$L_1 \cap L_2 = \{0\} \} \Rightarrow V = L_1 + L_2$

- Gausov metod rješavanja sistema linearnih jednači

1. $x_1 + 3x_2 + 2x_4 = 2$

$3x_1 + 7x_2 - x_3 + 2x_4 = 3$

$x_1 - x_2 + 5x_3 - 3x_4 = 4$

$2x_1 + 4x_2 - x_3 = 1$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 2 \\ 3 & 7 & -1 & 2 & 3 \\ 1 & -1 & 5 & -3 & 4 \\ 2 & 4 & -1 & 0 & 1 \end{array}\right) \cdot(-3) \quad \cdot(-1) \quad \cdot(-2)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 2 \\ 0 & -2 & -1 & -4 & -3 \\ 0 & -4 & 5 & -5 & 2 \\ 0 & -2 & -1 & -4 & -3 \end{array}\right) \cdot(-2) \quad \cdot(-1) \sim \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 2 \\ 0 & -2 & -1 & -4 & -3 \\ 0 & 0 & 7 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Dati sistem je ekvivalentan sistemu:

$x_1 + 3x_2 + 2x_4 = 2$

$-2x_2 - x_3 - 4x_4 = -3$

$7x_3 + 3x_4 = 8$

$$x_3 = \frac{8 - 3x_4}{7}$$

$$x_2 = \frac{3 - x_3 - 4x_4}{2} = \frac{3 - \frac{8-3x_4}{7} - 4x_4}{2} = \frac{13 - 25x_4}{14}$$

$$x_1 = 2 - 3x_2 - 2x_4 = 2 - 3 \cdot \frac{13 - 25x_4}{14} - 2x_4 = \frac{-11 + 47x_4}{14}$$

Sistem je saglasan i neodređen. Opšte rešenje je

$\left(\frac{-11+47x_4}{14}, \frac{13-25x_4}{14}, \frac{8-3x_4}{7}, x_4\right)$ x_4 - slobodna nepoznata

① $2x+y-2z=6$

$x-3y+2z=-7$

$-x+2y-z=4$

$x-3y+2z=-7$ /(-2)

$-x+2y-z=4$

$2x+y-2z=6$

x-3y+2z=-7

$-y+z=-3$ /7

$7y-6z=20$

$\boxed{z=-1}$

$y=3+z$

$\boxed{y=2}$

$\boxed{x=1}$

x-3y+2z=-3
-y+z=-3
z=-1

$(1,2,-1)=(x,y,z)$

JEDINSTVENO RJEŠENJE

DRUGI NAČIN

② $\left(\begin{array}{ccc|c} 2 & 1 & -2 & 6 \\ 1 & -3 & 2 & -7 \\ -1 & 2 & -1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & -7 \\ 2 & 1 & -2 & 6 \\ -1 & 2 & -1 & 4 \end{array}\right)$ (-2)

$\sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & -7 \\ 0 & 7 & -6 & 20 \\ 0 & -1 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & -7 \\ 0 & -1 & 1 & -3 \\ 0 & 7 & -6 & 20 \end{array}\right)$ (·7)

$\left(\begin{array}{ccc|c} 1 & -3 & 2 & -7 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$

$x-3y+2z=-7$

$-y+z=-3$

$\boxed{z=-1}$

a) $x+2y-az=1$
$ax+2y-z=2$
$x+z=3$

$$\left(\begin{array}{ccc|c} 1 & 2 & -a & 1 \\ a & 2 & -1 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 1 & -a & 1 \\ 2 & a & -1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -a & 1 \\ 0 & 1 & 1 & 3 \\ 2 & a & -1 & 2 \end{array}\right) \cdot(-1) \sim \left(\begin{array}{ccc|c} 2 & 1 & -a & 1 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & a-1 & 1 \end{array}\right) \cdot(1-a) \sim \left(\begin{array}{ccc|c} 2 & 1 & -a & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 4-3a \end{array}\right)$$

Sistem je ekvivalentan sistemu:

$2y+x-az=1$ I Ako je $a \neq \frac{4}{3}$ sistem je nesaglasan

$x+z=3$ II Ako je $a=\frac{4}{3}$ sistem je ekvivalentan siste-

$0=4-3a$ mu $2y+x-\frac{4}{3}z=1$

$x+z=3$

$x=3-z$

$$y=\frac{1-x+\frac{4}{3}z}{2}=\frac{1-3+z+\frac{4}{3}z}{2}=\frac{-6+7z}{6}$$

Sistem je saglasan i neodređen $\left(3-z, \frac{-6+7z}{6}, z\right)$

z - slobodna nepoznata

b) $ax-y+3z=a-1$
$x+ay-z=1$
$4x+3y+z=3$

$$\left(\begin{array}{ccc|c} a & -1 & 3 & a-1 \\ 1 & a & -1 & 1 \\ 4 & 3 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & -1 & a & a-1 \\ -1 & a & 1 & 1 \\ 1 & 3 & 4 & 3 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ -1 & a & 1 & 1 \\ 3 & -1 & a & a-1 \end{array}\right) \cdot(-3) \sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & a+3 & 5 & 4 \\ 0 & -10 & a-12 & a-10 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -10 & a-12 & a-10 \\ 0 & a+3 & 5 & 4 \end{array}\right) \cdot \frac{a+3}{10} \sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -10 & a-12 & a-10 \\ 0 & 0 & \frac{(a-2)(a-7)}{10} & \frac{(a-2)(a-5)}{10} \end{array}\right) \cdot 10$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -10 & a-12 & a-10 \\ 0 & 0 & (a-2)(a-7) & (a-2)(a-5) \end{array}\right)$$

