ekvivalentni.

Baza i dimenzija prostora

1) $V = P^n$ P - polje

$V = \{(x_1 \dots x_n) \mid x_i \in P \quad i = \overline{1,n}\}$

$= \{x_1(1,0 \dots 0) + x_2(0,1 \dots 0) + \dots + x_n(0,0 \dots 1) \mid x_1, x_2, \dots x_n \in P\}$

$= \mathcal{L}(e_1, e_2, \dots e_n)$ gdje je $e_1 = (1,0 \dots 0)$ $e_2 = (0,1 \dots 0)$ $e_n = (0,0, \dots 1)$

$\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n = 0$

$(\alpha_1, \alpha_2 \dots \alpha_n) = (0,0 \dots 0)$

$\Downarrow$

$\alpha_1 = 0$

$\alpha_2 = 0$

$\vdots$

$\alpha_n = 0$

Sistem $e_1 \dots e_n$ linearno nezavisan

1° $\mathcal{L}(e_1 \dots e_n) = V$

2° $\{e_1 \dots e_n\}$ je LN

$\Rightarrow$ baza u V je $\{e_1 \dots e_n\}$

$\dim V = n$

2) $V = P_{\leq n}$ - skup polinoma stepena $\leq n$, $P = R$

$V = \{a_0 + a_1 t + \dots + a_n t^n \mid a_0 \dots a_n \in R\}$

$= \{a_0 \cdot f_0(t) + a_1 \cdot f_1(t) + \dots + a_n \cdot f_n(t) \mid a_0, a_1 \dots a_n \in R\}$

$= \mathcal{L}(f_0, f_1, \dots f_n)$ $f_0(t) = 1$ $f_1(t) = t$ $f_2(t) = t^2$ $f_3(t) = t^3 \dots$

$\alpha_0 f_0 + \alpha_1 f_1 + \dots + \alpha_n f_n = 0$

$\alpha_0 \cdot 1 + \alpha_1 \cdot t + \dots + \alpha_n \cdot t^n = 0$

$\Downarrow$

$\alpha_0 = 0$ $\alpha_1 = 0$

$P_{\leq 4}$

$\dim P_{\leq 4} = 5$

$\{1, t, t^2, t^3, t^4\}$

$p(t) = 1 - 2t + t$

$p = (1, -2, 0, 0, 1)$

Sistem $\mathcal{L}(f_0, f_1 \dots f_n)$ je linearno nezavisan tj. Baza u $\{1, t, \dots t^n\}$

$\dim P_{\leq n} = n+1$

1. $V = \mathbb{C}$

a) nad poljem $\mathbb{R}$

b) nad poljem $\mathbb{C}$

a) Neka je $z = a + bi$ proizvoljan vektor iz V. Tada je $z = a \cdot 1 + b \cdot i$

$z = a \cdot e_1 + b \cdot e_2$, gdje je $e_1 = 1$, $e_2 = i$

$$\left.\begin{array}{l} (\forall z \in V) \quad z \in \mathcal{L}(e_1, e_2) \text{ pa je } V \subseteq \mathcal{L}(e_1, e_2) \\ e_1, e_2 \in V \Rightarrow \mathcal{L}(e_1, e_2) \subseteq V \end{array}\right\} \Rightarrow V = \mathcal{L}(e_1, e_2)$$

Pokažimo da su $\{e_1, e_2\}$ linearno nezavisan sistem

$$\begin{array}{l} \alpha \cdot e_1 + \beta \cdot e_2 = 0 \\ \alpha \cdot 1 + \beta \cdot i = 0 \end{array} \Rightarrow \alpha = 0 \land \beta = 0 \qquad \left.\begin{array}{l} 1^\circ \ V = \mathcal{L}(e_1, e_2) \\ 2^\circ \ \{e_1, e_2\} \text{ L.N.} \end{array}\right\} \begin{array}{l} \{e_1, e_2\} \text{ je ba-} \\ \text{za u } V \\ \dim V = 2 \end{array}$$

b) Neka je $z = a + bi$ proizvoljan vektor iz V.

