Matrica operatora

$A: X \to Y$ $\{e_1 \dots e_m\}$ baza u X $\{f_1 \dots f_n\}$ baza u Y

$Ae_1 = \alpha_{11} f_1 + \alpha_{21} f_2 + \dots + \alpha_{n1} f_1$

$Ae_2 = \alpha_{12} f_1 + \alpha_{22} f_2 + \dots + \alpha_{n2} f_2$

$\vdots$

$Ae_m = \alpha_{1m} f_1 + \alpha_{2m} f_2 + \dots + \alpha_{nm} f_m$

$$A_{ef} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & & \alpha_{2m} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & & \alpha_{nm} \end{pmatrix}$$ Matrica A u odnosu na baze e i f

$x \in X$ $x = x_1 e_1 + x_2 e_2 + \dots + x_m e_m$ $x_i \in P$

$Ax = y$ $y = y_1 f_1 + y_2 f_2 + \dots + y_n f_m$ $$[x]_e = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$A_{ef} \cdot [x]_e = [y]_f$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} \\ \vdots & & & \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$ $$[y]_f = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

VJEŽBE

1. Odrediti matricu operatora $\mathcal{A}: R^3 \to R^2$ $\mathcal{A}(x_1,x_2,x_3) = (x_1+2x_2-3x_3, -x_1+x_3)$ u odnosu na stand. baze prost. R^3 i R^2.

$\{e_1=(1,0,0), e_2=(0,1,0), e_3=(0,0,1)\}$ baza u R^3

$\{f_1=(1,0), f_2=(0,1)\}$ baza u R^2

$\mathcal{A}e_1 = \mathcal{A}(1,0,0) = (1,-1) = 1\cdot f_1 + (-1)\cdot f_2$

$\mathcal{A}e_2 = \mathcal{A}(0,1,0) = (2,0) = 2\cdot f_1 + 0\cdot f_2$

$\mathcal{A}e_3 = \mathcal{A}(0,0,1) = (-3,1) = -3\cdot f_1 + 1\cdot f_2$

Matrica oper. $\mathcal{A}$ u odn. na baze e i f je:

$$A_{ef} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 0 & 1 \end{pmatrix}$$

Napomena 1° Operator je određen svojom matricom. Da je bila zadata samo matrica A_{ef} mi bismo znali kako oper. $\mathcal{A}$ djeluje

$x \in R^3$ $\mathcal{A}x = ?$ $A_{ef} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 0 & 1 \end{pmatrix}$ $x=(x_1,x_2,x_3)$

$[x]_e = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $[y]_f = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\mathcal{A}x = y$ $y \in R^2$ $y=(y_1,y_2)$

$A_{ef}\cdot[x]_e = [y]_f$

$\begin{pmatrix} 1 & 2 & -3 \\ -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\begin{pmatrix} x_1+2x_2-3x_3 \\ -x_1+x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $y_1 = x_1+2x_2-3x_3$
$y_2 = -x_1+x_3$

$\mathcal{A}(x_1,x_2,x_3) = (y_1,y_2)$ $\mathcal{A}(x_1,x_2,x_3) = (x_1+2x_2-3x_3, -x_1+x_3)$

2° $x=(-2,3,-4)$ $\mathcal{A}x = ?$

I način $\mathcal{A}x = \mathcal{A}(-2,3,-4) = (-2+6+12, 2-4) = (16,-2)$

II način $[x]_e = \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$ $\mathcal{A}[x]_e = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 16 \\ -2 \end{pmatrix}$

2. Dokazati da je oper. $\mathcal{A}: R^{2\times2} \to R^{2\times2}$ zadat sa $\mathcal{A}X = X\cdot\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ linearan, odrediti mu po jednu bazu za jezgro i sliku, rang i defekt, kao i matricu u odn. na stand. bazu

I 1° $X, Y \in R^{2\times2}$

$\mathcal{A}(X+Y) = (X+Y)\cdot\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = X\cdot\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + Y\cdot\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathcal{A}(X) + \mathcal{A}(Y)$

2° $\alpha \in R$, $X \in R^{2\times 2}$ $\quad \mathcal{A}(\alpha \cdot X) = (\alpha \cdot X)\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \alpha \cdot X\begin{pmatrix}1&1\\0&0\end{pmatrix} = \alpha \cdot \mathcal{A}(X)$

Iz 1° i 2° $\Rightarrow$ $\mathcal{A}$ je linearan

II $N_{\mathcal{A}} = \{X \in R^{2\times 2} \mid \mathcal{A}X = 0\} = \left\{\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix} \,\middle|\, \begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}, x_1,x_2,x_3,x_4 \in R\right\} = \left\{\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix} \,\middle|\, \begin{pmatrix}x_1&x_1\\x_3&x_3\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}, x_1,x_2,x_3,x_4 \in R\right\} = \left\{\begin{pmatrix}0&x_2\\0&x_4\end{pmatrix} \,\middle|\, x_2,x_4 \in R\right\}$

$= \left\{x_2\begin{pmatrix}0&1\\0&0\end{pmatrix} + x_4\begin{pmatrix}0&0\\0&1\end{pmatrix} \,\middle|\, x_2,x_4 \in R\right\} = \mathcal{L}(F_1,F_2)$

$F_1 = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\quad F_2 = \begin{pmatrix}0&0\\0&1\end{pmatrix}$ $\quad \alpha_1 F_1 + \alpha_2 F_2 = 0$ $\quad \begin{pmatrix}0&\alpha_1\\0&\alpha_2\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ $\quad \alpha_1 = \alpha_2 = 0$

$\Rightarrow$ lin. nezav. i čini bazu u $N_{\mathcal{A}}$ $\quad \dim N_{\mathcal{A}} = 2$ $\quad \operatorname{def}\mathcal{A} = 2$

* Napomena: $F_1 = (0,1,0,0)$ u stand. bazi $E_1 = \begin{pmatrix}1&0\\0&0\end{pmatrix}$, $E_2 = \begin{pmatrix}0&1\\0&0\end{pmatrix}$,