$$\frac{(a-12)(a+3)}{10}+5=\frac{a^2-9a-36+50}{10}=\frac{a^2-9a+14}{10}=\frac{(a-2)(a-7)}{10}$$

$$\frac{(a-10)(a+3)}{10}+4=\frac{a^2-7a+10}{10}=\frac{(a-2)(a-5)}{10}$$

10

Sistem je ekvivalentan sistemu:

$z + 3y + 4x = 3$

$-10y + (a-12)x = a-10$

$(a-2)(a-7)x = (a-2)(a-5)$

I $a \neq 2 \quad a \neq 7$

$x = \frac{(a-2)(a-5)}{(a-2)(a-7)} = \frac{a-5}{a-7}$

$y = \frac{10-a+(a-2)x}{10} = \frac{1}{7-a}$

$z = 3-3y-4x = \frac{2-a}{a-7}$

Sistem je saglasan i određen i jedinstveno rešenje je $\left(\frac{a-5}{a-7}, \frac{1}{7-a}, \frac{2-a}{a-7}\right)$

II $a=2$

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -10 & -10 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Sistem je ekvivalentan sistemu:

$z + 3y + 4x = 3$

$-10y - 10z = -8$

$y = \frac{4-5x}{5}$

$z = 3-3y-4x = \frac{3-5x}{5}$

Sistem je saglasan i neodređen

Opšte rešenje $\left(x, \frac{4-5x}{5}, \frac{3-5x}{5}\right)$ x-slobodna nepoznat

III $a=7$

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -10 & -5 & -3 \\ 0 & 0 & 0 & 10 \end{array}\right)$$

$z + 3y + 4x = 3$

$-10y - 5z = -3$

$0 = 10 \perp$

Sistem je nesaglasan

Matrice i determinante

Primjer 1: Neka je $A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \\ -4 & 3 \end{pmatrix}$ i $B = \begin{pmatrix} 0 & -2 \\ 1 & 5 \\ -3 & 4 \end{pmatrix}$. Naći matricu $C = 4 \cdot A - 3 \cdot B$

$$C = 4 \cdot \begin{pmatrix} -1 & 2 \\ 0 & 3 \\ 4 & 3 \end{pmatrix} - 3 \cdot \begin{pmatrix} 0 & -2 \\ 1 & 5 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ 0 & 12 \\ -16 & 12 \end{pmatrix} - \begin{pmatrix} 0 & -6 \\ 3 & 15 \\ -9 & 12 \end{pmatrix} = \begin{pmatrix} -4 & 14 \\ -3 & -3 \\ -7 & 0 \end{pmatrix}$$

Primjer 2: Neka je $A = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix}$ i $B = \begin{pmatrix} 2 & 0 & -4 \\ 3 & 1 & 5 \end{pmatrix}$ Naći matricu

$C = 3 \cdot A \cdot B^T - B \cdot A^T$

$A = (a_{ij})_{m \times n} \quad A^T = (a_{ji})_{n \times m}$

$$A^T = \begin{pmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 4 \end{pmatrix} \quad C = 3 \cdot \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ -4 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & -4 \\ 3 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 4 \end{pmatrix}$$

Broj kolona prve jednak broju vrsta druge

$$C = 3\cdot\begin{pmatrix} 2+(-1)\cdot 0+2\cdot(-4) & 1\cdot 3+(-1)\cdot 1+2\cdot 5 \\ 0\cdot 2+3\cdot 0+4\cdot(-4) & 0\cdot 3+3\cdot 1+4\cdot 5 \end{pmatrix} - \begin{pmatrix} 2\cdot 1+0\cdot(-1)+(-4)\cdot 2 & 2\cdot 0+0\cdot 3+(-4)\cdot 4 \\ 3\cdot 1+1\cdot(-1)+5\cdot 2 & 3\cdot 0+1\cdot 3+5\cdot 4 \end{pmatrix}$$

$$C = 3\cdot\begin{pmatrix} -6 & 12 \\ -16 & 23 \end{pmatrix} - \begin{pmatrix} -6 & -16 \\ 12 & 23 \end{pmatrix} = \begin{pmatrix} -18 & 36 \\ -48 & 69 \end{pmatrix} - \begin{pmatrix} -6 & -16 \\ 12 & 23 \end{pmatrix} = \begin{pmatrix} -12 & 52 \\ -60 & 46 \end{pmatrix}$$