Tada je $z = (a + b \cdot i) \cdot 1$

$z = (a + b \cdot i) \cdot e$

$z = \alpha \cdot e$, $\alpha \in \mathbb{C}$, $e = 1 \in V$

$(\forall z \in V)$ $z \in \mathcal{L}(e) \Rightarrow V \subseteq \mathcal{L}(e)$

$e \in V \Rightarrow \mathcal{L}(e) \subseteq V \Rightarrow V = \mathcal{L}(e)$

$e \neq 0 \Rightarrow \{e\}$ L.N.

$$\left.\begin{array}{l} 1^\circ \ V = \mathcal{L}(e) \\ 2^\circ \ \{e\} \text{ L.N.} \end{array}\right\} \Rightarrow \begin{array}{l} e \text{ je baza u } V \\ \dim V = 1 \end{array}$$

2. Naći koordinate vektora $x=(1,1,1)$ u bazi $\{e_1, e_2, e_3\}$, gdje je $e_1=(2,2,-1)$, $e_2=(2,-1,-2)$, $e_3=(-1,2,2)$

$$x=\alpha_1\cdot e_1+\alpha_2\cdot e_2+\alpha_3\cdot e_3$$

$$(1,1,1)=\alpha_1\cdot(2,2,-1)+\alpha_2\cdot(2,-1,-2)+\alpha_3\cdot(-1,2,2)$$

$$\begin{aligned} 2\alpha_1+2\alpha_2-\alpha_3&=1\\ 2\alpha_1-\alpha_2+2\alpha_3&=1\\ -\alpha_1-2\alpha_2+2\alpha_3&=1 \end{aligned}$$

$$\begin{aligned} -\alpha_1-2\alpha_2+2\alpha_3&=1\\ 2\alpha_1+2\alpha_2-\alpha_3&=1 \quad \cdot 2\\ 2\alpha_1-\alpha_2+2\alpha_3&=1 \quad \cdot 2 \end{aligned}$$

$$\begin{aligned} -\alpha_1-2\alpha_2+2\alpha_3&=1\\ -2\alpha_2+3\alpha_3&=3 \quad /\cdot 5\\ -5\alpha_2+6\alpha_3&=3 \quad /\cdot(-2) \end{aligned}$$

$$\begin{aligned} -\alpha_1-2\alpha_2+2\alpha_3&=1\\ -10\alpha_2+15\alpha_3&=15 \quad +\\ 10\alpha_2-12\alpha_3&=-6 \end{aligned}$$

$$\begin{aligned} -\alpha_1-2\alpha_2+2\alpha_3&=1\\ -2\alpha_2+3\alpha_3&=3\\ 3\alpha_3&=9 \end{aligned}$$

$\alpha_3=3$, $\alpha_2=3$, $\alpha_1=-$

$$x=(-1)\cdot e_3+3\cdot e_2+3\cdot e_3$$

$x=(-1,3,3)$ u bazi $\{e_1,e_2,e_3\}$

3. U prostoru R^4 naći dvije različite baze koje sadrže vektore $e_1=(1,1,0,0)$ $e_2=(0,0,1,1)$.

1, 1, 0, 0
0, 1, 1, 0
0, 0, 1, 1
0, 0, 0, 1

Tražimo $e_3=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ tako da sistem e_1, e_2, e_3 bude linearno nezavisan, tj. da iz $\alpha\cdot e_1+\beta\cdot e_2+\gamma\cdot e_3=0 \Rightarrow \alpha=0 \wedge \beta=0 \wedge \gamma=0$

$$\alpha\cdot(1,1,0,0)+\beta\cdot(0,0,1,1)+\gamma\cdot(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(0,0,0,0)$$

$$\begin{aligned} \alpha+\gamma\cdot\alpha_1&=0\\ \alpha+\gamma\cdot\alpha_2&=0\\ \beta+\gamma\cdot\alpha_3&=0\\ \beta+\gamma\cdot\alpha_4&=0 \end{aligned} \quad (*)$$

Koordinate vektora e_3 biramo tako da sistem $(*)$ ima samo trivijalno rešenje

Npr. za $\alpha_1=1$, $\alpha_2=0$, $\alpha_3=0$, $\alpha_4=0$ dobijamo sistem

$$\left.\begin{aligned} \alpha+\gamma&=0\\ \alpha&=0\\ \beta&=0 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha&=0\\ \beta&=0\\ \gamma&=0 \end{aligned}$$

$e_3'=(0,1,0,0)$

$e_3=(1,0,0,0)$

Tražimo $e_4 = (\beta_1, \beta_2, \beta_3, \beta_4)$ tako da sistem $\{e_1, e_2, e_3, e_4\}$ bude linearno nezavisan, tj. da iz

$\alpha_1 \cdot e_1 + \beta \cdot e_2 + \gamma \cdot e_3 + \delta \cdot e_4 = 0 \Rightarrow \alpha = 0 \wedge \beta = 0 \wedge \gamma = 0 \wedge \delta = 0$

$$\left.\begin{aligned} &\alpha + \gamma + \delta \cdot \beta_1 = 0 \\ &\alpha + \delta \cdot \beta_2 = 0 \\ &\beta + \delta \cdot \beta_3 = 0 \\ &\beta + \delta \cdot \beta_4 = 0 \end{aligned}\right\} (**)$$