$E_3 = \begin{pmatrix}0&0\\1&0\end{pmatrix}$, $E_4 = \begin{pmatrix}0&0\\0&1\end{pmatrix}$ $\quad F_2 = (0,0,0,1)$

$\{F_1, F_2\}$ podsistem $\{E_1, E_2, E_3, E_4\}$ $\quad \{F_1, F_2\}$ trapezni sistem

III $T_{\mathcal{A}} = \{\mathcal{A}X \mid X \in R^{2\times 2}\} = \left\{\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} \,\middle|\, x_1,x_2,x_3,x_4 \in R\right\}$

$= \left\{\begin{pmatrix}x_1&x_1\\x_3&x_3\end{pmatrix} \,\middle|\, x_1,x_3 \in R\right\} = \left\{x_1\begin{pmatrix}1&1\\0&0\end{pmatrix} + x_3\begin{pmatrix}0&0\\1&1\end{pmatrix} \,\middle|\, x_1,x_3 \in R\right\} = \mathcal{L}(G_1, G_2)$

$G_1 = \begin{pmatrix}1&1\\0&0\end{pmatrix}$ $\quad G_2 = \begin{pmatrix}0&0\\1&1\end{pmatrix}$ $\quad \{G_1, G_2\}$ lin. nezav. i čini bazu u $T_{\mathcal{A}}$

$\operatorname{rang}\mathcal{A} = \dim T_{\mathcal{A}} = 2$ $\quad n_A + r_A = 4$ $\quad \dim R^{2\times 2} = 4$

IV $\left\{E_1 = \begin{pmatrix}1&0\\0&0\end{pmatrix}, E_2 = \begin{pmatrix}0&1\\0&0\end{pmatrix}, E_3 = \begin{pmatrix}0&0\\1&0\end{pmatrix}, E_4 = \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ stand. baza u $R^{2\times 2}$

$\mathcal{A}E_1 = E_1\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}1&1\\0&0\end{pmatrix} = 1\cdot E_1 + 1\cdot E_2 + 0\cdot E_3 + 0\cdot E_4$

$\mathcal{A}E_2 = E_2\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix} = 0\cdot E_1 + 0\cdot E_2 + 0\cdot E_3 + 0\cdot E_4$

$\mathcal{A}E_3 = E_3\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\1&0\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\1&1\end{pmatrix} = 0\cdot E_1 + 0\cdot E_2 + 1\cdot E_3 + 1\cdot E_4$

$\mathcal{A}E_4 = E_4\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix} = 0\cdot E_1 + 0\cdot E_2 + 0\cdot E_3 + 0\cdot E_4$

Matrica operatora $\mathcal{A}$ u bazi E je:

$$A_E = \begin{pmatrix}1&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&1&0\end{pmatrix}$$

* Napomena:

I 1° $X \in \mathbb{R}^{2\times2}$ $X=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ $[X]_E=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$$A_E\cdot[X]_E=\begin{pmatrix} 1&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&1&0 \end{pmatrix}\cdot\begin{pmatrix} x_1\\x_2\\x_3\\x_4 \end{pmatrix}=\begin{pmatrix} x_1\\x_1\\x_3\\x_3 \end{pmatrix} \quad \mathcal{A}X=Y \quad [Y]_E=\begin{pmatrix} x_1\\x_1\\x_3\\x_3 \end{pmatrix}$$

$$\mathcal{A}X=\begin{pmatrix} x_1 & x_1 \\ x_3 & x_3 \end{pmatrix}$$

II Jezgro možemo naći koristeći matricu operatora. Tražimo $X\in\mathbb{R}^{2\times2}$ tako da je $\mathcal{A}X=0$, odn. t.d. $A_E\cdot[X]_E=0$, odnosno

$$\begin{pmatrix} 1&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&1&0 \end{pmatrix}\cdot\begin{pmatrix} x_1\\x_2\\x_3\\x_4 \end{pmatrix}=\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \quad \begin{pmatrix} x_1\\x_1\\x_3\\x_3 \end{pmatrix}=\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \Rightarrow x_1=0,\ x_3=0 \quad [X]_E=\begin{pmatrix} 0\\x_2\\0\\x_4 \end{pmatrix}.$$

$$X=\begin{pmatrix} 0 & x_2 \\ 0 & x_4 \end{pmatrix}$$

III $\mathcal{A}: X\to Y$ $x\in X$ $x=\alpha_1e_1+\alpha_2e_2+\dots+\alpha_ne_n$

$y\in T_{\mathcal{A}} \Rightarrow \exists x\in X$ t.d. $y=\mathcal{A}x$

$y=\mathcal{A}(\alpha_1e_1+\dots+\alpha_ne_n)=\alpha_1\mathcal{A}e_1+\dots+\alpha_n\mathcal{A}e_n\in\mathcal{L}(\mathcal{A}e_1,\dots,\mathcal{A}e_n)$

$$\left.\begin{array}{l} T_{\mathcal{A}}\subseteq\mathcal{L}(\mathcal{A}e_1\dots\mathcal{A}e_n) \\ \mathcal{L}(\mathcal{A}e_1\dots\mathcal{A}e_n)\subseteq T_{\mathcal{A}} \end{array}\right\} \Rightarrow T_{\mathcal{A}}=\mathcal{L}(\mathcal{A}e_1\dots\mathcal{A}e_n)$$

Baza u $T_{\mathcal{A}}$ biće baza sist. $\mathcal{A}e_1\dots\mathcal{A}e_n$

$\mathcal{A}E_1=\begin{pmatrix} 1&1\\0&0 \end{pmatrix}$ $\mathcal{A}E_2=\begin{pmatrix} 0&0\\0&0 \end{pmatrix}$ $\mathcal{A}E_3=\begin{pmatrix} 0&0\\1&1 \end{pmatrix}$ $\mathcal{A}E_4=\begin{pmatrix} 0&0\\0&0 \end{pmatrix}$. Baza u $T_{\mathcal{A}}$ je baza sistema $\left\{\begin{pmatrix} 1&1\\0&0 \end{pmatrix},\begin{pmatrix} 0&0\\0&0 \end{pmatrix},\begin{pmatrix} 0&0\\1&1 \end{pmatrix},\begin{pmatrix} 0&0\\0&0 \end{pmatrix}\right\}$ a to je $\left\{\begin{pmatrix} 1&1\\0&0 \end{pmatrix},\begin{pmatrix} 0&0\\1&1 \end{pmatrix}\right\}$