$A = (a_{ij})_{m\times n}$ $\quad B = (b_{ij})_{n\times p}$ $\quad C = A\cdot B$ $\quad C = (c_{ij})_{m\times p}$

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\cdot b_{kj} \qquad i = \overline{1,m} \quad j = \overline{1,p}$$

1. Naći sve matrice komutativne matrici $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad A\cdot B = B\cdot A$

$$A\cdot B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$$

$$B\cdot A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} a+3c & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$$

$A\cdot B = B\cdot A \iff$

$a+2c = a+3b$

$b+2d = 2a+4b$

$3a+4c = c+3d$

$\underline{3b+4d = 2c+4d}$

$3b-2c=0$

$2a+3b-2d=0$

$a+c-d=0$

$\underline{3b-2c=0}$

$c = \frac{3}{2}b$

$d = a+c = a+\frac{3}{2}b$

$$B = \begin{pmatrix} a & b \\ \frac{3}{2}b & a+\frac{3}{2}b \end{pmatrix}, \ a, b \in R$$

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(\alpha\cdot A)^T = \alpha\cdot A^T, \ \alpha\in P$

$(A\cdot B)^T = B^T\cdot A^T$

Def. Neka je A kvadratna matrica reda n. Trag matrice A je $tr(A) = \sum_{i=1}^{n} a_{ii}$

2. Dokazati da za matrice A i B iz $R^{n\times n}$ važi:

a) $tr(A+B) = tr(A) + tr(B)$

b) $tr(\alpha\cdot A) = \alpha\cdot tr(A), \ \alpha\in R$

c) $tr(A\cdot B) = tr(B\cdot A)$

a) $A=(a_{ij})_{n\times n}$ $B=(b_{ij})_{n\times n}$ $C=A+B$ $C=(c_{ij})_{n\times n}$

$c_{ij}=a_{ij}+b_{ij}$

$c_{ii}=a_{ii}+b_{ii}$

$tr(A+B)=tr(C)=\sum_{i=1}^{n} c_{ii}=\sum_{i=1}^{n}(a_{ii}+b_{ii})=\sum_{i=1}^{n} a_{ii}+\sum_{i=1}^{n} b_{ii}=tr(A)+tr(B)$

b) $A=(a_{ij})_{n\times n}$

$D=\alpha\cdot A$

$D=(d_{ij})_{n\times n}$ $\quad tr(\alpha A)=tr(D)=\sum_{i=1}^{n} d_{ii}=\sum_{i=1}^{n}\alpha\cdot a_{ii}$

$d_{ij}=\alpha\cdot a_{ij}$ $\quad =\alpha\cdot\sum_{i=1}^{n} a_{ii}=\alpha\cdot tr(A)$

$d_{ii}=\alpha\cdot a_{ii}$

c) $A=(a_{ij})_{n\times n}$ $B=(b_{ij})_{n\times n}$

$C=A\cdot B$ $\quad D=B\cdot A$

$C=(c_{ij})_{n\times n}$ $\quad D=(d_{ij})_{n\times n}$

$c_{ij}=\sum_{k=1}^{n} a_{ik}\cdot b_{kj}$ $\quad d_{ij}=\sum_{k=1}^{n} b_{ik}\cdot a_{kj}$

$c_{ii}=\sum_{k=1}^{n} a_{ik}\cdot b_{ki}$ $\quad d_{ii}=\sum_{k=1}^{n} b_{ik}\cdot a_{ki}$

$tr(A\cdot B)=\sum_{i=1}^{n} c_{ii}=\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}\cdot b_{ki}=\sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}\cdot b_{ki}=\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki}\cdot b_{ik}=\sum_{i=1}^{n}\underline{\sum_{k=1}^{n} b_{ik}\cdot a_{ki}}=\sum_{i=1}^{n} d_{ii}=tr(D)=tr(B\cdot A)$

3. $V=R^{2\times 2}$ Odredi bazu i dimenziju

$P=R$

$V=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \mid a,b,c,d\in R\right\}=\left\{\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix} 0 & b\\ 0 & 0\end{pmatrix}+\begin{pmatrix} 0 & 0\\ c & 0\end{pmatrix}+\begin{pmatrix} 0 & 0\\ 0 & d\end{pmatrix} \mid a,b,c,d\in R\right\}=\left\{a\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}+b\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}+c\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}+d\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix} \mid a,b,c,d\in R\right\}=\{a\cdot E_1+b\cdot E_2+c\cdot E_3+d\cdot E_4 \mid a,b,c,d\in R\}=\mathcal{L}(E_1,E_2,E_3,E_4)$

$E_1=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}, E_2=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, E_3=\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}, E_4=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}$