Koordinate vektora e_4 biramo tako da sistem $(**)$ ima samo trivijalno rešenje

Npr. $\beta_1 = 0, \beta_2 = 0, \beta_3 = 0, \beta_4 = 1$ dobijamo sistem:

$$\left.\begin{aligned} &\alpha + \gamma = 0 \\ &\alpha = 0 \\ &\beta = 0 \\ &\beta + \delta = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &\alpha = 0 \\ &\beta = 0 \\ &\gamma = 0 \\ &\delta = 0 \end{aligned}$$

$e_4' = (0, 0, 1, 0)$

$\{e_1, e_2, e_3, e_4\}$ baza u R^4

$\{e_1, e_2, e_3', e_4'\}$ baza u R^4

4. Neka je $L_1 = \{(a, b, c, d) \mid b + c + d = 0\}$ $L_2 = \{(a, b, c, d) \mid a + b = 0 \wedge c = 2d\}$

Naći bazu i dimenziju prostora $L_1, L_2, L_1 + L_2, L_1 \cap L_2$

Napomena: $\dim(L_1 + L_2) = \dim L_1 + \dim L_2 - \dim(L_1 \cap L_2)$

$L_1 = \{(a, b, c, d) \mid b + c + d = 0\} = \{(a, -c-d, c, d) \mid a, c, d \in R\}$

$= \{(a, 0, 0, 0) + (0, -c, c, 0) + (0, -d, 0, d) \mid a, c, d \in R\}$

$= \{a \cdot (1, 0, 0, 0) + c \cdot (0, -1, 1, 0) + d(0, -1, 0, 1) \mid a, c, d \in R\}$

$= \mathcal{L}(e_1, e_2, e_3)$ gdje je $e_1 = (1, 0, 0, 0)$, $e_2 = (0, -1, 1, 0)$, $e_3 = (0, -1, 0, 1)$

$$\begin{aligned} e_1 &= (1, 0, 0, 0) \\ e_2 &= (0, -1, 1, 0) \\ e_3 &= (0, -1, 0, 1) \end{aligned} \; (-1) \quad \sim \quad \begin{aligned} &(1, 0, 0, 0) \; b_1 \\ &(0, -1, 1, 0) \; b_2 \\ &(0, 0, -1, 1) \; b_3 \end{aligned}$$

$\{b_1, b_2, b_3\}$ je trapezni sistem nenultih vektora pa je linearno nezavisan. Dati sistem je dobijen elementarnim transformacijama sistema $\{e_1, e_2, e_3\}$ pa je i sistem $\{e_1, e_2, e_3\}$ linearno nezavisan, i čini bazu u L_1 $\dim L_1 = 3$

Napomena: $e_1 = (1, 0, 0, 0)$ $0 + 0 + 0 = 0$

$e_2 = (0, -1, 1, 0)$ $-1 + 1 + 0 = 0$ $\quad e_1, e_2, e_3 \in L_1$

$e_3 = (0, -1, 0, 1)$ $-1 + 0 + 1 = 0$

$L_2 = \{(a,b,c,d) \mid a+b=0 \land c=2d\} = \{(-b,b,2d,d) \mid b,d\in R\}$
$= \{(-b,b,0,0)+(0,0,2d,d) \mid b,d\in R\} = \{b(-1,1,0,0)+d(0,0,2,1)\}$
$= \mathcal{L}(f_1,f_2)$ gdje je $f_1=(-1,1,0,0)\in L_2$ $f_2=(0,0,2,1)\in L_2$

sistem $\{f_1,f_2\}$ je linearno nezavisan (trapezni) pa je baza u L_2 $\dim L_2 = 2$

$L_1 \subseteq L_1+L_2$

$L_2 \subseteq L_1+L_2$

Odredimo bazu potprostora L_1+L_2 kao maksimalan linearno nezavisan podsistem sistema $\{e_1,e_2,e_3,f_1,f_2\}$

$$\begin{matrix} e_1=(1,0,0,0) \\ e_2=(0,-1,1,0) \\ e_3=(0,-1,0,1) \\ f_1=(-1,1,0,0) \\ f_2=(0,0,2,1)\end{matrix} \sim \begin{matrix} (1,0,0,0) \\ (0,-1,1,0) \\ (0,-1,0,1) \\ (0,1,0,0) \\ (0,0,2,1)\end{matrix} \cdot(-1) \sim \begin{matrix} (1,0,0,0) \\ (0,-1,1,0) \\ (0,0,-1,1) \\ (0,0,1,0) \\ (0,0,2,1)\end{matrix} \; + \;\; \cdot 2$$

$$\sim \begin{matrix} (1,0,0,0) \\ (0,-1,1,0) \\ (0,0,-1,0) \\ (0,0,0,1) \\ (0,0,0,3)\end{matrix} \cdot(-3) \sim \begin{matrix} (1,0,0,0) & b_1 \\ (0,-1,1,0) & b_2 \\ (0,0,-1,1) & b_3 \\ (0,0,0,1) & b_4 \\ (0,0,0,0) & b_5\end{matrix}$$

Sistem $\{b_1,b_2,b_3,b_4,b_5\}$ je linearno zavisan jer sadrži nula vektor pa je i sistem $\{e_1,e_2,e_3,f_1,f_2\}$ linearno zavisan.