IV Prostor slika možemo posmatrati i kao potpr. generisan kolonama matrice operatora

$$A_E=\begin{pmatrix} 1&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&1&0 \end{pmatrix}$$

3. Neka je $\mathcal{A}\in\mathcal{L}(\mathbb{R}^{2\times2},\mathbb{R}^2)$ definisan sa $\mathcal{A}\left(\begin{pmatrix} a&b\\c&d \end{pmatrix}\right)=(a+d,\ b-c)$

a) Pokazati da je operator $\mathcal{A}$ linearan

b) Ispitati da li je operator $\mathcal{A}$ monomorfizam

c) Ispitati da li je operator $\mathcal{A}$ epimorfizam

d) Navedite proizv. $N\subseteq\mathbb{R}^{2\times2}$ tako da je $\mathbb{R}^{2\times2}=N\dotplus N_{\mathcal{A}}$

e) Naći matricu u odnosu na stand. baze

a) 1° $X,Y\in\mathbb{R}^{2\times2}$

$X=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $Y=\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}$ $X+Y=\begin{pmatrix}a+a_1&b+b_1\\c+c_1&d+d_1\end{pmatrix}$

$\mathcal{A}(X+Y)=(a+a_1+d+d_1,\ b+b_1-(c+c_1))=(a+d,\ b-c)+(a_1+d_1,\ b_1-c_1)$
$=\mathcal{A}X+\mathcal{A}Y$

2° $\alpha\in R$ $X\in R^{2\times2}$ $X=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $\alpha\cdot X=\begin{pmatrix}\alpha a&\alpha b\\\alpha c&\alpha d\end{pmatrix}$

$\mathcal{A}(\alpha X)=(\alpha a+\alpha d,\ \alpha b-\alpha c)=\alpha(a+d,\ b-c)=\alpha\cdot\mathcal{A}X$

Iz 1° i 2° $\Rightarrow$ $\mathcal{A}$ je linearan

b) Pokažimo da preslikavanje $\mathcal{A}$ nije injektivno. Uočimo 2 vektora

$X=(1,7,4,5)$ u bazi $\{E_1,E_2,E_3,E_4\}$ $X=\begin{pmatrix}1&7\\4&5\end{pmatrix}$ $Y=\begin{pmatrix}2&8\\5&4\end{pmatrix}$

$\mathcal{A}X=(6,3)$ $\mathcal{A}(Y)=(6,3)$. Uočili smo dva vektora takva da $X\neq Y$ i $\mathcal{A}X=\mathcal{A}Y$ pa $\mathcal{A}$ nije injektivno $\Rightarrow$ nije monomorfizam

c) $T_{\mathcal{A}}=\{\mathcal{A}X \mid X\in R^{2\times2}\}=\{\mathcal{A}\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid a,b,c,d\in R\}=\{(a+d,\ b-c) \mid a,b,c,$
$d\in R\}=\{a(1,0)+b(0,1)+c(0,-1)+d(1,0) \mid a,b,c,d\in R\}=\mathcal{L}(f_1,f_2,$
$f_3,f_4)$ $f_1=(1,0)$ $f_2=(0,1)$ $f_3=(0,-1)$ $f_4=(1,0)$
$=\mathcal{L}(f_1,f_2)$ $T_{\mathcal{A}}=\mathcal{L}((1,0),(0,1))=R^2$ $\mathcal{A}:R^{2\times2}\to R^2$

$T_{\mathcal{A}}=\mathcal{A}(R^{2\times2})=R^2 \Rightarrow \mathcal{A}$ je sirjektivno

1° $\mathcal{A}$ je linearan 2° $\mathcal{A}$ je sirjektivno $\Rightarrow$ epimorfizam

d) $N_{\mathcal{A}}=\{X\in R^{2\times2} \mid \mathcal{A}X=0\in R^{2\times1}\}=\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid (a+d,\ b-c)=(0,0);\ a,b,$
$c,d\in R\}=\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid a+d=0 \wedge b-c=0;\ a,b,c,d\in R\}=\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid$
$d=-a,\ c=b;\ a,b,c,d\in R\}=\{\begin{pmatrix}a&b\\b&-a\end{pmatrix} \mid a,b\in R\}=\{a\begin{pmatrix}1&0\\0&-1\end{pmatrix}+b\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$a,b\in R\}=\mathcal{L}(G_1,G_2)$ $G_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $G_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\{G_1,G_2\}$ lin. nez.

Baza u $N_{\mathcal{A}}$ je $\{G_1,G_2\}$

$R^{2\times2}=N\dot{+}N_{\mathcal{A}}$ $\dim R^{2\times2}=\dim N+\dim N_{\mathcal{A}}$

$4=\dim N+2$ $\dim N=2$

Tražimo N t.d. unija ~~potpr.~~ baza N i $N_{\mathcal{A}}$ daje bazu prostora $R^{2\times2}$

$G_1=(1,0,0,-1)$ $G_3=(0,0,1,0)$ $\{G_3,G_4\}$ baza u N

$G_2=(0,1,1,0)$ $G_4=(0,0,0,1)$

e) $\mathcal{A}E_1 = \mathcal{A}\begin{pmatrix}1&0\\0&0\end{pmatrix} = (1,0) = 1\cdot g_1 + 0\cdot g_2$ $\{g_1, g_2\}$ - stand. baza u R^2

$\mathcal{A}E_2 = \mathcal{A}\begin{pmatrix}0&1\\0&0\end{pmatrix} = (0,1) = 0\cdot g_1 + 1\cdot g_2$

