Neodređeni integrali

$\int f(x)\,dx = F(x) + C$, F primitivna f-ja f-je $f(x)$

$F'(x) = f(x)$

* Tablica osnovnih integrala *

1° $\int dx = x + C$

2° $\int 0 \cdot dx = const.$

3 $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C$

4 $\int \sin x \cdot dx = -\cos x + C$

5 $\int \cos x \cdot dx = \sin x + C$

6 $\int \frac{dx}{\cos^2 x} = tg\,x + C$

$\int \frac{dx}{\sin^2 x} = -ctg\,x + C$

8) $\int \frac{dx}{x} = \ln|x| + C$, $x \neq 0$

9) $\int e^x dx = e^x + C$

10 $\int a^x \cdot dx = \frac{a^x}{\ln a} + C$, $a > 0$, $a \neq 1$

11 $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$, $|x| < 1$

12 $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln|x + \sqrt{x^2 \pm 1}|$

(13) $$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$$

(14) $$\int \frac{dx}{1-x^2} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C$$

Osobine:

konstanta

(1) $$\int (c \cdot f)\, dx = c \cdot \int f(x)\, dx$$

(2) $$\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Osobine :

$$\int (c\, f(x))\, dx = c \cdot \int f(x)\,dx$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

- Metoda dekompozicije

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

1. $\int (2x^3 + 5x^2)\,dx = 2\int x^3 dx + 5\int x^2 dx =$

$= 2 \cdot \frac{x^4}{4} + 5 \cdot \frac{x^3}{3} + C = \frac{1}{2}x^4 + \frac{5}{3}x^3 + C$

2. $\int \frac{5x^4 + 2x^5 - x + 6}{x^3}\,dx = 5\int x\,dx + 2\int x^2 dx - \int \frac{1}{x^2}\,dx + 6\int \frac{1}{x^3}\,dx =$

$$= 5\,\frac{x^2}{2} + 2\,\frac{x^3}{3} - \int x^{-2}\,dx + 6\int x^{-3}\,dx = \frac{5}{2}x^2 + \frac{2}{3}x^3 - \frac{x^{-1}}{-1} + 6\,\frac{x^{-2}}{-2} + C$$

$$= \frac{5}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{x} - 3\,\frac{1}{x^2} + C$$

③ $$\int \frac{\sqrt{x} - 2\cdot\sqrt[3]{x^2} + 1}{\sqrt[4]{x}}\,dx = \int \frac{x^{\frac{1}{2}}}{x^{\frac{1}{4}}}\,dx - 2\int \frac{x^{\frac{2}{3}}}{x^{\frac{1}{4}}}\,dx + \int \frac{1}{x^{\frac{1}{4}}}\,dx =$$

$$\int x^{\frac{1}{4}}\,dx - 2\int x^{\frac{5}{12}}\,dx + \int x^{-\frac{1}{4}}\cdot dx = \frac{4}{5}\cdot x^{\frac{5}{4}} - 2\cdot\frac{12}{17}x^{\frac{17}{12}} +$$

$$\frac{4}{3}x^{\frac{3}{4}} + C = \frac{4}{5}\cdot\sqrt[4]{x^5} - \frac{24}{17}\cdot\sqrt[12]{x^{17}} + \frac{4}{3}\sqrt[4]{x^3} + C$$

④ $$\int\left(x^2 + x + \frac{1}{x} + \sin x\right)dx = \int x^2\,dx + \int x\,dx + \int \frac{1}{x}\,dx + \int \sin x\,dx =$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + \ln x + (-\cos x) + C$$

⑤ $$\int \frac{x^3 + x - 2}{x^2 + 1}\,dx = \int\left(\frac{x^3 + x}{x^2 + 1} - \frac{2}{x^2 + 1}\right)dx =$$

$$\int \frac{x(x^2 + 1)}{x^2 + 1}\,dx - 2\int \frac{dx}{x^2 + 1} = \frac{x^2}{2} - 2\cdot \operatorname{arctg} x + C$$

$$(x^3 + x - 2) : (x^2 + 1) = x$$

$$\underline{-x^3 \pm x}$$

$$-2$$

$$x^3 + x - 2 = x(x^2 + 1) - 2$$

$$\frac{x^3 + x - 2}{x^2 + 1} = x - \frac{2}{x^2 + 1}$$

$$\int \frac{(x^3 + x - 2)}{x^2 + 1}\,dx = \int x\,dx - 2\int \frac{dx}{x^2 + 1}$$