Sistem $\{b_1,b_2,b_3,b_4\}$ je linearno nezavisan trapezni sistem pa je i sistem od koga je nastao elementarnim transformacijama linearno nezavisan, a to je $\{e_1,e_2,e_3,f_1\}$ i on čini bazu u L_1+L_2 $\dim(L_1+L_2)=4$

$\dim(L_1+L_2) = \dim L_1 + \dim L_2 - \dim(L_1\cap L_2)$

$4 = 3+2-\dim(L_1\cap L_2)$ Neka je $x\in L_1\cap L_2$ proizvoljno

Tada $x\in L_1 \land x\in L_2$

$x=(a,b,c,d)$ $x\in L_1 \to b+c+d=0$

$x\in L_2 \to a+b=0 \land c=2d$

$b+c+d=0$ $b=-c-d=-3d$

$a+b=0$ $a=-b=3d$

$c=2d$ $x=(3d,-3d,2d,d)$

$L_1 \cap L_2 = \{(3d, -3d, 2d, d) \mid d \in R\} = \{d(3,-3,2,1) \mid d \in R\} = \mathcal{L}(e)$

$e = (3,-3,2,1)$ $e \in L_1$ $e \in L_2$ $\{e\}$ je baza u $L_1 \cap L_2$

5. Ispitati linearnu zavisnost vektora a_1, a_2, a_3, a_4, a_5. Naći bazu i dimenziju potprostora generisanog tim vektorima

$$\begin{array}{l} a_1 = (1,0,0,-1) \\ a_2 = (2,1,1,0) \\ a_3 = (1,1,1,1) \\ a_4 = (1,2,3,4) \\ a_5 = (0,1,2,3) \end{array} \begin{array}{c} \cdot(-2) \quad \cdot(-1) \\ + \quad + \end{array} \sim \begin{array}{l} (1,0,0,-1) \\ (0,1,1,2) \\ (0,1,1,2) \\ (0,2,3,5) \\ (0,1,2,3) \end{array} \begin{array}{c} \cdot(-1) \quad \cdot(-2) \\ + \quad + \end{array} \sim \begin{array}{l} (1,0,0,-1) \\ (0,1,1,2) \\ (0,0,0,0) \\ (0,0,1,1) \\ (0,0,1,1) \end{array} \cdot(-1)$$

$$\sim \begin{array}{l} (1,0,0,-1) \\ (0,1,1,2) \\ (0,0,0,0) \\ (0,0,1,1) \\ (0,0,0,0) \end{array} \sim \begin{array}{ll} (1,0,0,-1) & b_1 \\ (0,1,1,2) & b_2 \\ (0,0,1,1) & b_3 \\ (0,0,0,0) & b_4 \\ (0,0,0,0) & b_5 \end{array}$$

Sistem $\{b_1, b_2, b_3, b_4, b_5\}$ je linearno zavisan jer sadrži nula vektor. Onda je i sistem $\{a_1, a_2, a_3, a_4, a_5\}$ linearno zavisan. Sistem $\{b_1, b_2, b_3\}$ je linearno nezavisan trapezni sistem nenultih vektora $\{a_1, a_2, a_4\}$ linearno nezavisan i čini bazu u $\mathcal{L}(a_1, a_2, a_3, a_4, a_5)$ pa je $\mathcal{L}(a_1, a_2, a_4) = \mathcal{L}(a_1, a_2, a_3, a_4, a_5)$

$\dim \mathcal{L}(a_1, \dots, a_5) = 3$

6. U prostoru $P_{\leq n}$ dat je skup polinoma koji zadovoljavaju uslov $f(0)=0$. Ispitati da li je dati skup potprostor prostora $P_{\leq n}$ nad poljem R i odrediti mu bazu i dimenziju $L = \{f \mid f \in P_{\leq n} \land f(0) = 0\}$

1° $f, g \in L$

$$\left.\begin{array}{l} f+g \in P_{\leq n} \\ (f+g)(0) = f(0) + g(0) = 0+0 = 0 \end{array}\right\} \Rightarrow f+g \in L$$

2° $f \in L$, $\alpha \in \mathbb{R}$

$$\left.\begin{array}{l} \alpha \cdot f \in P_{\le n} \\ (\alpha \cdot f)(0) = \alpha \cdot f(0) = \alpha \cdot 0 = 0 \end{array}\right\} \Rightarrow \alpha \cdot f \in L$$