$\mathcal{A}E_3 = \mathcal{A}\begin{pmatrix}0&0\\1&0\end{pmatrix} = (0,-1) = 0\cdot g_1 + (-1)\cdot g_2$

$\mathcal{A}E_4 = \mathcal{A}\begin{pmatrix}0&0\\0&1\end{pmatrix} = (1,0) = 1\cdot g_1 + 0\cdot g_2$

Matrica operatora $\mathcal{A}$ u odn. na E je $A_E = \begin{pmatrix}1&0&0&1\\0&1&-1&0\end{pmatrix}$

4. Neka je $\mathcal{A} : P_{\leq 2} \to P_{\leq 2}$ definisan sa $\mathcal{A}p = -2p + (3x-1)\cdot p'$

a) Pokazati da je $\mathcal{A}$ lin. operator

b) Ispitati da li je $\mathcal{A}$ regularan

c) Za polinom $q(x) = 7x^2 - 2x - 3$ odrediti prikaz vektora $\mathcal{A}q$ u standardnoj bazi koristeći matricu operatora

a) 1° $p, q \in P_{\leq 2}$

$\mathcal{A}(p+q) = -2\cdot(p+q) + (3x-1)(p+q)'$

$= -2p - 2q + (3x-1)(p'+q') = -2p + (3x-1)p' - 2q + (3x-1)q'$

$= \mathcal{A}p + \mathcal{A}q$

2° $\alpha \in R$ $p \in P_{\leq 2}$ $\mathcal{A}(\alpha p) = -2(\alpha\cdot p) + (3x-1)(\alpha p)'$

$= \alpha(-2p) + \alpha\cdot(3x-1)p' = \alpha(-2p + (3x-1)p') = \alpha\cdot\mathcal{A}p$

Iz 1° i 2° $\Rightarrow$ $\mathcal{A}$ lin. operator

b) $N_{\mathcal{A}} = \{p \in P_{\leq 2} \mid \mathcal{A}p = 0\} = \{a_0 + a_1x + a_2x^2 \mid -2(a_0 + a_1x + a_2x^2) + (3x-1)(a_1 + 2a_2x) = 0$ (kao nula polinom) $a_0, a_1, a_2 \in R$

$= \{a_0 + a_1x + a_2x^2 \mid -2a_0 - a_1 + (a_1 - 2a_2)x + 4a_2x^2 = 0 \mid a_0, a_1, a_2 \in R\}$

$-2a_0 - a_1 = 0$ $\quad a_2 = 0$

$a_1 - 2a_2 = 0$ $\quad a_1 = 0$

$4a_2 = 0$ $\quad a_0 = 0$

$N_{\mathcal{A}} = \{0\} \Rightarrow$ Operator $\mathcal{A}$ je regularan

c) Odredimo matricu operatora u odn. na stand. bazu

$\{e_1(x) = 1,\ e_2(x) = x,\ e_3(x) = x^2\}$ - stand. baza u $P_{\leq 2}$

$\mathcal{A}e_1 = -2e_1 + (3x-1)e_1' = -2\cdot 1 + (3x-1)\cdot 0 = -2 = -2\cdot e_1 + 0\cdot e_2 + 0\cdot e_3$

$\mathcal{A}e_2 = -2e_2 + (3x-1)e_2' = -2x + (3x-1)\cdot 1 = -1 + x = -1\cdot e_1 + 1\cdot e_2 + 0\cdot e_3$

$\mathcal{A}e_3 = -2e_3 + (3x-1)e_3' = -2x^2 + (3x-1)\cdot 2x = 0\cdot e_1 - 2\cdot e_2 + 4\cdot e_3$

$$Ae = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{pmatrix} \quad Ag = ? \quad [g]_e = \begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix}$$

$$Ae \cdot [g]_e = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ -16 \\ 21 \end{pmatrix} \quad Ag = 8 - 16x + 21x^2$$

5. Operator $A: V \to V$ koji djeluje u trodim. prostoru V vektore a_1, a_2, a_3 prevodi u vektore b_1, b_2, b_3. Naći matricu oper. A u odn. na bazu $\{a_1, a_2, a_3\}$

$a_1 = (5,3,1) \qquad b_1 = (-2,1,0)$

$a_2 = (1,-3,-2) \qquad b_2 = (-1,3,0)$

$a_3 = (1,2,1) \qquad b_3 = (2,-3,0)$

Tražimo $\quad Aa_1 = b_1 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$

$(-2,1,0) = \alpha_1(5,3,1) + \alpha_2(1,-3,-2) + \alpha_3(1,2,1)$

$$\begin{aligned} 5\alpha_1 + \alpha_2 + \alpha_3 &= -2 \\ 3\alpha_1 - 3\alpha_2 + 2\alpha_3 &= 1 \\ \alpha_1 - 2\alpha_2 + \alpha_3 &= 0 \end{aligned} \Rightarrow \begin{aligned} \alpha_1 &= 6 \\ \alpha_2 &= 17 \\ \alpha_3 &= -5 \end{aligned} \qquad Aa_1 = 6a_1 + 17a_2 - 5a_3$$

$Aa_2 = b_2 = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3$

$$\begin{aligned} 5\beta_1 + \beta_2 + \beta_3 &= -1 \\ 3\beta_1 - 3\beta_2 + 2\beta_3 &= 3 \\ \beta_1 - 2\beta_1 + \beta_3 &= 0 \end{aligned} \Rightarrow \begin{aligned} \beta_1 &= -10 \\ \beta_2 &= 13 \\ \beta_3 &= 36 \end{aligned} \qquad Aa_2 = -10a_1 + 13a_2 + 36a_3$$

$Aa_3 = b_3 = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3$

$$\begin{aligned} 5\gamma_1 + \gamma_2 + \gamma_3 &= 2 \\ 3\gamma_1 - 3\gamma_2 + 2\gamma_3 &= -3 \\ \gamma_1 - 2\gamma_1 + \gamma_3 &= 0 \end{aligned} \qquad Aa_3 = 11a_1 - 14a_2 - 39a_3$$