⑥ $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx = \int \frac{2^{x+1}}{10^x} dx - \int \frac{5^{x-1}}{10^x} dx =$

$\int \frac{2^x \cdot 2}{10^x} \cdot dx - \int \frac{5^x}{5 \cdot 10^x} dx = 2 \cdot \int \left(\frac{1}{5}\right)^x dx - \frac{1}{5} \int \left(\frac{1}{2}\right)^x \cdot dx =$

$2 \cdot \frac{\left(\frac{1}{5}\right)^x}{\ln\frac{1}{5}} - \frac{1}{5} \frac{\left(\frac{1}{2}\right)^x}{\ln\frac{1}{2}} + C = -2 \cdot \frac{\left(\frac{1}{5}\right)^x}{\ln 5} + \frac{1}{5} \cdot \frac{\left(\frac{1}{2}\right)^x}{\ln 2} + C$

⑦ $\int tg^2 x\, dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx - \int dx = tg\,x - x + C$

Metoda zamjene

① $\int (7-4x)^6 dx = \begin{bmatrix} 7-4x = t \\ -4dx = 1 \cdot dt \\ dx = -\frac{1}{4} dt \end{bmatrix} = \int t^6 \cdot \left(-\frac{1}{4} dt\right) =$

$-\frac{1}{4} \int t^6 dt = -\frac{1}{4} \cdot \frac{t^7}{7} + C = -\frac{1}{28} \cdot (7-4x)^7 + C$

② $\int \frac{x+5}{\sqrt[5]{1-2x}} dx = \begin{bmatrix} 1-2x = t \\ -2dx = dt \\ dx = -\frac{1}{2} dt \\ \underbrace{x = \frac{1-t}{2}} \end{bmatrix} = -\frac{1}{2} \int \frac{\frac{1-t}{2} + 5}{\sqrt[5]{t}} dt = -\frac{1}{2} \int$

$\frac{\frac{11-t}{2}}{\sqrt[5]{t}} dt = -\frac{1}{4} \int \frac{11-t}{\sqrt[5]{t}} dt = -\frac{11}{4} \int t^{-\frac{1}{5}} \cdot dt + \frac{1}{4} \int t^{\frac{4}{5}} \cdot dt =$

$= -\frac{11}{4} \cdot \frac{5}{4} \cdot t^{\frac{4}{5}} + \frac{1}{4} \cdot \frac{5}{9} \cdot t^{\frac{9}{5}} + C =$

$= -\frac{55}{16} \cdot \sqrt[5]{(1-2x)^4} + \frac{5}{36} \cdot \sqrt[5]{(1-2x)^9} + C$

③ $\int \frac{dx}{2+3x^2} = \int \frac{dx}{2(1+\frac{3}{2}x^2)} = \frac{1}{2} \int \frac{dx}{1+\frac{3}{2}x^2} = \begin{bmatrix} \sqrt{\frac{3}{2}}\, x = t \\ \sqrt{\frac{3}{2}}\, dx = dt \\ dx = \sqrt{\frac{2}{3}}\, dt \end{bmatrix} =$

$\int \frac{dx}{1+x^2} = arctg\,x + C$

$\int \frac{dt}{1+t^2} = arctg\,t + C$

$$= \frac{1}{2}\int \frac{\sqrt{\frac{2}{3}}\,dt}{1+t^2} = \frac{1}{2}\sqrt{\frac{2}{3}}\int \cdot \frac{dt}{1+t^2} = \frac{1}{\sqrt{6}} \cdot \operatorname{arctg} t + C = \frac{1}{\sqrt{6}} \operatorname{arctg}\left(\sqrt{\frac{3}{2}} \cdot x\right) + C$$

4\. a)