Iz 1° i 2° slijedi da je L potprostor prostora $P_{\le n}$ nad poljem $\mathbb{R}$

Neka je $f \in L$ proizvoljan vektor. Tada je $f(0) = 0$

$f(t) = a_0 + a_1 t + \ldots + a_n t^n$

$f(0) = 0 \Rightarrow a_0 = 0$

$f(t) = a_1 \cdot f_1 + a_2 \cdot f_2 + \ldots + a_n \cdot f_n$

$f_1(t) = t \in L$ $\quad f_n(t) = t^n \in L$

$f_2(t) = t^2 \in L$ $\quad (\forall f \in L)\ f \in \mathcal{L}(f_1, \ldots, f_n)$ pa je $L \subseteq \mathcal{L}(f_1, \ldots, f_n)$

$f_1 \ldots f_n \in L \Rightarrow \mathcal{L}(f_1 \ldots f_n) \subseteq L$

$\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n = 0$

$(\alpha_1 f_1 + \ldots + \alpha_n f_n)(t) = 0, \forall t \in \mathbb{R}$

$\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_n \cdot t^n = 0, \forall t \in \mathbb{R} \Rightarrow \alpha_1 = 0 \ldots \alpha_n = 0$

Sistem $\{f_1 \ldots f_n\}$ je linearno nezavisan

Baza u L je $\{f_1 \ldots f_n\}$ $\dim L = n$

7. Odrediti bazu i dimenziju potprostora $L = \{f \mid f \in P_{\le n} \wedge f(1) = 0\}$

$f(t) = a_0 + a_1 t + \ldots + a_n t^n$ proizvoljan vektor iz L

$f(1) = 0 \Rightarrow a_0 + a_1 + \ldots + a_n = 0$

$a_0 = -a_1 - a_2 \ldots - a_n$

$f(t) = -a_1 - a_2 - \ldots - a_n + a_1 t + a_2 t^2 + \ldots + a_n t^n$

$f(t) = a_1(t-1) + a_2(t^2-1) + \ldots + a_n(t^n-1)$

$f(t) = a_1 \cdot g_1(t) + a_2 g_2(t) + \ldots + a_n g_n(t)$

$g_i(t) = (t^i - 1), i = \overline{1,n}$

Uočimo $g_i(1) = 0$ $\forall i = \overline{1,n} \Rightarrow g_i \in L$

$(\forall f \in L)$ $f \in \mathcal{L}(g_1 \ldots g_n) \Rightarrow \alpha \subseteq \mathcal{L}(g_1, \ldots, g_n)$

$g_1 \ldots g_n \in L \Rightarrow \mathcal{L}(g_1 \ldots g_n) \subseteq L \Rightarrow L = \mathcal{L}(g_1 \ldots g_n)$

$\alpha_1 g_1 + \ldots + \alpha_n g_n = 0$ $\quad \alpha_1 g_1(t) + \ldots + \alpha_n g_n(t) = 0$ $\forall t$

$\alpha_1(t-1)+\alpha_2(t^2-1)+\dots+\alpha_n(t^n-1)=0 \quad \forall t$ $\quad \alpha_1=0$

$(-\alpha_1-\alpha_2-\dots-\alpha_n)+\alpha_1 t+\alpha_2 t^2+\dots+\alpha_n t^n=0$ $\quad \alpha_n=0$

$\{1,t\dots t^n\}$ linearno nezavisan $\Rightarrow -\alpha_1 \dots -\alpha_n=0$

$\{g_1\dots g_n\}$ je linearno nezavisan i čini bazu u L

8. Dokazati da je $\{x^2+x, x^2-x, x+1\}$ baza vektorskog prostora $P\leq 2$. Naći koordinate vektora $-x^2+2x+3$ u toj bazi.

$f_1(x)=x^2+x$

$f_2(x)=x^2-x$

$f_3(x)=x+1 \quad \dim P\leq 2=3$

Dovoljno je pokazati da je sistem linearno nezavisan

$\alpha_1\cdot f_1+\alpha_2\cdot f_2+\alpha_3\cdot f_3=0$

$\alpha_1(x^2+x)+\alpha_2(x^2-x)+\alpha_3(x+1)=0$

$\alpha_3+(\alpha_1-\alpha_2+\alpha_3)x+(\alpha_1+\alpha_2)x^2=0$

$\{1,x,x^2\}$ linearno nezavisan $\Rightarrow \alpha_3=0 \quad \alpha_1-\alpha_2+\alpha_3=0$