Matrica oper. A u bazi a je

$$Aa = \begin{pmatrix} 6 & -10 & 11 \\ 17 & 13 & -14 \\ -5 & 36 & -39 \end{pmatrix}$$

Napomena: Naći Ax, ako je $x = (-1,5,7)$ $\quad V = R^3$

$$[x]_a = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad Aa \cdot [x]_a \qquad x = \alpha a_1 + \beta a_2 + \gamma a_3$$

6. Neka je $\mathcal{B}$ lin. operator $\mathcal{B}: \mathbb{R}^4 \to \mathbb{R}$ takav da $\mathcal{B}(1,0,0,0)=1$, $\mathcal{B}(1,-1,1,0)=1$, $\mathcal{B}(1,-1,0,0)=0$ i $\mathcal{B}(1,-1,1,-1)=0$. Odrediti $\mathcal{B}(a,b,c,d)$

I način $f_1=(1,0,0,0)$ $f_3=(1,-1,0,0)$

$f_2=(1,-1,1,0)$ $f_4=(1,-1,1,-1)$

Razložimo vektor (a,b,c,d) preko vektora f_1, f_2, f_3, f_4

$(a,b,c,d)= \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4$

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3+\alpha_4&=a \qquad & \alpha_1&=a+b \\ -\alpha_2-\alpha_3-\alpha_4&=b & \alpha_2&=c+d \\ \alpha_2+\alpha_4&=c & \alpha_3&=-b-c \\ -\alpha_4&=d & \alpha_4&=-d \end{aligned}$$

$(a,b,c,d)=(a+b)f_1+(c+d)f_2+(-b-c)f_3+(-d)f_4$

$\mathcal{B}(a,b,c,d)=(a+b)\mathcal{B}f_1+(c+d)\mathcal{B}f_2+(-b-c)\mathcal{B}f_3+(-d)\mathcal{B}f_4 = a+b+c+d$

II način. Odredimo slike baznih vektora

$\mathcal{B}f_1 = \mathcal{B}(1\cdot e_1+0\cdot e_2+0\cdot e_3+0\cdot e_4)=1 \qquad \mathcal{B}e_1=1$

$\mathcal{B}f_2 = \mathcal{B}(1\cdot e_1-1\cdot e_2+1\cdot e_3+0\cdot e_4)=1 \qquad \mathcal{B}e_1-\mathcal{B}e_2+\mathcal{B}e_3=1$

$\mathcal{B}f_3 = \mathcal{B}(1\cdot e_1-1\cdot e_2+0\cdot e_3+0\cdot e_4)=0 \qquad \mathcal{B}e_2=\mathcal{B}e_3$

$\mathcal{B}f_4 = \mathcal{B}(1\cdot e_1-1\cdot e_2+1\cdot e_3-1\cdot e_4)=0 \qquad \mathcal{B}e_1-\mathcal{B}e_2=0$

$\mathcal{B}e_1-\mathcal{B}e_2+\mathcal{B}e_3-\mathcal{B}e_4=0 \quad \mathcal{B}e_4=1 \qquad \mathcal{B}e_2=1=\mathcal{B}e_3$

$(a,b,c,d)=a\cdot e_1+b\cdot e_2+c\cdot e_3+d\cdot e_4$

$\mathcal{B}(a,b,c,d)=a\mathcal{B}e_1+b\mathcal{B}e_2+c\mathcal{B}e_3+d\mathcal{B}e_4=a\cdot 1+b\cdot 1+c\cdot 1+d\cdot 1=a+b+c+d$

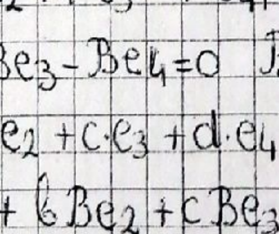
7. Neka je operator $\mathcal{A}$ operator rotacije prostora $\mathbb{R}^3$ oko x-ose za usmjereni ugao φ. Naći matricu tog operatora u stand. bazi.

$\mathcal{A}: \mathbb{R}^3 \to \mathbb{R}^3$

$e_1=(1,0,0)$

$e_2=(0,1,0)$

$e_3=(0,0,1)$



$\mathcal{A}e_1 = e_1 = 1\cdot e_1+0\cdot e_2+0\cdot e_3$

$\cos\varphi = \frac{a}{1} \Rightarrow a=\cos\varphi$

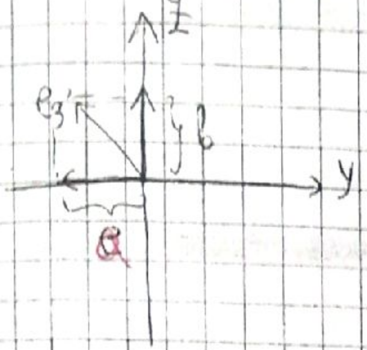
$\sin\varphi = \frac{b}{1} \Rightarrow b=\sin\varphi$

$\mathcal{A}e_2 = e_2'$

$e_2' = a\cdot e_2 + b\cdot e_3$

$e_2' = a\cdot e_2 + b\cdot e_3$

$\mathcal{A} e_2 = 0\cdot e_1 + \cos\varphi\cdot e_2 + \sin\varphi\, e_3$



$\mathcal{A} e_3 = e_3'$

$e_3' = a\cdot e_2 + b\cdot e_3$

$\cos\varphi = \frac{b}{1} \Rightarrow b = \cos\varphi$

$\sin\varphi = \frac{a}{1} \Rightarrow a = \sin\varphi$

$\mathcal{A} e_3 = 0\cdot e_1 - \sin\varphi\, e_2 + \cos\varphi\, e_3$

$$Ae = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

* Promjena matrice operatora pri promjeni baze *

$$X \xrightarrow{\mathcal{A}} Y$$

$$\begin{array}{ccc} \{e_1 \dots e_m\} & A_{ef} & \{f_1 \dots f_n\} \\ \downarrow P & & \downarrow Q \\ \{g_1 \dots g_m\} & A_{gh} & \{h_1 \dots h_n\} \end{array}$$