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \left\lceil \begin{matrix} 1-x^2 = t \\ -2x\,dx = dt \\ x\,dx = -\frac{1}{2}dt \end{matrix} \right\rfloor = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2} \cdot 2 \cdot \sqrt{t} + C =$$

$$= -\sqrt{1-x^2} + C$$

b)

$$\int \frac{x-3}{x^2-6x+7}\,dx = \left\lceil \begin{matrix} x^2-6x+7 = t \\ (2x-6)dx = dt \\ 2(x-3)dx = dt \\ (x-3)\,dx = \frac{dt}{2} \end{matrix} \right\rfloor = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C =$$

$$\frac{1}{2}\ln|x^2-6x+7) + C$$

c)

$$\int \frac{\ln^3 x}{x}\,dx = \left\lceil \begin{matrix} \ln x = t \\ \frac{1}{x}dx = dt \end{matrix} \right\rfloor = \int t^3 \cdot dt = \frac{t^4}{4} + C = \frac{\ln^4 x}{4} + C$$

d)

$$\int \frac{\operatorname{arctg}^3 x}{1+x^2}\,dx = \left\lceil \begin{matrix} \operatorname{arctg} x = t \\ \frac{1}{1+x^2} \cdot dx = dt \end{matrix} \right\rfloor = \int t^3 . dt = \frac{t^4}{4} + C =$$

$$\frac{\operatorname{arctg}^4 x}{4} + C$$

5\.

$$\int \frac{\cos x + 1}{\sin x + x}\,dx = \left\lceil \begin{matrix} \sin x + x = t \\ (\cos x + 1)\,dx = dt \end{matrix} \right\rfloor = \int \frac{dt}{t} = \ln|t| + C =$$

$$\ln|\sin x + x| + C$$

6\.

$$\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx = \left\lceil \begin{matrix} \sin x = t \\ \cos x\,dx = dt \end{matrix} \right\rfloor = 3 \cdot \sqrt[3]{\sin x} + C$$

7\.

$$\int \frac{e^{2x}}{1-3\cdot e^{2x}}\,dx = \left\lceil \begin{matrix} 1-3\cdot e^{2x} = t \\ -3\cdot e^{2x}\cdot 2 \cdot dx = dt \\ e^{2x}.\ dx = -\frac{1}{6}dt \end{matrix} \right\rfloor = -\frac{1}{6}\int \frac{dt}{t} = -\frac{1}{6}\ln|t| + C =$$

$$= -\frac{1}{6}\ln|1-3\cdot e^{2x}| + C$$

⑧ $\int \frac{x^3}{x+3}\,dx = \int \frac{x^3+27-27}{x+3}\,dx = \int \frac{x^3+27}{x+3}\,dx - 27\int \frac{dx}{x+3} =$

$\int \frac{(x+3)(x^2-3x+9)}{x+3}\,dx - 27\int\frac{dx}{x+3} = \left\ulcorner \begin{matrix} x+3=t \\ dx=dt \end{matrix} \right\lrcorner = \frac{x^3}{3} - 3\cdot\frac{x^2}{2} + 9x - 27\int\frac{dt}{t} =$

$= \frac{x^3}{3} - \frac{3}{2}x^2 + 9x - 27\ln|x+3|$

$\int \frac{dx}{\sqrt{1-x^2}} = \cancel{\ln|x+\sqrt{1}}$ arcsin x

⑨ $\int \frac{dx}{\sqrt{3-5x^2}} = \int\frac{dx}{\sqrt{3\cdot(1-\frac{5}{3}x^2)}} = \frac{1}{\sqrt3}\cdot\int\frac{dx}{\sqrt{1-\frac{5}{3}x^2}} =$

$\left\ulcorner \begin{matrix} \sqrt{\frac{5}{3}}\,x = t \\ \sqrt{\frac{5}{3}}\,dx = dt \\ dx = \sqrt{\frac{3}{5}}\cdot dt \end{matrix} \right\lrcorner = \frac{1}{\sqrt3}\cdot\int\frac{\sqrt{\frac{3}{5}}\cdot dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt3}\cdot\frac{\sqrt3}{\sqrt5}\int\frac{dt}{\sqrt{1-t^2}} =$

