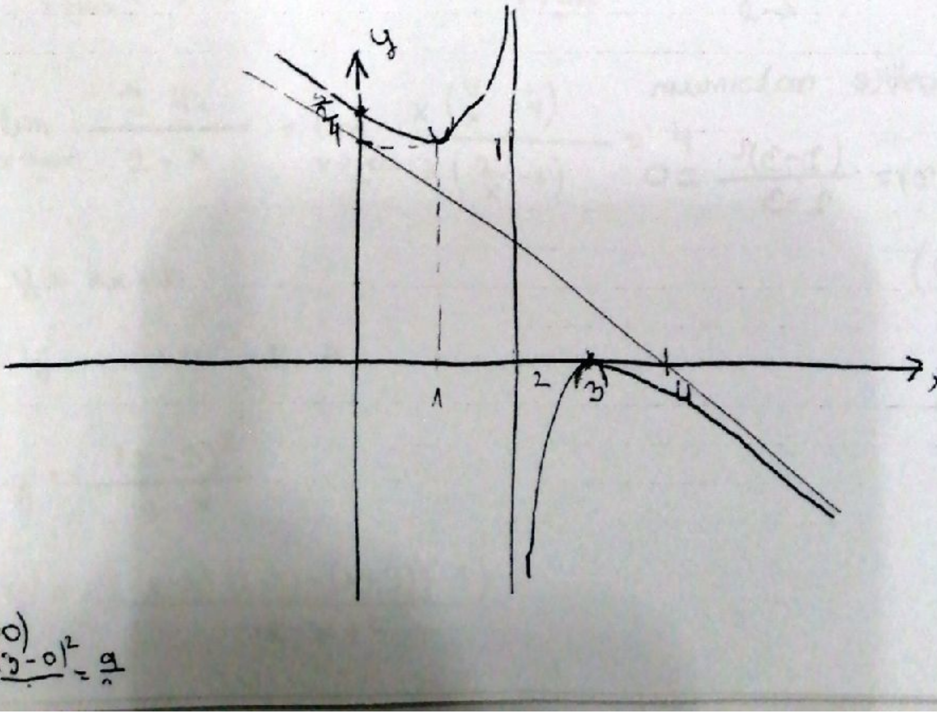
y'' nije definisano za $x=2$, ali $x=2 \notin Df$

$y''>0 \iff 2-x>0$
$\iff x<2$

$y''<0 \iff 2-x<0$
$x>2$

y'' + 2 −

Nemamo prevojnu tačku.



$\cap Oy\ (x=0)$
$y(0) = \frac{(3-0)^2}{\cdot} = \frac{9}{\cdot}$

(*)

	$-\infty$		1		2		3		$+\infty$
y		+		+	nedef.	—	0	—	
y'		—	0	+	nedef.	+	0	—	
y''		+		+	nedef.	—		—	

(5.) $y = \dfrac{\ln x + 1}{x}$

1. $Df: \ x \neq 0 \ \wedge \ x > 0$

$Df: (0, +\infty)$

2. nit parna nit neparna
$\Downarrow$ x-osu $\Downarrow$ koordinantni početak

3. $y = 0 \iff \ln x + 1 = 0$
$\iff \ln x = -1$
$\iff x = e^{-1}$
$x = \frac{1}{e}$

$y > 0 \iff \dfrac{\ln x + 1}{\underset{>0 \ \forall x \in Df}{x}} > 0$

$y > 0 \iff \ln x + 1 > 0$
$\iff \ln x > -1$
$\iff \ln x > \ln e^{-1}$
$\iff x > e^{-1}$
$\iff x \in \left(\frac{1}{e}, +\infty\right)$

$y < 0 \iff \ln x + 1 < 0$
$\iff \ln x < -1$
$\iff \ln x < \ln e^{-1}$
$\iff x < \frac{1}{e}$
$x \in \left(0, \frac{1}{e}\right)$

4. $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\ln x+1}{x} = \lim_{x\to 0+} \underbrace{(\ln x+1)}_{-\infty} \cdot \underbrace{\frac{1}{x}}_{+\infty} = -\infty$