$A_{gh} = Q^{-1} A_{ef} P$

* Napomena: Ako je $X = Y$

$$X \xrightarrow{\mathcal{A}} X$$

$$\begin{array}{ccc} \{e_1 \dots e_m\} & A_e & \{e_1 \dots e_m\} \\ \downarrow P & & \downarrow P \\ \{g_1 \dots g_m\} & A_g & \{g_1 \dots g_m\} \end{array}$$

$A_g = P^{-1} A_e P$

$x = \alpha_1 e_1 + \dots + \alpha_m e_m$

$\{e_1 \dots e_m\}$ baza u V

$x = \beta_1 g_1 + \dots + \beta_m g_m$

$\{g_1 \dots g_m\}$ baza u V

$[x]_e = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \quad [x]_g = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$ P-matrica prelaska sa baze e na bazu g

$[x]_e = P\cdot [x]_g \qquad [x]_g = P^{-1} [x]_e$

$g_1 = \alpha_{11} e_1 + \alpha_{21} e_2 + \dots + \alpha_{m1} e_m$

$g_2 = \alpha_{12} e_1 + \alpha_{22} e_2 + \dots + \alpha_{m2} e_m$

$\vdots$

$g_m = \alpha_{1m} e_1 + \alpha_{2m} e_2 + \dots + \alpha_{mm} e_m$

$$P = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} \\ \vdots & & & \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mm} \end{pmatrix}$$

Vježbe

$Ap = -2p + (3x-1)p'$

I način

$\operatorname{Im} A = \{Ap \mid p \in P_{\leq 2}\} = \{Ap \mid p(x) = a_0 + a_1x + a_2x^2, a_0, a_1, a_2 \in R\}$

$= \{-2(a_0 + a_1x + a_2x^2) + (3x-1)(a_1 + 2a_2x) \mid a_0, a_1, a_2 \in R\}$

$= \{-2a_0 - 2a_1x - 2a_2x^2 + 3a_1x + 6a_2x^2 - a_1 - 2a_2x \mid a_0, a_1, a_2 \in R\}$

$= \{a_0(-2) + a_1(x-1) + a_2(4x^2 - 2x) \mid a_0, a_1, a_2 \in R\}$

$= \mathcal{L}(f_1, f_2, f_3) \quad f_1(x) = -2 \quad f_2(x) = x-1 \quad f_3(x) = 4x^2 - 2x$

$\{f_1, f_2, f_3\}$ - lin. nezavisan sistem - baza u $\operatorname{Im} A$

II Baza u $\operatorname{Im} A$ je baza sistema $\{Aa_1, Aa_2, Aa_3\}$

$a_1(x) = 1 \quad a_2(x) = x \quad a_3(x) = x^2$

$Aa_1 = -2e_1 = -2 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3$

$Ae_2 = -2e_2 + (3x-1) = -e_1 + e_2 + 0 \cdot e_3$

$Ae_3 = -2x^2 + 6x^2 - 2x = 0 \cdot e_1 - 2 \cdot e_2 + 4 e_3$

1. Matrica operatora A u stand. bazi je

$$A_e = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Odrediti matricu operatora A u bazi: $\{f_1 = (0,-1,-1), f_2 = (1,-1,1), f_3 = (-1,1,0)\}$

R: $A: V \to V$

$\{e_1, e_2, e_3\} \xrightarrow{P} \{f_1, f_2, f_3\}$ $\quad A_e \ \{e_1, e_2, e_3\} \xrightarrow{P} \{f_1, f_2, f_3\}$

$A_f = P^{-1} A_e P$

P - matrica prelaza sa baze e na bazu f

$f_1 = (0,-1,-1) = 0 \cdot e_1 - 1 \cdot e_2 - 1 \cdot e_3$

$f_2 = 1 \cdot e_1 - 1 \cdot e_2 + 1 \cdot e_3$

$f_3 = -1 \cdot e_1 + 1 \cdot e_2 + 0 \cdot e_3$

$$P = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$$

$A_f = P^{-1} A_e P$

$$A_f = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -3 & 3 \\ 0 & -2 & 2 \end{pmatrix}$$

$B_1 = \{e_1, \dots, e_m\}$ $B_2 = \{f_1, \dots, f_m\}$ $B_3 = \{g_1, \dots, g_m\}$ baze u X

1° P_1 - matrica prelaza sa B_1 na B_2

P_2 - matrica prelaza sa B_2 na B_3

P_3 - matrica prelaza sa B_1 na B_3

$P_3 = P_1 \cdot P_2$

2° Q matrica prelaza sa B_2 na B_1 $\quad Q = P_1^{-1}$

Def. A i B su ekviv. matrice ako postoje regulare matrice R i S takve da je $B = RAS$

Napomena: $A_{gh} = Q^{-1} A_{ef} P$

Matrice jednog oper. su međusobno ekviv.

Važi i obrnuto, ako su matrice A i B ekviv. onda su one matrice jednog operatora

Def. Matrica B je slična matrici A ako postoji reg. matrica P t.d. $B = P^{-1}AP$.

Uočimo: $Q = P^{-1}$ $\quad A = Q^{-1}BQ$ pa je A slična sa B

Napomena: 1° ako su A i B slične, one su kvadratne i istog reda.