$= \frac{1}{\sqrt5}\arcsin\left(\sqrt{\frac{5}{3}}\,x\right) + C$

$\int\frac{dx}{\sqrt{x^2-1}} = \ln|x+\sqrt{x^2-1}|$

⑩ $\int\frac{dx}{\sqrt{7x^2-4}} = \int\frac{dx}{\sqrt{4(\frac{7}{4}x^2-1)}} = \frac{1}{2}\int\frac{dx}{\sqrt{\frac{7}{4}x^2-1}} = \left\ulcorner \begin{matrix} \sqrt{\frac{7}{4}}\,x = t \\ \frac{\sqrt7}{2}\,dx = dt \\ dx = \frac{2}{\sqrt7}\,dt \end{matrix} \right\lrcorner$

$= \cancel{\int}$

$= \frac{1}{2}\cdot\frac{2}{\sqrt7}\int\frac{dt}{\sqrt{t^2-1}} = \frac{1}{\sqrt7}\cdot\ln|t+\sqrt{t^2-1}| + C =$

$= \frac{1}{\sqrt7}\cdot\ln\left|\frac{\sqrt7}{2}x + \sqrt{\frac{7}{4}x^2-1}\right| + C$

⑪ $\int\frac{dx}{\sqrt{3-x-x^2}} = \int\frac{dx}{\sqrt{-1\cdot(x^2+x-3)}} = \int\frac{dx}{\sqrt{-1\cdot(x^2+x+\frac{1}{4}-3-\frac{1}{4})}} = \int\frac{dx}{\sqrt{(-1)\cdot[(x+\frac{1}{2})^2-\frac{13}{4}]}} =$

$= \int\frac{dx}{\sqrt{\underbrace{\frac{13}{4}-(x+\frac{1}{2})^2}_{\frac{13}{4}t^2}}} = \left\ulcorner \begin{matrix} x+\frac{1}{2} = \frac{\sqrt{13}}{2}\cdot t \\ dx = \frac{\sqrt{13}}{2}\,dt \\ t = \frac{2x+1}{\sqrt{13}} \end{matrix} \right\lrcorner =$

$$\int \frac{\frac{\sqrt{13}}{2}\,dt}{\sqrt{\frac{13}{4}-\frac{13}{4}t^2}} = \frac{\sqrt{13}}{2}\cdot\frac{2}{\sqrt{13}}\int\frac{dt}{\sqrt{1-t^2}} = \arcsin t + C =$$

$$\arcsin\left(\frac{2x+1}{\sqrt{13}}\right) + C$$

→ Metoda parcijalne integracije

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

① $\int x \cdot \sin x\, dx = \left[\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \sin x dx \Rightarrow v = \int \sin x dx = -\cos x \end{array}\right] =$

$= x\cdot(-\cos x) - \int(-\cos x)\cdot dx = -x\cdot\cos x + \int \cos x dx = -x\cdot\cos x + \sin x + C$

② $\int x^2 \cdot e^x \cdot dx = \left[\begin{array}{l} u = x^2 \Rightarrow du = 2x dx \\ dv = e^x \cdot dx \Rightarrow v = \int e^x \cdot dx = e^x \end{array}\right] =$

$= x^2\cdot e^x - \int e^x \cdot 2x dx = x^2\cdot e^x - 2\int x\cdot e^x\cdot dx = \left[\begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x dx \Rightarrow v = \int e^x \cdot dx = e^x \end{array}\right] =$

$= x^2\cdot e^x - 2\cdot\left(x\cdot e^x - \int e^x \cdot dx\right) = x^2\cdot e^x - 2\cdot x\cdot e^x + 2\cdot e^x + C$

③ $\int x \cdot \ln x\, dx = \left[\begin{array}{l} u = \ln x \Rightarrow du = \frac{1}{x}dx \\ dv = x\, dx \Rightarrow v = \int x\cdot dx = \frac{x^2}{2} \end{array}\right] =$

$\ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}\cdot\ln x - \frac{1}{2}\int x dx = \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + C$

④ $\int \arcsin x dx = \left[\begin{array}{l} u = \arcsin x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}} \\ dv = dx \Rightarrow v = \int dx = x \end{array}\right] =$

$= x\cdot\arcsin x - \int x\cdot\frac{dx}{\sqrt{1-x^2}} = x\cdot\arcsin x - \int\frac{x}{\sqrt{1-x^2}}dx =$