$\alpha_1 E_1+\alpha_2 E_2+\alpha_3 E_3+\alpha_4 E_4=0\in R^{2\times 2}$

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} \alpha_1 = 0 & \alpha_2 = 0 \\ \alpha_3 = 0 & \alpha_4 = 0 \end{matrix} \Big\} \Rightarrow$$ vektori E_1, E_2, E_3, E_4 su LN s tem i čine bazu u $\mathbb{R}^{2\times 2}$ - standardna baza

$\dim \mathbb{R}^{2\times 2} = 4$

4. $V = \mathbb{C}^{2\times 2}$

a) $P = \mathbb{R}$ b) $P = \mathbb{C}$

$\mathbb{C}^{2\times 2} = \left\{ \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \middle| z_1, z_2, z_3, z_4 \in \mathbb{C} \right\}$

a) $\mathbb{C}^{2\times 2} = \left\{ \begin{pmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{pmatrix} \middle| a_i, b_i \in \mathbb{R}, i = \overline{1,4} \right\}$

$= \left\{ a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b_1 \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right.$

$\left. + b_3 \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} + b_4 \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} + a_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| a_i, b_i \in \mathbb{R} \right\}$

$= \{ a_1 E_1 + b_1 E_2 + a_2 E_3 + b_2 E_4 + a_3 E_5 + b_3 E_6 + a_4 E_7 + b_4 E_8$

$a_i, b_i \in \mathbb{R} \; i = \overline{1,4} \} = \mathcal{L}(E_1, \dots, E_8)$

$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$E_6 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, E_7 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E_8 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$ Dokazati da je $\{E_1, \dots, E_8\}$ LN

Sistem $\{E_1, \dots, E_8\}$ je baza u $\mathbb{C}^{2\times 2}$ nad poljem $\mathbb{R}$

$\dim \mathbb{C}^{2\times 2} = 8$, nad poljem $\mathbb{R}$

$A = \begin{pmatrix} 3-5i & -2 \\ -7i & 8+9i \end{pmatrix}$ $A = (3, -5, -2, 0, 0, -7, 8, 9)$ u bazi $\{E_1, \dots, E_8\}$

$2 - 7x + x^3 \Rightarrow p = (2, -7, 0, 1)$

b) $\mathbb{C}^{2\times 2} = \left\{ \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \middle| z_1, z_2, z_3, z_4 \in \mathbb{C} \right\} = \left\{ z_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right.$

$\left. + z_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + z_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| z_i \in \mathbb{C} \right\} = \mathcal{L}(E_1, E_2, E_3, E_4)$

$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Pokazati da je sistem $\{E_1, E_2, E_3, E_4\}$ L.N.

$\{E_1, E_2, E_3, E_4\}$ je baza u $\mathbb{C}^{2\times 2}$ nad poljem $\mathbb{C}$

$\dim \mathbb{C}^{2\times 2} = 4$ nad poljem $\mathbb{C}$

$A = (3-5i, -2, -7i, 8+9i)$ u bazi $\{E_1, E_2, E_3, E_4\}$

5. U prostoru $R^{2\times2}$ zadate su matrice

$A=\begin{pmatrix}1&\lambda\\0&0\end{pmatrix}$ $B=\begin{pmatrix}0&0\\1&\lambda\end{pmatrix}$ $\lambda\in R$ Da li je $M=\{X\in R^{2\times2} \mid AX=XB\}$ potprostor prostora $R^{2\times2}$ nad poljem R?

Ako jeste, odrediti neku bazu za M (u zavisnosti od λ)

1° $X,Y\in M$

$A\cdot(X+Y)=A\cdot X+A\cdot Y=X\cdot B+Y\cdot B=(X+Y)\cdot B \Rightarrow X+Y\in M$

2° $\alpha\in R$ $X\in M$

$A\cdot(\alpha\cdot X)=\alpha\cdot(A\cdot X)=\alpha\cdot(X\cdot B)=(\alpha\cdot X)\cdot B\Rightarrow \alpha X\in M$

Iz 1° i 2° $\Rightarrow$ M je potprostor prostora $R^{2\times2}$

Neka je $X=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ proizvoljna matrica iz M. Tada je $A\cdot X=X\cdot B$

$A\cdot X=\begin{pmatrix}1&\lambda\\0&0\end{pmatrix}\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a+\lambda c&b+\lambda d\\0&0\end{pmatrix}$

$X\cdot B=\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot\begin{pmatrix}0&0\\1&\lambda\end{pmatrix}=\begin{pmatrix}b&b\lambda\\d&d\lambda\end{pmatrix}$

$A\cdot X=X\cdot B\Rightarrow$
$$a+\lambda c=b$$
$$b+\lambda d=b\lambda$$
$$d=0$$
$$d\lambda=0$$

$$\begin{cases}a-b+\lambda c=0\\(1-\lambda)\cdot b+\lambda d=0\\d=0\end{cases}$$

$$\begin{cases}a-b+\lambda c=0\\(1-\lambda)\cdot b=0\end{cases}$$

I Ako $\lambda\neq1$ onda $b=0$

$a=-\lambda c$

$d=0$

$X=\begin{pmatrix}-\lambda c&0\\c&0\end{pmatrix}$

$M=\left\{\begin{pmatrix}-\lambda c&0\\c&0\end{pmatrix}\mid c\in R\right\}=\left\{c\begin{pmatrix}-\lambda&0\\1&0\end{pmatrix}\mid c\in R\right\}=\mathcal{L}(E)$