2° Matrice oper. $\mathcal{A}: V \to V$ u odn. na različite baze su međusobno slične

2. U prostoru $\mathbb{R}^3$ date su baze:

G: $g_1 = (8, -6, 7)$ $\quad$ H: $h_1 = (1, -2, 1)$

$g_2 = (-16, 7, -13)$ $\quad$ $h_2 = (3, -1, 2)$

$g_3 = (9, -3, 7)$ $\quad$ $h_3 = (2, 1, 2)$

Ako je matrica op. $\mathcal{A}: \mathbb{R}^3 \to \mathbb{R}^3$, u bazi G:

$$A_G = \begin{pmatrix} -1 & -18 & 15 \\ -1 & -22 & 20 \\ -1 & -25 & 22 \end{pmatrix}$$, odred. matricu op. u bazi H

I način: P matrica prelaza sa G na H

$A_H = P^{-1} A_G P$

$h_1 = \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3$

$h_2 = \beta_1 g_1 + \beta_2 g_2 + \beta_3 g_3$

$h_3 = \gamma_1 g_1 + \gamma_2 g_2 + \gamma_3 g_3$

$$P = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}$$

II način

P - matrica prelaza sa G na H

P_1 - matr. prelaza sa E na G

P_2 - matr. prelaza sa E na H

E - stand. baza u $\mathbb{R}^3$

$[x]_E = P_1 \cdot [x]_G$ (1)

$[x]_E = P_2 \cdot [x]_H$ (2)

$[x]_G = P[x]_H$ (3)

$(1) \Rightarrow [x]_G = P_1^{-1} \cdot [x]_E \overset{(2)}{=} P_1^{-1} \cdot P_2 [x]_H$ (4)

$[x]_G = P[x]_H$
$[x]_G = P_1^{-1} P_2 [x]_H$ $\Rightarrow P = P_1^{-1} P_2$

$$P_1 = \begin{pmatrix} 8 & -16 & 9 \\ -6 & 7 & -3 \\ 7 & -13 & 7 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 3 & 2 \\ -2 & -1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$A_H = P^{-1} A_G P$

3. Zadato je preslikavanje $\mathcal{D}: P_{\leq 2} \to P_{\leq 2}$

za $(\mathcal{D}p)(x) = \dfrac{p(x+a) - p(x)}{a}$, $a \in \mathbb{R}$, $a \neq 0$

Dokazati da je $\mathcal{D}$ lin. oper. odrediti mu rang i defekt te po jednu bazu za sliku i jezgro. Naći matricu op. u bazi $\{1, x+1, x^2\}$

1° $\mathcal{D}(p+g) = \mathcal{D}p + \mathcal{D}g$

$$(\mathcal{D}(p+g))(x) = \frac{(p+g)(x+a) - (p+g)(x)}{a}$$

$$(\mathcal{D}(p+g))(x) = \frac{p(x+a) + g(x+a) - (p(x) + g(x))}{a}$$

$$= \frac{p(x+a) - p(x)}{a} + \frac{g(x+a) - g(x)}{a} = (\mathcal{D}p)(x) + (\mathcal{D}g)(x) = (\mathcal{D}p + \mathcal{D}g)(x) \ \forall x$$

$$\mathcal{D}(p+g) = \mathcal{D}p + \mathcal{D}g$$

2° $\mathcal{D}(\alpha p) = \alpha \cdot \mathcal{D}p$

$$(\mathcal{D}(\alpha p))(x) = \frac{(\alpha p)(x+a) - (\alpha p)(x)}{a} = \frac{\alpha (p)(x+a) - p(x))}{a}$$

$= \alpha \cdot (\mathcal{D}p(x))$ $\quad \mathcal{D}(\alpha p) = \alpha \cdot \mathcal{D}p$

Iz 1° i 2° $\mathcal{D}$ je lin. operator

Odredimo matricu op. $\mathcal{D}$ u odnosu na stand. bazu
$\{e_1(x)=1,\ e_2(x)=x,\ e_3(x)=x^2\}$

$$(\mathcal{D}e_1)(x)=\frac{e_1(x+a)-e_1(x)}{a}=\frac{1-1}{a}=0$$

$$\mathcal{D}e_1=0\cdot e_1+0\cdot e_2+0\cdot e_3$$

$$(\mathcal{D}e_2)(x)=\frac{e_2(x+a)-e_2(x)}{a}=\frac{\cancel{x}+a-\cancel{x}}{a}=1$$

$$\mathcal{D}e_2=1\cdot e_1+0\cdot e_2+0\cdot e_3$$

$$(\mathcal{D}e_3)(x)=\frac{e_3(x+a)-e_3(x)}{a}=\frac{(x+a)^2-x^2}{a}=2x+a$$

$$D(e)=\begin{pmatrix}0&1&a\\0&0&2\\0&0&0\end{pmatrix}\qquad \mathcal{D}e_3=a\cdot e_1+2\cdot e_2+0\cdot e_3$$

$$\operatorname{Ker}\mathcal{D}=\{x \mid \mathcal{D}x=0\}$$

$$\begin{pmatrix}0&1&a\\0&0&2\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$x_2+ax_3=0$$
$$2x_3=0$$
$$x_3=0$$
$$x_2=0$$

$$\operatorname{Ker}\mathcal{D}=\{(x_1,0,0)\mid x_1\in\mathbb{R}\}$$
$$=\{x_1(1,0,0)\mid x\in\mathbb{R}\}$$
$$=\mathcal{L}(f)\quad f=(1,0,0)\qquad n_A=\dim\operatorname{Ker}\mathcal{D}=1\quad f=1\cdot e_1+0\cdot e_2+0\cdot e_3$$

$$\operatorname{Im}\mathcal{D}=\{\mathcal{D}p \mid p\in P_{\le 2}\}$$

$$\begin{pmatrix}0&1&a\\0&0&2\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_2+ax_3\\2x_3\\0\end{pmatrix}$$

$$\operatorname{Im}\mathcal{D}=\{(x_2+ax_3,2x_3,0)\mid x_2,x_3\in\mathbb{R}\}$$
$$=\{x_2(1,0,0)+x_3(a,2,0)\mid x_2,x_3\in\mathbb{R}\}$$
$$=\mathcal{L}(f_1,f_2)\quad f_1=(1,0,0)\quad f_2=(a,2,0)$$
$$r_A=\dim\operatorname{Im}\mathcal{D}=2$$

P - matricea prelaza sa e na B

$B = \{1, X+1, X^2\}$

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_B = P^{-1} D_e P$$

$$D_B = \begin{pmatrix} 0 & 1 & a-3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

4. a) Zadat je polinom $p(x) = 2 + x + x^2$. Koristeći matricu prelaza napisati $p(x)$ kao lin. komb. polinoma $e(x) = 1 + x$, $f(x) = 2 + x^2$, $g(x) = 1 + x + 2x^2$