$\left[\begin{array}{l} 1-x^2 = t \\ -2x dx = dt \\ x dx = -\frac{1}{2}dt \end{array}\right] = x\cdot\arcsin x - \frac{1}{2}\int\frac{dt}{\sqrt{t}} = x\cdot\arcsin x - \frac{1}{2}\cdot 2\sqrt{t} + C =$

$= x \cdot \arcsin x - \sqrt{1-x^2} + C$

Integracija racionalnih f-ja

I $\int \frac{Ax+B}{\underbrace{ax^2+bx+c}_{\text{nesvodljiv}}} dx$ $\qquad \int \frac{dx}{ax^2+bx+c}$ ne treba korijen

(1.) $\int \frac{4x-3}{3x^2-4x+5} dx = \int \frac{\frac{4}{6}(6x-4)-3+\frac{8}{3}}{3x^2-4x+5} dx = \frac{2}{3} \int \frac{6x-4}{3x^2-4x+5} dx - \frac{1}{3} \int \frac{dx}{3x^2-4x+5} =$

$\left\lceil 3x^2-4x+5 = t \right.$
$\left. (6x-4)dx = dt \right\rfloor$

$\lceil 3x^2-4x+5 = 3(x^2 - \frac{4}{3}x + \frac{5}{3})$
$3 \cdot (x^2 - \frac{4}{3}x + \frac{5}{3} - \frac{4}{9}) =$
$3 \cdot [(x-\frac{2}{3})^2 + \frac{11}{9}] \rfloor$

$= \frac{2}{3} \int \frac{dt}{t} - \frac{1}{3} \int \frac{dx}{3 \cdot [(x-\frac{2}{3})^2 + \frac{11}{9}]} = \frac{2}{3} \ln|t| - \frac{1}{9} \cdot \int \frac{dx}{\underbrace{(x-\frac{2}{3})^2}_{\frac{11}{9}z^2} + \frac{11}{9}} =$

$= \left\lceil x - \frac{2}{3} = \frac{\sqrt{11}}{3} \cdot z \right.$
$dx = \frac{\sqrt{11}}{3} dz$
$\left. z = \frac{3x-2}{\sqrt{11}} \right\rfloor$
$= \frac{2}{3} \ln|t| - \frac{1}{9} \cdot \frac{\sqrt{11}}{3} \int \frac{dz}{\frac{11}{9}z^2 + \frac{11}{9}} =$

$= \frac{2}{3} \ln|t| - \frac{\sqrt{11}}{27} \cdot \frac{9}{11} \cdot \int \frac{dz}{z^2+1} = \frac{2}{3} \ln|3x^2-4x+5| - \frac{\sqrt{11}}{33} \operatorname{arctg}\left(\frac{3x-2}{\sqrt{11}}\right) + C$

II $\int \frac{3x+2}{x^2-5x+4}\,dx$, metod neodređenih koef.

$\frac{P(x)}{Q(x)}$, st P < st Q

$x^2-5x+4=0$

$x_{12} = \frac{5 \pm \sqrt{25-16}}{2}$

$x_1 = 4 \qquad x_2 = 1$

$x^2-5x+4 = (x-4)(x-1)$

$\frac{3x+2}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$

$\frac{3x+2}{(x-4)(x-1)} = \frac{A(x-1)+B(x-4)}{(x-4)(x-1)}$

$3x+2 = (A+B)x - A - 4B \Rightarrow$

$A+B=3$

$-A-4B=2$

$-3B=5$

$B=-\frac{5}{3}$

$A = 3-B = 3-\left(-\frac{5}{3}\right) = \frac{14}{3}$

$\int\left(\frac{14}{3}\cdot\frac{1}{x-4} + \left(-\frac{5}{3}\right)\cdot\frac{1}{x-1}\right)dx = \frac{14}{3}\int\frac{dx}{x-4} - \frac{5}{3}\int\frac{dx}{x-1} = \frac{14}{3}\ln|x-4| - \frac{5}{3}\ln|x-1| + C$