$E=\begin{pmatrix}-\lambda&0\\1&0\end{pmatrix}$

Baza u M je $\{E\}$ $\dim M=1$

II Ako je $\lambda=1$

$c=b-a$

$d=0$

$X=\begin{pmatrix}a&b\\b-a&0\end{pmatrix}$

$M=\left\{\begin{pmatrix}a&b\\b-a&0\end{pmatrix}\mid a,b\in R\right\}=\left\{a\begin{pmatrix}1&0\\-1&0\end{pmatrix}+b\begin{pmatrix}0&1\\1&0\end{pmatrix}\mid a,b\in R\right\}=\mathcal{L}(E_1,E_2)$

$E_1=\begin{pmatrix}1&0\\-1&0\end{pmatrix}$ $E_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ Pokazati da je $\{E_1,E_2\}$ L.N.

-||- -||- $E_1,E_2\in M$

Baza u M je $\{E_1,E_2\}$

$\dim M=2$

* Vježbe *

1. Zadat je skup $S = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathbb{C}^{2\times 2} : a_1 - 2\overline{a_2} + a_3 = 0 \wedge a_1 + \overline{a_2 + a_3} + a_4 = 0 \right\}$. Ispitati da li je S potprostor prostora $\mathbb{C}^{2\times2}$ nad poljem $\mathbb{R}$? Ako jeste odrediti bazu i dimenziju i ispitati da li je S potprostor prostora $\mathbb{C}^{2\times2}$ nad poljem $\mathbb{C}$.

I $A, B \in S$

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \quad A+B = \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix} \in \mathbb{C}^{2\times2}$

a) $a_1 + b_1 - 2(\overline{a_2 + b_2}) + a_3 + b_3 = 0$?

$a_1 + b_1 - 2\overline{a_2} - 2\overline{b_2} + a_3 + b_3 = 0$

$a_1 - 2\overline{a_2} + a_3 + b_1 - 2\overline{b_2} + b_3 = 0$

$0 + 0 = 0 \quad T$

b) $a_1 + b_1 + \overline{a_2 + b_2 + a_3 + b_3} + a_4 + b_4 = 0$?

$a_1 + b_1 + \overline{a_2} + \overline{b_2} + \overline{a_3} + \overline{b_3} + a_4 + b_4 = 0$

$a_1 + \overline{a_2} + \overline{a_3} + a_4 + b_1 + \overline{b_2} + \overline{b_3} + b_4 = 0$

$a_1 + \overline{a_2 + a_3} + a_4 + b_1 + \overline{b_2 + b_3} + b_4 = 0$

$0 + 0 = 0 \quad T \qquad$ Iz a) i b) $\Rightarrow A+B \in S$

II $\alpha \in \mathbb{R} \quad A \in S$

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \alpha A = \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix} \in \mathbb{C}^{2\times2}$

a) $\alpha a_1 - 2\overline{\alpha a_2} + \alpha a_3 = 0$?

$\alpha a_1 - 2\overline{\alpha} \cdot \overline{a_2} + \alpha a_3 = 0$

$\alpha a_1 - 2\alpha \overline{a_2} + \alpha a_3 = 0$

$\alpha(a_1 - 2\overline{a_2} + a_3) = 0$

$\alpha \cdot 0 = 0 \quad T$

b) $\alpha a_1 + \overline{\alpha a_2 + \alpha a_3} + \alpha a_4 = 0$?

$\alpha a_1 + \overline{\alpha a_2} + \overline{\alpha a_3} + \alpha a_4 = 0$

$\alpha a_1 + \alpha \overline{a_2} + \alpha \overline{a_3} + \alpha a_4 = 0$

$\alpha(a_1 + \overline{a_2 + a_3} + a_4) = 0$

$\alpha \cdot 0 = 0 \quad T$

Iz I i II $\Rightarrow$ S je potprostor prostora $\mathbb{C}^{2\times2}$ nad poljem $\mathbb{R}$

III Ako je $\alpha \in \mathbb{C}$ a $A \in S$ onda bismo imali $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$

$\alpha \cdot A = \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix} \qquad \alpha a_1 - 2\overline{\alpha a_2} + \alpha a_3 = 0$?

$\alpha a_1 - 2 \cdot \overline{\alpha} \cdot \overline{a_2} + \alpha a_3 = 0$?

-uočavamo da neće važiti za $(\forall \alpha \in \mathbb{C})(\forall A \in S)(\alpha A \in S)$

S nije potprostor prostora $\mathbb{C}^{2\times 2}$ nad poljem $\mathbb{C}$.