$B = \{e, f, g\}$ ovo je takođe baza u $P_{\leq 2}$

P - matr. prelaza sa E na B E - stand. baza

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad [p]_E = P \cdot [p]_B \quad P^{-1} = \frac{1}{-4} \cdot \begin{pmatrix} -1 & -3 & 2 \\ -2 & 2 & 0 \\ -1 & -1 & -2 \end{pmatrix}$$

$$[p]_B = P^{-1} \cdot [p]_E$$

$$P^{-1} \cdot [p]_E = \frac{1}{-4} \cdot \begin{pmatrix} 1 & -3 & 2 \\ -2 & 2 & 0 \\ -1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/2 \\ 1/4 \end{pmatrix}$$

$$p(x) = \frac{3}{4} e(x) + \frac{1}{2} f(x) + \frac{1}{4} g(x)$$

b) Na prostoru R^3 fiksiramo bazu $\{f_1 = (1, 0, -1), f_2 = (0, 2, 0), f_3 = (1, 2, 3)\}$ a na prostoru R^2 fik stand. bazu. Lin. op. $A: R^3 \to R^2$ u tom paru baza prikazan je matricom

$$A_{fe} = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \end{pmatrix} \quad \{e_1 = (1, 0), e_2 = (0, 1)\}$$

Odr. matr. op. A u paru stand. baza prostora R^3 i R^2 i izračunati $A(-3, 2, 5)$

$$\begin{array}{ccc} R^3 & \xrightarrow{A} & R^2 \\ \{f_1, f_2, f_3\} & A_{fe} & \{e_1, e_2\} \\ \downarrow P & & \downarrow Q \\ \{g_1, g_2, g_3\} & A_{ge} & \{e_1, e_2\} \end{array} \qquad A_{ge} = Q^{-1} A_{fe} P$$

P - matrica prelaza sa f na g

$g_1 = (1, 0, 0)$ $g_2 = (0, 1, 0)$ $g_3 = (0, 0, 1)$

Ako je P_1 matrica prelaza sa g na f

$$P = P_1^{-1} \quad P_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ -1 & 0 & 3 \end{pmatrix} \quad P = \frac{1}{8} \begin{pmatrix} 6 & 0 & -2 \\ -2 & 4 & -2 \\ 2 & 0 & 2 \end{pmatrix}$$

Q - matrica prelaza sa e na e $Q = E \Rightarrow Q^{-1} = E$

$$A_{ge} = E \cdot A_{fe} \cdot P = A_{fe} \cdot P = \frac{1}{8} \begin{pmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 & 0 & -2 \\ -2 & 4 & -2 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 5/2 & -1/2 & 1/2 \\ 2 & 1/2 & -1 \end{pmatrix}$$

5. U prostoru polinoma $P_{\leq 2}$ zadata su 2 operatora. Op. $\mathcal{F}$ svakom polinomu $a_0 + a_1x + a_2x^2$ pridružuje polinom $a_0 + a_1x$, a op. $\mathcal{D}$ polinomima x^2+x, x^2+1, x^2+x+1 pridr. redom polinome x^2+1, x^2+x+1 i nula polinom. Naći matr. op. $\mathcal{D}\cdot\mathcal{F}$ u odn. na bazu

$B = \{x^2, x^2+x, x^2+x+1\}$

Neka je $\mathcal{A} = \mathcal{D}\mathcal{F}$. Odredimo matr. op. $\mathcal{A}$ u odn. na stand. bazu $\{e_1(x)=1, e_2(x)=x, e_3(x)=x^2\}$

1° $\mathcal{F}e_1 = \mathcal{F}1 = 1 = 1\cdot e_1 + 0\cdot e_2 + 0\cdot e_3$

$\mathcal{F}e_2 = \mathcal{F}x = 0\cdot e_1 + 1\cdot e_2 + 0\cdot e_3$

$\mathcal{F}e_3 = \mathcal{F}x^2 = 0 = 0\cdot e_1 + 0\cdot e_2 + 0\cdot e_3$

Matrica op. $\mathcal{F}$ u bazi e je

$$F_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2° $\mathcal{D}(x^2+x) = x^2+1$ $\qquad \mathcal{D}(e_2+e_3) = e_1+e_3$

$\mathcal{D}(x^2+1) = x^2+x+1$ $\qquad \mathcal{D}(e_1+e_3) = e_1+e_2+e_3$

$\mathcal{D}(x^2+x+1) = 0$ $\qquad \mathcal{D}(e_1+e_2+e_3) = 0$

$d_i = \mathcal{D}e_i \quad i=\overline{1,3}$

$d_2 + d_3 = e_1 + e_3$ $\qquad d_1 = -e_1 - e_3$ $\qquad \mathcal{D}e_1 = -e_1 - e_3$

$d_1 + d_3 = e_1 + e_2 + e_3$ $\qquad d_2 = -e_1 - e_2 - e_3$ $\qquad \mathcal{D}e_2 = -e_1 - e_2 - e_3$

$d_1 + d_2 + d_3 = 0$ $\qquad d_3 = 2e_1 + e_2 + 2e_3$ $\qquad \mathcal{D}e_3 = 2e_1 + e_2 + 2e_3$

Matrica op. $\mathcal{D}$ u bazi e je

$$D_e = \begin{pmatrix} -1 & -1 & 2 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{pmatrix}$$

Matr. op. $\mathcal{A}$ u bazi e je

$$A_e = D_e \cdot F_e = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$

A_B - matrica op. A u bazi B

$A_B = P^{-1} A_e \cdot P$ P - matr. prelaza sa e na B

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad A_B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix}$$

* Neka je A lin. op. na prostoru R^2 koji vektor prvo rotira za ugao $\pi/3$ oko koord. početka u poz. smjeru, a zatim ga reflektuje (sim. presl.) u odn. na pravac $y=x$. Naći matricu op. u bazi $B = \{(1,1), (1,-1)\}$. Odrediti koordinate vektora Av u odn. na tu bazu, gdje je v proizvoljan vektor u R^2 *

... vektori *