2. $\int \frac{x^4-3x}{x^3+1}\,dx$

$(x^4-3x):(x^3+1) = x$

$\underline{-x^4 \mp x}$

$-4x$

$x^4-3x = x(x^3+1)-4x$

Integracija iracionalnih f-ja

I $\displaystyle\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx \quad , \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}$

ax^2+bx+c → nije potpun kvadrat

1. $$\int \frac{x+3}{\sqrt{4x^2+4x+3}}\,dx = \int \frac{\frac{1}{8}(8x+4)+3-\frac{1}{2}}{\sqrt{4x^2+4x+3}}\,dx = \frac{1}{8}\int \frac{8x+4}{\sqrt{4x^2+4x+3}}\,dx + \frac{5}{2}\int \frac{dx}{\sqrt{4x^2+4x+3}}$$

$$= \left[\begin{array}{l} 4x^2+4x+3 = t \\ (8x+4)\,dx = dt \end{array}\right] \quad \left[\begin{array}{l} 4x^2+4x+3 = 4\left(x^2+x+\frac{3}{4}\right) = 4\left(x^2+x+\frac{1}{4}+\frac{3}{4}-\frac{1}{4}\right) = \\ = 4\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{2}\right) \end{array}\right]$$

$$= \frac{1}{8}\int \frac{dt}{\sqrt{t}} + \frac{5}{2}\cdot\frac{1}{2}\int \frac{dx}{\sqrt{\underbrace{\left(x+\frac{1}{2}\right)^2}_{\frac{1}{2}z^2}+\frac{1}{2}}} = \left[\begin{array}{l} x+\frac{1}{2} = \frac{1}{\sqrt{2}}z \Rightarrow z = \frac{2x+1}{\sqrt{2}} \\ dx = \frac{1}{\sqrt{2}}\,dz \end{array}\right] =$$

$$= \frac{1}{8}\cdot 2\sqrt{t} + \frac{5}{4}\cdot\frac{1}{\cancel{\sqrt{2}}}\cancel{\sqrt{2}}\int \frac{dz}{\sqrt{z^2+1}} = \frac{1}{4}\sqrt{t} + \frac{5}{4}\ln\left|z+\sqrt{z^2+1}\right| + C =$$

$$= \frac{1}{4}\sqrt{4x^2+4x+3} + \frac{5}{4}\ln\left|\frac{2x+1}{\sqrt{2}} + \sqrt{\frac{(2x+1)^2}{2}+1}\right| + C$$

(*) $$\int \frac{dx}{\sqrt{1-x-3x^2}}$$

26. februar

II a) $\int R\left(x, x^{\frac{m_1}{n_1}}, \dots, x^{\frac{m_s}{n_s}}\right) dx$

$x = t^k$, $k = NZS(n_1, \dots, n_s)$

1. $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \left[\begin{array}{l} x^{\frac{1}{2}}, x^{\frac{1}{3}} \\ x = t^6 \\ dx = 6t^5 dt \\ t = \sqrt[6]{x} \end{array} \right] = \int \frac{6t^5 dt}{t^3 + t^2} = 6 \int \frac{t^{\cancel{5}\,3} dt}{\cancel{t^2}(t+1)} = 6 \int \frac{t^3}{t+1} dt$

$= 6 \int \frac{t^3 + 1 - 1}{t+1} dt = 6 \int (t^2 - t + 1) dt - 6 \int \frac{dt}{t+1} = 2t^3 - 3t^2 + 6t - 6 \ln|t+1| + C$

$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln|\sqrt[6]{x} + 1| + C$

b) $\int R\left(x, (ax+b)^{\frac{m_1}{n_1}}, \dots, (ax+b)^{\frac{m_s}{n_s}}\right) dx$

$ax + b = t^k$, $k = NZS(n_1, \dots, n_s)$

2. $\int \frac{x + \sqrt{1+x}}{\sqrt[3]{1+x}} dx = \left[\begin{array}{ll} x, (1+x)^{\frac{1}{2}}, (1+x)^{\frac{1}{3}} & \\ 1 + x = t^6 & t = (1+x)^{\frac{1}{6}} \\ dx = 6t^5 dt & x = t^6 - 1 \\ (1+x)^{\frac{1}{2}} = t^3, (1+x)^{\frac{1}{3}} = t^2 & \end{array} \right] = \int \frac{t^6 - 1 + t^3}{t^2} 6t^5 dt =$

$= 6 \int (t^9 - t^3 + t^6) dt = \frac{3}{5} t^{10} - \frac{3}{2} t^4 + \frac{6}{7} t^7 + C = \frac{3}{5} \sqrt[3]{(1+x)^5} - \frac{3}{2} \sqrt[3]{(1+x)^2} +$