Odredimo bazu i dimenziju potprostora S nad poljem $\mathbb{R}$

Neka je $A \in S$ proizvoljan vektor

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad a_i \in \mathbb{C} \quad i = \overline{1,4}$

$$A \in S \Rightarrow \left.\begin{array}{l} a_1 - 2\overline{a_2} + a_3 = 0 \\ \overline{a_1 + a_2 + a_3 + a_4 = 0} \end{array}\right\} \begin{array}{l} a_1 - 2\overline{a_2} + a_3 = 0 \\ a_1 + \overline{a_2} + \overline{a_3} + a_4 = 0 \end{array} \cdot (-1)$$

$$\begin{array}{ll} a_1 - 2\overline{a_2} + a_3 = 0 & a_1 = 2\overline{a_2} - a_3 \\ 3\overline{a_2} + \overline{a_3} - a_3 + a_4 = 0 & a_4 = a_3 - 3\overline{a_2} - \overline{a_3} \end{array}$$

$a_1 = x_1 + y_1 i$

$a_2 = x_2 + y_2 i$

$a_3 = x_3 + y_3 i$

$a_4 = x_4 + y_4 i$

$$A = \begin{pmatrix} 2(x_2 + y_2 i) - x_3 - y_3 i & x_2 + y_2 i \\ x_3 + y_3 i & -3(x_2 - y_2 i) - (x_3 - y_3 i) + x_3 + y_3 i \end{pmatrix}$$

$$= \begin{pmatrix} 2x_2 - 2y_2 i - x_3 - y_3 i & x_2 + y_2 i \\ x_3 + y_3 i & -3x_2 + 3y_2 + 2y_3 i \end{pmatrix}$$

$$= x_2 \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} + y_2 \begin{pmatrix} -2i & i \\ 0 & 3i \end{pmatrix} + x_3 \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + y_3 \begin{pmatrix} -i & 0 \\ i & 2i \end{pmatrix} \Rightarrow S = \mathcal{L}(E_1, E_2, E_3, E_4)$$

$E_1 = \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix}, E_2 = \begin{pmatrix} -2i & i \\ 0 & 3i \end{pmatrix}, E_3 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} -i & 0 \\ i & 2i \end{pmatrix}$

Pokažimo da je sistem $\{E_1, E_2, E_3, E_4\}$ linearno nezavisan

$\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 = 0 \in \mathbb{C}^{2\times 2}$

$$\begin{pmatrix} 2\alpha_1 - 2\alpha_2 i - \alpha_3 - i\alpha_4 & \alpha_1 + \alpha_2 i \\ \alpha_3 + i\alpha_4 & -3\alpha_1 + 3\alpha_2 i + 2\alpha_4 i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} 2\alpha_1 - \alpha_3 + (-2\alpha_2 - \alpha_4) i = 0 \\ \alpha_1 + \alpha_2 i = 0 \\ \alpha_3 + \alpha_4 i = 0 \\ -3\alpha_1 + (3\alpha_2 + 2\alpha_4) i = 0 \end{array}\right\} \Rightarrow \begin{array}{ll} \alpha_1 = 0 & \alpha_2 = 0 \\ \alpha_3 = 0 & \alpha_4 = 0 \end{array}$$

$\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 = 0 \Rightarrow \alpha_1 = 0 \land \alpha_2 = 0 \land \alpha_3 = 0 \land \alpha_4 = 0$

$\Rightarrow \{E_1, E_2, E_3, E_4\}$ lin. nezavisan

$$\left.\begin{array}{l} 1.\ \mathcal{L}(E_1, E_2, E_3, E_4) = S \\ 2.\ \{E_1 \ldots E_4\}\ \text{L.N} \end{array}\right\} \Rightarrow \begin{array}{l} \{E_1, E_2, E_3, E_4\}\ \text{baza u } S \\ \dim S = 4 \text{ nad poljem } \mathbb{R} \end{array}$$

* $A_{n\times n}$: $A^0 = E$ $\quad A^1 = A$ $\quad A^{p+1} = A^p \cdot A$

$A_{n\times n}$, $p, q \in \mathbb{N}_0$: $A^p \cdot A^q = A^{p+q}$ $\quad (A^p)^q = A^{p\cdot q}$

A i B komutativne matrice, tada $\forall p, q \in \mathbb{N}_0$ $A^p \cdot B^q = B^q \cdot A^p$

$A^p \cdot B^p = (A \cdot B)^p$

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} \cdot B^k$$

$A_{n\times n}$: Ako $\exists k \in \mathbb{N}$ tako da $A^{k+1} = A$, A je periodično

Ako je $A^2 = A$, A je idempotentno

Ako je $A^2 = E$, A je involutivno

Ako $\exists k \in \mathbb{N}$ tako da $A^k = 0$, A je nilpotentna, najmanje takvo k je index nilpotentnosti

