1. a) Zadat je polinom $p(x)=2+x+x^2$. Koristeći matricu prelaza napisati $p(x)$ kao lin. komb. polinoma $e(x)=1+x$, $f(x)=2+x^2$, $g(x)=1+x+2x^2$

$B=\{e,f,g\}$ ovo je takođe baza u $P_{\leq 2}$

P - matr. prelaza sa E na B   E - stand. baza

$$P=\begin{pmatrix}1&2&1\\1&0&1\\0&1&2\end{pmatrix} \qquad [p]_E=P\cdot[p]_B \qquad P^{-1}=\frac{1}{-4}\cdot\begin{pmatrix}-1&-3&2\\-2&2&0\\-1&-1&-2\end{pmatrix}$$

$$[p]_B=P^{-1}\cdot[p]_E$$

$$P^{-1}\cdot[p]_E=\frac{1}{-4}\cdot\begin{pmatrix}1&-3&2\\-2&2&0\\-1&-1&-2\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}3/4\\1/2\\1/4\end{pmatrix}$$

$$p(x)=\frac{3}{4}\cdot e(x)+\frac{1}{2}f(x)+\frac{1}{4}g(x)$$

b) Na prostoru R^3 fiksiramo bazu

$\{f_1=(1,0,-1),\ f_2=(0,2,0),\ f_3=(1,2,3)\}$ a na prostoru R^2 fik stand. bazu. Lin. op. $A: R^3\to R^2$ u tom paru baza prikazan je matricom

$$A_{fe}=\begin{pmatrix}2&-1&3\\3&1&0\end{pmatrix} \qquad (\{e_1=(1,0),\ e_2=(0,1)\})$$

Odr. matr. op. A u paru stand. baza prostora R^3 i R^2 i izračunati $A(-3,2,5)$

$$\begin{array}{ccc} R^3 & \xrightarrow{A} & R^2 \\ \{f_1,f_2,f_3\} & A_{fe} & \{e_1,e_2\} \\ \downarrow P & & \downarrow Q \\ \{g_1,g_2,g_3\} & A_{ge} & \{e_1,e_2\} \end{array} \qquad A_{ge}=Q^{-1}A_{fe}P$$

P - matrica prelaza sa f na g $g_1=(1,0,0)$ $g_2=(0,1,0)$ $g_3=(0,0,1)$

Ako je P_1 matrica prelaza sa g na f

$$P=P_1^{-1} \qquad P_1=\begin{pmatrix}1&0&1\\0&2&2\\-1&0&3\end{pmatrix} \qquad P=\frac{1}{8}\begin{pmatrix}6&0&-2\\-2&4&-2\\2&0&2\end{pmatrix}$$

Q - matrica prelaza sa e na e $Q=E\Rightarrow Q^{-1}=E$

$$A_{ge}=E\cdot A_{fe}\cdot P=A_{fe}\cdot P=\frac{1}{8}\begin{pmatrix}2&-1&3\\3&1&0\end{pmatrix}\begin{pmatrix}6&0&-2\\-2&4&-2\\2&0&2\end{pmatrix}=\begin{pmatrix}5/2&-1/2&1/2\\2&1/2&-1\end{pmatrix}$$

5. U prostoru polinoma $P_{\leq 2}$ zadata su 2 operatora. Op. $\mathcal{F}$ svakom polinomu $a_0+a_1x+a_2x^2$ pridružuje polinom a_0+a_1x, a op. $\mathcal{D}$ polinomima x^2+x, x^2+1, x^2+x+1 pridr. redom polinome x^2+1, x^2+x+1 i nula polinom. Naći matr. op. $\mathcal{D}\cdot\mathcal{F}$ u odn. na bazu

$B=\{x^2, x^2+x, x^2+x+1\}$

Neka je $\mathcal{A}=\mathcal{D}\mathcal{F}$. Odredimo matr. op. $\mathcal{A}$ u odn. na stand. bazu $\{e_1(x)=1, e_2(x)=x, e_3(x)=x^2\}$

1° $\mathcal{F}e_1=\mathcal{F}1=1=1\cdot e_1+0\cdot e_2+0\cdot e_3$

$\mathcal{F}e_2=\mathcal{F}x=0\cdot e_1+1\cdot e_2+0\cdot e_3$

$\mathcal{F}e_3=\mathcal{F}x^2=0=0\cdot e_1+0\cdot e_2+0\cdot e_3$

Matrica op. $\mathcal{F}$ u bazi e je

$$F_e=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

2° $\mathcal{D}(x^2+x)=x^2+1$ $\qquad$ $\mathcal{D}(e_2+e_3)=e_1+e_3$

$\mathcal{D}(x^2+1)=x^2+x+1$ $\qquad$ $\mathcal{D}(e_1+e_3)=e_1+e_2+e_3$

$\mathcal{D}(x^2+x+1)=0$ $\qquad$ $\mathcal{D}(e_1+e_2+e_3)=0$

$d_i=\mathcal{D}e_i \quad i=\overline{1,3}$

$d_2+d_3=e_1+e_3$ $\qquad$ $d_1=-e_1-e_3$ $\qquad$ $\mathcal{D}e_1=-e_1-e_3$

$d_1+d_3=e_1+e_2+e_3$ $\qquad$ $d_2=-e_1-e_2-e_3$ $\qquad$ $\mathcal{D}e_2=-e_1-e_2-e_3$