$+ \frac{6}{7} \sqrt[6]{(1+x)^7} + C$

Integracija trigonometrijskih f-ja

I $\int R(\sin x, \cos x)\,dx \qquad tg\frac{x}{2}=t \quad , \quad \sin x = \frac{2t}{1+t^2}$

$$dx = \frac{2\,dt}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

1. $$\int \frac{dx}{(2+\cos x)\sin x} = \begin{bmatrix} tg\frac{x}{2}=t & \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2\,dt}{1+t^2} & \cos x = \frac{1-t^2}{1+t^2} \end{bmatrix} = \int \frac{\frac{\cancel{2}dt}{\cancel{1+t^2}}}{\left(2+\frac{1-t^2}{1+t^2}\right)\cdot\frac{\cancel{2}t}{\cancel{1+t^2}}} =$$

$$= \int \frac{1+t^2}{t(3+t^2)}\,dt = \frac{1}{3}\int\frac{dt}{t} + \frac{2}{3}\int\frac{t}{3+t^2}\,dt = \begin{bmatrix} 3+t^2 = z \\ t\,dt = \frac{1}{2}dz \end{bmatrix} =$$

$$\frac{1+t^2}{t(3+t^2)} = \frac{A}{t} + \frac{Bt+C}{3+t^2} \quad \dots \quad A=\frac{1}{3},\ B=\frac{2}{3},\ C=0$$

$$= \frac{1}{3}\ln|t| + \frac{2}{3}\cdot\frac{1}{2}\int\frac{dz}{z} = \frac{1}{3}\ln|t| + \frac{1}{3}\ln|z| + C =$$

$$= \frac{1}{3}\ln\left|tg\frac{x}{2}\right| + \frac{1}{3}\ln\left|3+tg^2\frac{x}{2}\right| + C$$

$\int R(\sin x, \cos x)\,dx$

R neparna po $\sin x$, $\cos x = t$

R neparna po $\cos x$, $\sin x = t$

2. $\int \frac{\cos^3 x}{2+\sin x}\,dx$

$R(\sin x, \cos x) = \frac{\cos^3 x}{2+\sin x}$

$R(\sin x, -\cos x) = \frac{(-\cos x)^3}{2+\sin x} = -\frac{\cos^3 x}{2+\sin x} = -R(\sin x, \cos x) \Rightarrow$

$\Rightarrow$ R je neparna po $\cos x$

$$\int \frac{\cos^3 x}{2+\sin x}\,dx = \int \frac{\cos^2 x \cos x}{2+\sin x}\,dx = \begin{bmatrix} \sin x = t \\ \cos x\,dx = dt \end{bmatrix} = \int \frac{1-t^2}{2+t}\,dt =$$

$$= -\int \frac{t^2-1}{t+2}\,dt = -\int \frac{t^2-4+4-1}{t+2}\,dt = -\int (t-2)\,dt - 3\int \frac{dt}{t+2} =$$

$$= -\frac{t^2}{2} + 2t - 3\ln|t+2| + C = -\frac{1}{2}\sin^2 x + 2\sin x - 3\ln|\sin x + 2| + C$$

III $\int R(\sin x, \cos x)\,dx$ R parna po $\sin x$ i po $\cos x$

$\operatorname{tg} x = t$ $\qquad \sin x = \frac{t}{\sqrt{1+t^2}}$

$dx = \frac{dt}{1+t^2}$ $\qquad \cos x = \frac{1}{\sqrt{1+t^2}}$

3. $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx =$

$R(\sin x, \cos x) =$

$$R(-\sin x, -\cos x) = \frac{-\sin x - (-\cos x)}{-\sin x + (-\cos x)} = \frac{-(\sin x - \cos x)}{-(\sin x + \cos x)} = R(\sin x, \cos x)$$