$\alpha_1+\alpha_2=0 \quad \alpha_1-\alpha_2=0$

$2\alpha_1=0\Rightarrow\alpha_1=0$

$\alpha_2=-\alpha_1=0$

Sistem $\{f_1,f_2,f_3\}$ je linearno nezavisan i čini bazu u $P\leq 2$

$p(x)=-x^2+2x+3$

$p=\alpha_1 f_1+\alpha_2\cdot f_2+\alpha_3\cdot f_3$

$-x^2+2x+3=\alpha_1(x^2+x)+\alpha_2(x^2-x)+\alpha_3(x+1)$

$-x^2+2x+3=\alpha_3+(\alpha_1-\alpha_2+\alpha_3)x+(\alpha_1+\alpha_2)x^2$

$\alpha_3=3$

$\alpha_1-\alpha_2+\alpha_3=2$

$\alpha_1+\alpha_2=-1$

$\alpha_1=-1$

$\alpha_2=0$

$\alpha_3=3$

$p=-1\cdot f_1+0\cdot f_2+3\cdot f_3$

$p=(-1,0,3)$ u bazi $\{f_1,f_2,f_3\}$

$p=(3,2,-1)$ u bazi $\{1,x,x^2\}$

9. U vektorskom prostoru R^3 dat je potprostor L_1 svih vektora čija je prva koordinata 0 i L_2 potprostor generisan vektorima $(1,1,1)$ i $(2,3,0)$. Odrediti bazu potpro

stora $L_1 \cap L_2$.

$L_1 = \{(0, \alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$

$L_2 = \mathcal{L}((1,1,1),(2,3,0)) = \{\beta_1(1,1,1) + \beta_2(2,3,0) \mid \beta_1, \beta_2 \in \mathbb{R}\}$

$= \{\beta_1 + 2\beta_2, \beta_1 + 3\beta_2, \beta_1 \mid \beta_1 + \beta_2 \in \mathbb{R}\}$

Neka je $x \in L_1 \cap L_2$ proizvoljan vektor

$x \in L_1 \Rightarrow$ postoji α_1, α_2 tako da $x = (0, \alpha_1, \alpha_2)$

$x \in L_2 \Rightarrow$ postoji β_1, β_2 tako da $x = (\beta_1 + 2\beta_2, \beta_1 + 3\beta_2, \beta_1)$

$\Rightarrow \beta_1 + 2\beta_2 = 0$ $\qquad \beta_1 = -2\beta_2$ $\qquad$ Baza u $L_1 \cap L_2$ je

$\beta_1 + 3\beta_2 = \alpha_1$ $\qquad x = (0, \beta_2, -2\beta_2)$ $\qquad \{e\}$ $e = (0, 1, -2)$

$\beta_1 = \alpha_2$ $\qquad x = \beta_2 \cdot (0, 1, -2)$

10. U vektorskom prostoru V dat je sistem vektora $a_1 \dots a_n$. Ako se svaki vektor vektorskog prostora V može predstaviti kao linearna kombinacija vektora $a_1 \dots a_n$ i ako postoji vektor $b \in V$ koji se na samo jedan način može predstaviti kao linearna kombinacija vektora $a_1 \dots a_n$ tada je $\{a_1 \dots a_n\}$ baza u V. Dokazati:

Sistem $\{a_1 \dots a_n\}$ je baza akko se svaki vektor iz V na jedinstven način predstaviti kao linearna kombinacija vektora $a_1 \dots a_n$. Pretpostavimo suprotno tj. da postoji x iz V tako da se na dva načina razlaže duž vektora $a_1 \dots a_n$

$x = \alpha_1 a_1 + \dots + \alpha_n a_n$

$x = \beta_1 a_1 + \dots + \beta_n a_n$

$(\alpha_1 - \beta_1) a_1 + \dots + (\alpha_n - \beta_n) a_n = 0$

$b = \gamma_1 a_1 + \dots + \gamma_n a_n$ (b se na jedinstven način razlaže duž vektora $a_1 \dots a_n$)

$b = b + 0 = \gamma_1 a_1 + \dots + \gamma_n a_n + (\alpha_1 - \beta_1) a_1 + \dots + (\alpha_n - \beta_n) a_n$

$\gamma_1 a_1 + \dots + \gamma_n a_n = (\gamma_1 + \alpha_1 - \beta_1) a_1 + \dots + (\gamma_n + \alpha_n - \beta_n) a_n$ $(*)$

Kako se b na jedinstven način razlaže to iz * slijedi da

$\alpha_1-\beta_1=0 \quad \alpha_n-\beta_n=0 \Rightarrow \alpha_1=\beta_1 \quad \alpha_n=\beta_n$

Vektor x se na jedinstven način razlaže duž vektora $a_1 \dots a_n$. To važi za svako x iz V pa je $\{a_1 \dots a_n\}$ baza u V