$x=0$ vertikalna asimptota

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln x+1}{x} \overset{\frac{\infty}{\infty}}{\underset{LP}{=}} \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0$

$x\to\infty)$ $y=0$ H.A. $x\to+\infty$

Nema kose

5. $y = \frac{\ln x+1}{x}$

$y' = \frac{\frac{1}{x}\cdot x - (\ln x+1)\cdot 1}{x^2}$

$y' = \frac{1-\ln x-1}{x^2}$

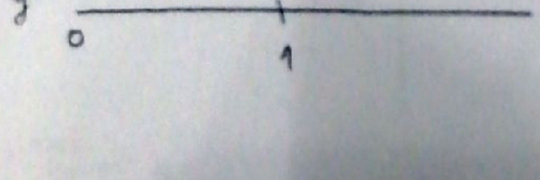
$y' = -\frac{\ln x}{x^2}$

$y'=0 \iff \ln x=0$
$\iff x=1$

y' nije definirana za $x=0$ ali $x=0 \notin D_f$

$y'>0 \iff \ln x<0$
$\iff x<1$
$x\in(0,1)$

$y'<0 \iff \ln x>0$
$\iff x>1$



U točki $x=1$ f postiže maksimum

$y_{max} = f(1) = \frac{\ln 1+1}{1} = \frac{0+1}{1} = 1$

$M(1,1)$

→ 4.

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} \frac{x}{\underbrace{(x-1)}_{0+}(x+1)} = +\infty$$

$x = 1$ vertikalna A.

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} \frac{x}{\underbrace{(x-1)}_{0-}(x+1)} = -\infty$$

$$\lim_{x\to -1+} f(x) = \lim_{x\to -1+} \frac{x}{(x-1)\underbrace{(x+1)}_{0+}} = +\infty$$

$$\lim_{x\to -1-} f(x) = \lim_{x\to -1-} \frac{x}{(x-1)\underbrace{(x+1)}_{0-}} = -\infty$$

$\frac{-1}{-1(0)+1} \to -1$

$x = -1$ V.A.

$$\lim_{x\to \pm\infty} f(x) = \lim_{x\to \pm\infty} \frac{x}{x^2-1} = \lim_{x\to \pm\infty} \frac{x}{x^2(1-\frac{1}{x^2})} = \lim_{x\to \pm\infty} \overbrace{\frac{1}{x}}^{0} \cdot \frac{1}{1-\underbrace{\frac{1}{x^2}}_{0}} = 0$$

$y = 0$ H.A., $x \to +\infty$

također $\lim_{x\to -\infty} f(x) = 0$, $y = 0$ H.A, $x \to -\infty$

Nema kose asimptote!

→ 5. $y = \frac{x}{x^2-1}$

$$y' = \frac{x^2-1-x\cdot 2x}{(x^2-1)^2}$$

$$y' = \frac{-x^2-1}{(x^2-1)^2}$$

$$y' = -\frac{x^2+1}{(x-1)^2(x+1)^2}$$

$y' = 0 \Leftrightarrow x^2+1 = 0$ nemoguće

y' nije def. u $x=1$ i $x=-1$, $x=1 \notin D_f$
$x=-1 \notin D_f$

Uočimo $y' = - \dfrac{\overbrace{(x^2+1)}^{>0}}{\underbrace{(x-1)^2}_{>0}\underbrace{(x+1)^2}_{>0}}$ $\forall x \in D_f$

$y' < 0$, $\forall x \in D_f$

y stalno opada i ne postiže ekstremum.