R je parna i po $\sin x$ i po $\cos x$

$$\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = \begin{bmatrix} tg\,x = t & \sin x = \frac{t}{\sqrt{1+t^2}} \\ dx = \frac{dt}{1+t^2} & \cos x = \frac{1}{\sqrt{1+t^2}} \end{bmatrix} =$$

$$= \int \frac{\frac{t}{\sqrt{1+t^2}} - \frac{1}{\sqrt{1+t^2}}}{\frac{t}{\sqrt{1+t^2}} + \frac{1}{\sqrt{1+t^2}}} \cdot \frac{dt}{1+t^2} = \int \frac{t-1}{t+1} \cdot \frac{dt}{1+t^2} = \int \frac{t-1}{(t+1)(t^2+1)}\,dt =$$

$$\frac{t-1}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$$

$$t-1 = (A+B)t^2 + (B+C)t + A + C$$

$A+B=0 \rightarrow A=-B$

$B+C=1 \rightarrow C=1-B$

$A+C=-1$

$-B+1-B=-1$

$\underline{B=1}$ $\underline{A=-1}$ $C=0$

$$= -\int \frac{dt}{t+1} + \int \frac{t}{t^2+1}\,dt = -\ln|t+1| + \frac{1}{2}\ln|t^2+1| + C =$$

$$= -\ln|tg\,x+1| + \frac{1}{2}\ln|tg^2x+1| + C$$

$\underline{IV}$ $\int \sin^m x \cos^n x\,dx$

m - neparno, n - parno $\rightarrow$ smjena $\cos x = t$

m - parno, n - neparno $\rightarrow$ smjena $\sin x = t$

4. $$\int \sin^3 x \cos^4 x\,dx = \int \sin^2 x \sin x \cos^4 x\,dx = \begin{bmatrix} \cos x = t \\ \sin x\,dx = -dt \end{bmatrix} =$$

$$= -\int (1-t^2)t^4\,dt = -\frac{t^5}{5} + \frac{t^7}{7} + C = -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

V $\int \sin^m x \cos^n x \, dx$

n, m - parno $\quad \sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$

1. $\int \sin^2 x \cos^4 x \, dx = \int \frac{1-\cos 2x}{2} \left(\frac{1+\cos 2x}{2} \right)^2 dx = \frac{1}{8} \int (1-\cos^2 2x)(1+\cos$

$= \frac{1}{8} \int dx + \frac{1}{8} \int \cos 2x \, dx - \frac{1}{8} \int \cos^2 2x \, dx - \frac{1}{8} \int \cos^3 2x \, dx =$

$= \begin{bmatrix} \sin 2x = t \\ \cos 2x dx = \frac{1}{2} dt \end{bmatrix} = \frac{1}{8} x + \frac{1}{8} \cdot \frac{1}{2} \sin 2x - \frac{1}{8} \int \frac{1+\cos 4x}{2} dx - \frac{1}{8} \frac{1}{2} \int (1-t^2) dt$

$= \frac{1}{8} x + \frac{1}{16} \sin 2x - \frac{1}{16} x - \frac{1}{16} \cdot \frac{1}{4} \sin 4x - \frac{1}{16} t + \frac{1}{16} \frac{t^3}{3} + C =$

$= \frac{1}{16} x - \frac{1}{64} \sin 4x + \frac{1}{48} \sin^3 2x + C$

VI $\int \frac{\sin^m x}{\cos^n x} dx$

m, n - parno $\quad$ smjena $tg x = t$

2. $\int \frac{\sin^2 x}{\cos^6 x} dx = \begin{bmatrix} tg x = t & \sin x = \frac{t}{\sqrt{1+t^2}} \\ dx = \frac{dt}{1+t^2} & \cos x = \frac{1}{\sqrt{1+t^2}} \end{bmatrix} = \int \frac{\frac{t^2}{1+t^2}}{\frac{1}{(1+t^2)^3}} \cdot \frac{dt}{1+t^2} =$

$= \int t^2 (1+t^2) dt = \frac{t^3}{3} + \frac{t^5}{5} + C = \frac{tg^3 x}{3} + \frac{tg^5 x}{5} + C$

VII $\int \sin \alpha x \cos \beta x \, dx$, adicione formule

3. $\int \sin 5x \cos x \, dx = \frac{1}{2} \int (\sin 6x + \sin 4x) dx = \frac{1}{2} \cdot \left(-\frac{1}{6}\right) \cos 6x + \frac{1}{2} \left(-\frac{1}{4}\right) c$

$+$