11. Dokazati da kompleksni broj $x=2+3i$, $y=1-2i$ generišu prostor $V=\mathbb{C}$ kompleksnih brojeva nad poljem $\mathbb{R}$

x i $y\in V \quad \dim V=2$ nad poljem $\mathbb{R}$

Pokažimo da je sistem $\{x,y\}$ linearno nezavisan

$\alpha\cdot x+\beta\cdot y=0$

$\alpha\cdot(2+3i)+\beta\cdot(1-2i)=0, \ \alpha,\beta\in\mathbb{R}$

$2\alpha+\beta+(3\alpha-2\beta)\cdot i=0$

$\left.\begin{array}{l}2\alpha+\beta=0\\3\alpha-2\beta=0\end{array}\right\} \Rightarrow \begin{array}{l}7\alpha=0\\ \alpha=0 \ \beta=0\end{array}$

$\{x,y\}$ je linearno nezavisan i čini bazu u V, a to znači i da je generator u V.

12. $V=\mathbb{C}^2$

a) $P=\mathbb{R}$ b) $P=\mathbb{C}$

a) $x\in V$

$x=(a+bi, c+di)$

$x=(a,0)+(bi,0)+(0,c)+(0,di)$

$x=a\cdot(1,0)+b\cdot(i,0)+c\cdot(0,1)+d(0,i)$

$V=\mathcal{L}(e_1,e_2,e_3,e_4)$

$e_1=(1,0)$

$e_2=(i,0)$

$e_3=(0,1)$

$e_4=(0,i)$

$\alpha\cdot e_1+\beta\cdot e_2+\gamma\cdot e_3+\delta\cdot e_4=0\in\mathbb{C}^2$

$(\alpha+\beta i, \gamma+\delta i)=(0,0)$

$\Downarrow$

$\left.\begin{array}{l}\alpha+\beta i=0\\ \gamma+\delta i=0\end{array}\right\} \Rightarrow \begin{array}{l}\alpha=0 \wedge \beta=0\\ \gamma=0\wedge\delta=0\end{array}$

$\{e_1,e_2,e_3,e_4\}$ je linearno nezavisan

$\left.\begin{array}{l}1^\circ \ \mathcal{L}(e_1,e_2,e_3,e_4)=V\\ 2^\circ \ \{e_1,e_2,e_3,e_4\} \text{ LN}\end{array}\right\} \Rightarrow$

$\{e_1,e_2,e_3,e_4\}$ baza u V $\dim V=4$ nad poljem $\mathbb{R}$

$x=(2-5i, -3+7i)$

$x=(2,-5,-3,7)$ u bazi $\{e_1,e_2,e_3,e_4\}$

1. Neka je L potprostor prostora V tako da je $\dim L = m$, a $\dim V = n$ $(m<n)$. Dokazati da postoji baza u V koja ne sadrži nijedan vektor iz L.

Neka je $\{e_1, \dots, e_m\}$ baza u L

Dopunimo ovaj sistem do baze u V

Neka je $\{e_1, \dots, e_m, e_{m+1}, \dots, e_n\}$ baza u V

$e_{m+1} \neq L$ jer ako bi pripadao $\{e_1, \dots, e_m, e_{m+1}\}$ linearno nezavisan sistem (kao podsistem $L N \{e_1, \dots e_m, \dots e_n\}$) u prostoru L čija je dimenzija m, što nije moguće.

Slično zaključujemo da $e_{m+2} \notin L, \dots, e_n \notin L$

Uočimo $e_1 + e_{m+1} \notin L$ jer ako bi $a = e_1 + e_{m+1} \in L$, tada bi i $e_{m+1} = a - e_1 \in L$, što nije tačno

Slično $e_2 + e_{m+1} \notin L, \dots, e_m + e_{m+1} \notin L$

$\{e_1 + e_{m+1}, \dots e_m + e_{m+1}, e_{m+1}, \dots, e_n\}$ linearno nezavisan (nastao elementarnim transformacijama sistema $\{e_1, \dots e_m, \dots e_n\}$) nijedan vektor nije $\cap L$, a ima ih n, koliko je dimenzija prostora V. Dakle, ovaj sistem je baza u V.