→ 6. $y' = - \dfrac{x^2+1}{(x^2-1)^2}$

$y'' = - \dfrac{2x(x^2-1) - (x^2+1)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4}$

$y'' = -2 \dfrac{x\cancel{(x^2-1)}\cdot[x^2-1-2(x^2+1)]}{(x^2-1)^{\cancel{4}\,3}}$

$y'' = -2 \cdot \dfrac{x\cdot(x^2-1-2x^2-2)}{(x^2-1)^3} = -2 \cdot \dfrac{x\cdot(-x^2-3)}{(x^2-1)^3}$

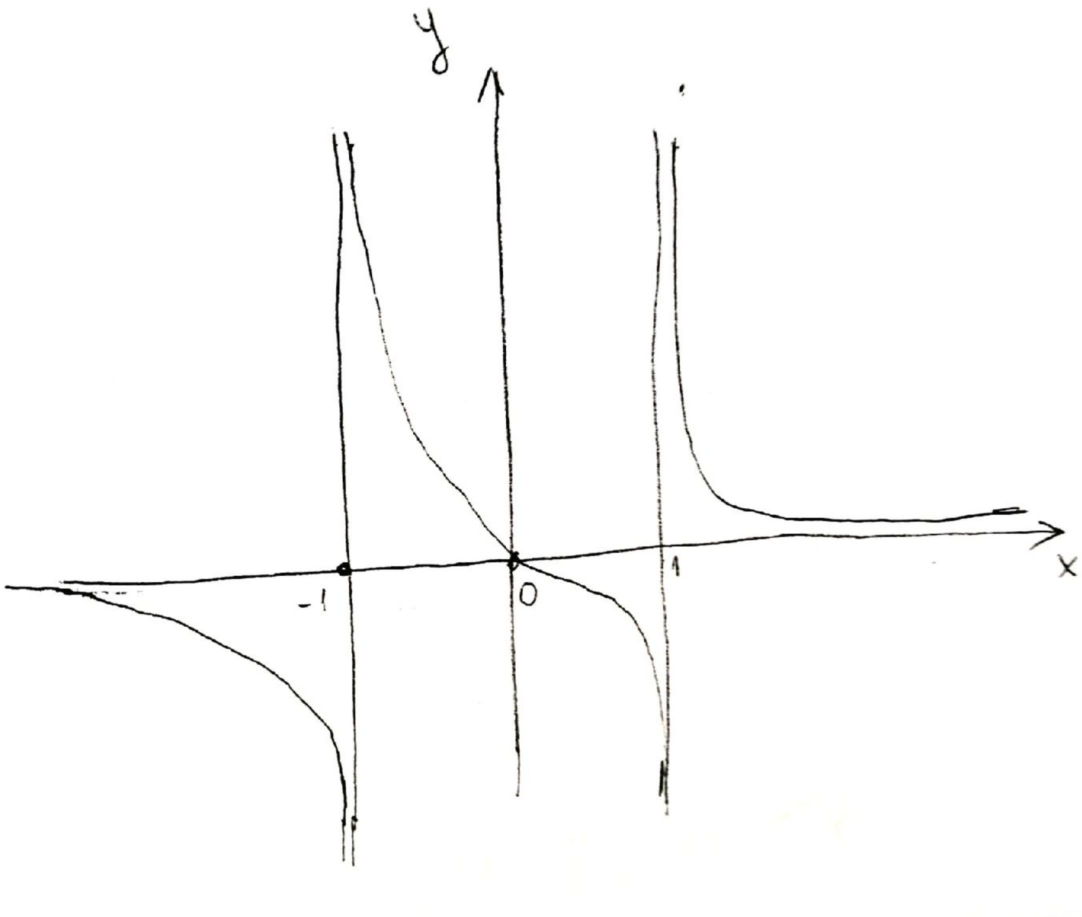
$y'' = 2 \cdot \dfrac{x(x^2+3)}{(x-1)^3\cdot(x+1)^3}$

$y''=0 \iff x=0$

y'' nije definisano u $x=1$ i $x=-1$ ali $x=1 \notin D_f$
$x=-1 \notin D_f$

	$-\infty$	-1		0		1		$-\infty$
x	$-$		$-$	0	$+$		$+$	$+$
$x-1$	$-$		$-$		$-$	0	$+$	$+$
$x+1$	$-$	0	$+$		$+$		$+$	$+$
y''	$-$	nedef.	$+$	0	$-$	nedef.	$+$	
y		nedef.		prevoj		nedef.		

⇓
nemamo prevoj


y
x
-1
0
1