2. Kvadratna matrica A je nilpotentna, i indexa 2. Dokazati da je $A \cdot (E+A)^n = A$ $\forall n \in \mathbb{N}$

Kako su matrice A i E komutativne, to važi

$(E+A)^n = \sum_{k=0}^{n} \binom{n}{k} E^{n-k} A^k$ $\quad$ A je nilpotentna, indexa 2 pa je $A^2 = 0$

$A^3 = A^2 \cdot A = 0 \cdot A = 0$ $\qquad A^k = 0, k \geq 2$

$$(E+A)^n = \binom{n}{0} E^n \cdot A^0 + \binom{n}{1} E^{n-1} A^1 + \binom{n}{2} E^{n-2} \underset{0}{\underset{\shortparallel}{A^2}} + \quad + \binom{n}{n} E^0 \underset{0}{\underset{\shortparallel}{A^n}}$$

$= E + nA$

$A \cdot (E+A)^n = A \cdot (E+nA) = A \cdot E + n \cdot A^2 = A + n \cdot 0 = A$

3. Neka je $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Naći A^n.

$A^2 = A \cdot A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$

$A^3 = A^2 \cdot A$

$= \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix}$

Uočimo da je $A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$. Dokažimo ovo matematičkom indukcijom

1) Neka je $n = 1$

$A^1 = \begin{pmatrix} 1 & 1\cdot a \\ 0 & 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ tvrđenje je tačno za $n = 1$

2) Pretpostavimo da je tvrđenje tačno za $n = k$, tj. da važi

$A^k = \begin{pmatrix} 1 & ka \\ 0 & 1 \end{pmatrix}$. Dokažimo da je tada tvrđenje tačno i za $n=k+1$ tj dokažimo da je $A^{k+1} = \begin{pmatrix} 1 & (k+1)a \\ 0 & 1 \end{pmatrix}$

$$A^{k+1} = A^k \cdot A = \begin{pmatrix} 1 & ka \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (k+1)a \\ 0 & 1 \end{pmatrix}$$

Iz 1) i 2) na osnovu principa mat. indukcije $\Rightarrow A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$ $\forall n \geq 1$

4. Odrediti A^n $(n \in N)$ ako je:

$$A = \begin{pmatrix} -2 & 0 & 0 \\ a & -2 & 0 \\ 0 & a & -2 \end{pmatrix}, a \in R$$

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & a & 0 \end{pmatrix} \quad A = -2E + B, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & a & 0 \end{pmatrix}$$

Matrice $-2E$ i B su komutativne pa važi

$$A^n = (-2E+B)^n = \sum_{k=0}^{n} (-2E)^{n-k} B^k$$

$$B^2 = B \cdot B = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & a & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a^2 & 0 & 0 \end{pmatrix}$$

$$B^3 = B^2 \cdot B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a^2 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B^k = 0, k \geq 3$$

$$A^n = \binom{n}{0} (-2E)^n \cdot B^0 + \binom{n}{1} (-2E)^{n-1} \cdot B^1 + \binom{n}{2} (-2E)^{n-2} \cdot B^2$$

$$A^n = (-2)^n \cdot E + n \cdot (-2)^{n-1} \cdot B + \frac{n(n-1)}{2} \cdot (-2)^{n-2} \cdot B^2$$

$$A^n = (-2)^n \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n(-2)^{n-1} \cdot \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & a & 0 \end{pmatrix} + \frac{n(n-1)}{2} (-2)^{n-2} \cdot$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a^2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} (-2)^n & 0 & 0 \\ n(-2)^{n-1} \cdot a & (-2)^n & 0 \\ \frac{n(n-1)}{2} \cdot (-2)^{n-2} a^2 & n(-2)^{n-1} a & (-2)^n \end{pmatrix}$$

5. Odrediti A^n $(n \in N)$ ako je $A = \begin{pmatrix} a & 0 & 0 \\ a & a & 0 \\ a & a & a \end{pmatrix}$

$$A = a \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad A = a \cdot B, \; B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad A^n = a^n \cdot B^n$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad B = E + C, \; C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$ E i C su komutativne pa je

$$B^n = (E+C)^n = \sum_{k=0}^{n} \binom{n}{k} E^{n-k} \cdot C^k$$

$$C^2 = C \cdot C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$C^3 = C^2 \cdot C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C^k = O \quad k \geq 3$$

$$B^n = \binom{n}{0} E^n \cdot C^0 + \binom{n}{1} E^{n-1} C^1 + \binom{n}{2} E^{n-2} C^2$$

$$B^n = E + n \cdot C + \frac{n(n-1)}{2} C^2$$

$$B^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} + \frac{n(n-1)}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

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$$B^n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ n + \frac{n(n-1)}{2} & n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ \frac{n(n+1)}{2} & n & 1 \end{pmatrix}$$

$$A^n = \begin{pmatrix} a^n & 0 & 0 \\ n \cdot a^n & a^n & 0 \\ \frac{n(n+1)}{2} a^n & n \cdot a^n & a^n \end{pmatrix}$$