Ako su L_1 i L_2 potprostori konačnodimenzionalnog vektorskog prostora V tada su sledeće tvrdnje ekvivalentne:

a) $V = L_1 \dot{+} L_2$ b) za svaki vektor $a \in V$ postoje jedinstveni vektori $x_1 \in L_1$, $x_2 \in L_2$ takvi da je $a = x_1 + x_2$ c) ako je $\{a_1, \dots, a_k\}$ baza u L_1 i $\{b_1, \dots b_l\}$ baza u L_2 tada je $\{a_1, \dots, a_k, b_1, \dots b_l\}$ baza u V

d) $\dim V = \dim L_1 + \dim L_2$ $V = L_1 + L_2$

2. Neka su L_1 i L_2 konačnodimenzionalni potprostori prostora V takvi da je $L_1 \subseteq L_2$ i $\dim L_1 = \dim L_2$. Dokazati da je $L_1 = L_2$

Neka je $\{a_1, \dots, a_k\}$ baza u L_1

$\{a_1,\dots,a_k\} \subseteq L_1 \subseteq L_2$
$\{a_1,\dots,a_k\}$ LN $\subseteq L_2$ $\Big\} \Rightarrow \{a_1,\dots,a_k\}$ je baza u L_2
$\{a_1,\dots,a_k\}$ LN jer je baza u L_1

$\dim L_2 = \dim L_1 = k$

$\{a_1 \dots a_k\}$ baza u $L_1 \Rightarrow \mathcal{L}(a_1,\dots,a_k) = L_1$
$\{a_1 \dots a_k\}$ baza u $L_2 \Rightarrow \mathcal{L}(a_1,\dots,a_k) = L_2$ $\Big\} \Rightarrow L_1 = L_2$

3. U vektorskom prostoru R^R svih realnih funkcija, dokaži da su potprostori L_1 - svih parnih f-ja i L_2 - svih neparnih f-ja potprostori prostora $V - R^R$ i da je $V = L_1 \dot{+} L_2$

I 1° $f, g \in L_1$

$f + g \in V$

$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x) \quad \forall x \Rightarrow f+g \in L_1$

2° $f \in L_1$, $\alpha \in R$

$\alpha \cdot f \in V$

$(\alpha \cdot f)(-x) = \alpha \cdot f(-x) = \alpha \cdot f(x) = (\alpha \cdot f)(x) \Rightarrow \alpha \cdot f \in L_1$

Iz 1° i 2° $\Rightarrow L_1$ je potprostor prostora V nad poljem $\mathbb{R}$

II 1° $f, g \in L_2$

$f + g \in V$

$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f(x) + g(x)) = -(f+g)(x)$
$\forall x \in \mathbb{R} \Rightarrow f + g \in L_2$

2° $f \in L_2$, $\alpha \in R$

$\alpha \cdot f \in V$

$(\alpha \cdot f)(-x) = \alpha \cdot f(-x) = \alpha \cdot (-f(x)) = -\alpha \cdot f(x) = -(\alpha \cdot f)(x) \Rightarrow \alpha f \in L_2$

Iz 1° i 2° $\Rightarrow L_2$ je potprostor prostora V nad poljem $\mathbb{R}$

III 1° Neka je $f \in V$ proizvoljno. Neka je $g(x) = \frac{f(x) + f(-x)}{2}$
$h(x) = \frac{f(x) - f(-x)}{2}$. Tada je $g(-x) = \frac{f(-x) + f(x)}{2} = g(x) \quad \forall x \Rightarrow g \in$
$h(-x) = -\frac{f(x) - f(-x)}{2} = -h(x), \ \forall x \Rightarrow h \in L_2$

$$g(x)+h(x)=\frac{f(x)+f(-x)+f(x)-f(-x)}{2}=f(x)\quad \forall x\in R \Rightarrow g+h=f$$

$(\forall f\in V)\ f\in L_1+L_2$ pa je $V\subseteq L_1+L_2$ $\quad L_1\subseteq V\quad L_2\subseteq V$

$\quad L_1+L_2\subseteq V$

$V=L_1+L_2$

Neka je $f\in L_1\cap L_2$

$$\left.\begin{array}{l} f\in L_1 \Rightarrow f(-x)=f(x) \\ f\in L_2 \Rightarrow f(-x)=-f(x)\end{array}\right\} \Rightarrow f(x)=-f(x)$$

$2\cdot f(x)=0,\ \forall x\quad f\equiv 0$

$$\left.\begin{array}{l} V=L_1+L_2 \\ L_1\cap L_2=\{0\}\end{array}\right\} \Rightarrow V=L_1\dotplus L_2$$

- Gausov metod rješavanja sistema linearnih jednačina

1. $x_1+3x_2+2x_4=2$

$3x_1+7x_2-x_3+2x_4=3$

$x_1-x_2+5x_3-3x_4=4$

$2x_1+4x_2-x_3=1$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 2 \\ 3 & 7 & -1 & 2 & 3 \\ 1 & -1 & 5 & -3 & 4 \\ 2 & 4 & -1 & 0 & 1 \end{array}\right) \cdot(-3) \quad \cdot(-1) \quad \cdot(-2)$$

$$\sim\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 2 \\ 0 & -2 & -1 & -4 & -3 \end{array}\right) \quad \sim\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 2 \\ 0 & -2 & -1 & -4 & -3 \end{array}\right)$$