⌐ $A_{n \times n}$:
Ako je $A^T = A$, A je simetrična matrica
Ako je $A^T = -A$, A je antisimetrična matrica ⌙

6. Jedina matrica koja je i simetrična i antisimetrična je nula matrica. Dokazati

A - simetrična $\Rightarrow A^T = A$
A - antisimetrična $\Rightarrow A^T = -A$

$$\left.\begin{matrix} A^T = A \\ A^T = -A \end{matrix}\right\} \quad A = -A \quad 2A = 0 \quad A = 0$$

7. Svaka kvadratna matrica A se može predstaviti kao suma simetrične i antisimetrične matrice. Dokazati.

⌐ $A = B + C$ $\quad$ B - simetrična
$A^T = B^T + C^T$ $\quad$ C - antisimetrična
$A^T = B - C$

$$\left.\begin{matrix} A = B + C \\ A^T = B - C \end{matrix}\right\} \quad \begin{matrix} B = \frac{1}{2}(A + A^T) \\ C = \frac{1}{2}(A - A^T) \end{matrix}$$

Kvadratnu matricu A moguće je predstaviti kao sumu $A = B + C$, $B = \frac{1}{2}(A + A^T)$, $C = \frac{1}{2}(A - A^T)$ gdje je B simetrična matrica, C antisimetrična ⌙

$$B + C = \frac{1}{2}(A + A^T + A - A^T) = A$$

$$B^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = B \Rightarrow B \text{ je simetrična}$$

$$C^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -C \Rightarrow C \text{ je antisimetrična}$$

$*\quad A\cdot B = B\cdot A = E \qquad A^{-1}$ inverzna matrica matrice A

$B = A^{-1}$

$A\cdot A^{-1} = A^{-1}\cdot A = E$

$(A^{-1})^{-1} = A$

$(A\cdot B)^{-1} = B^{-1}\cdot A^{-1}$

$(A^T)^{-1} = (A^{-1})^T$

$(A+B)^{-1} \neq A^{-1} + B^{-1}$

$(A^p)^{-1} = (A^{-1})^p$

8. Dokazati sledeću jednakost

$(E_n - I_n)^{-1} = E_n - \frac{1}{n-1}\cdot I_n$, gdje je E_n - jedinična matrica

I_n - matrica čiji su elementi jednaki jedan

$(E_n - I_n)\cdot\left(E_n - \frac{1}{n-1}\cdot I_n\right) = E_n - \frac{1}{n-1} I_n - I_n + \frac{1}{n-1} I_n^2$

$= E_n - \left(\frac{1}{n-1} + 1\right) I_n + \frac{1}{n-1} I_n^2 = E_n - \frac{n}{n-1} I_n + \frac{1}{n-1} I_n^2 = E_n \quad (1)$

$I_n^2 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} n & n & \dots & n \\ n & n & \dots & n \\ n & n & \dots & n \end{pmatrix} = n\cdot I_n$

Slično $\left(E_n - \frac{1}{n-1} I_n\right)\cdot(E_n - I_n) = E_n \quad (2)$

Iz 1 i 2 slijedi da je $(E_n - I_n)^{-1} = E_n - \frac{1}{n-1} I_n$

9. Ako je $B = X^{-1}\cdot A\cdot X$ tada je $B^n = X^{-1}\cdot A^n \cdot X$ - mat. ind.

10. Dat je polinom $P(t) = -t^2 + t + 2$. Ako su A i B kvadratne matrice i B regularna matrica: a) dokazati jednakost $P(B^{-1}\cdot A\cdot B) = B^{-1}\cdot P(A)\cdot B$ b) naći $[P(B^{-1}AB)]^n$ ako je $A = \begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix}$

$P(A) = -A^2 + A + 2\cdot E$

a) $P(B^{-1}AB) = -(B^{-1}AB)^2 + B^{-1}AB + 2E = -B^{-1}AB\cdot B^{-1}AB + B^{-1}AB$
$+ 2E = -B^{-1}\cdot A^2\cdot B + B^{-1}AB + 2E = -B^{-1}A^2 B + B^{-1}AB + 2B^{-1}\cdot B =$
$= B^{-1}(-A^2 B + AB + 2B) = B^{-1}(-A^2 + A + 2E)B = B^{-1}\cdot P(A)\cdot B$

b) $[P(B^{-1}AB)]^n \overset{a)}{=} [B^{-1}\cdot P(A)\cdot B]^n \overset{9.}{=} B^{-1}\cdot (P(A))^n \cdot B$

$P(A) = -\begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = -\begin{pmatrix} 13 & 1 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} 5 & 1 \\ 4 & 0 \end{pmatrix}$

$= \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix} \quad P(A) = -8\cdot E \quad (P(A))^n = (-8)^n\cdot E$

$[P(B^{-1}AB)]^n = B^{-1}\cdot(-8)^n\cdot E\, B = (-8)^n\cdot E = \begin{pmatrix} (-8)^n & 0 \\ 0 & (-8)^n \end{pmatrix}$