$d_1+d_2+d_3=0$ $\qquad$ $d_3=2e_1+e_2+2e_3$ $\qquad$ $\mathcal{D}e_3=2e_1+e_2+2e_3$

Matrica op. $\mathcal{D}$ u bazi e je

$$D_e=\begin{pmatrix}-1&-1&2\\0&-1&1\\-1&-1&2\end{pmatrix}$$

Matr. op. $\mathcal{A}$ u bazi e je

$$A_e=D_e\cdot F_e=\begin{pmatrix}-1&-1&0\\0&-1&0\\-1&-1&0\end{pmatrix}$$

A_B - matrica op. A u bazi B

$A_B = P^{-1} A_e P$ P - matr. prelaza sa e na B

$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. $P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $A_B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix}$

* Neka je A lin. op. na prostoru $\mathbb{R}^2$ koji vektor prvo ro tira za ugao $\pi/3$ oko koord. početka u poz. smjeru, a zatim ga reflektuje (sim. presl.) u odn. na pravac $y=x$. Naći matricu op. u bazi $B = \{(1,1), (1,-1)\}$. Odrediti koordinate vektora Av u odn. na tu bazu, gdje je v proizvoljan vektor u $\mathbb{R}^2$ *

* Svojstv. vrij. i svojstv. vektori *

$A: V \to V$ V - vekt. pr. nad K

$\{e_1 \dots e_n\}$ baza u V A_e - matrica op. u bazi e

Def. $\lambda \in K$ je svojstv. vrij. op. A ako postoji $x \neq 0 \in V$ t.d.

$Ax = \lambda x$ x - svojstv. vektor

Teorema: $\lambda \in K$ je svojstv. vrijednost akko je λ korijen jednačine $\det(A_e - \lambda E) = 0$

$x = x_1 e_1 + \dots + x_n e_n$ je pripadni svojstv. vektor, i dobija se kao rešenje jednačine

$$(A_e - \lambda E)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$p(\lambda) = \det(A_e - \lambda E)$ - karakt. polinom op. A

$a(\lambda)$ - algebarska višestrukost svojstv. vrij. λ

$a(\lambda_0) = \nu_0$

$L_\lambda = \{x \in V \mid Ax = \lambda x\}$ - svojstv. potpr., invarijantan u odnosu na A

L_λ - sadrži sve svoj. vektore pripadne svoj. vrij. λ i nula vektor

$\dim L_\lambda \geq 1$

$g(\lambda) = \dim L_\lambda$ - geom. višestrukost

$g(\lambda) \leq a(\lambda)$ - algebarska -II-

λ - svoj. vrij.

$\mathcal{A}x = \lambda x$

$\mathcal{A}x - \lambda_0 x = 0$

$\mathcal{A}x - \lambda_0 J x = 0$

$(\mathcal{A} - \lambda_0 J)x = 0$

$x \in Ker(\mathcal{A} - \lambda_0 J)$ $\quad L_{\lambda_0} = Ker(\mathcal{A} - \lambda_0 J)$

$\lambda_1, \dots \lambda_m$ - razl. svojstv. vrij. $m \leq n$

$v_j = a(\lambda_j)$

$p(\lambda) = \det(A_e - \lambda E) = g(\lambda) \cdot \prod_{j=1}^{n} (\lambda - \lambda_j)^{v_i}$

$g(\lambda)$ - polinom sa koef. iz $\mathbb{R}$ koji u n nema nula

Teor: $\lambda_1, \dots, \lambda_m$ - razl. svojstv. vrij.

$V = L_{\lambda_1} \dot{+} L_{\lambda_2} \dot{+} \dots \dot{+} L_{\lambda_m}$ akko $g(\lambda) = (-1)^n$ i $g(\lambda_j) = a(\lambda_j)$, $1, m$

Baza prostora V je sastavljena od svoj. vektora i dobija se kao unija baza pp-a $L_{\lambda_1} \dots L_{\lambda_m}$. U odn. na tu bazu f op. $\mathcal{A}$ ima

Na dijag. su svojstv. vrijednosti sa v_j - potpr.

Teor.: Postoji baza svoj. vektora akko $g(\lambda) = (-1)^n$, $a(v_j) = g(v_j)$

Def. $\mathcal{A}$ je proste strukture ako postoji baza u v. pr. V sastavljena od svojstvenih vektora

Teor. $\mathcal{A}$ u odnosu na bazu f ima dijagonalnu matricu akko je baza sastavljena od svojstvenih vektora operatora $\mathcal{A}$

Def. Potprostor L vektorskog prostora V naziva se invarijantnim potprostorom ako je $Ax \in L$ za $\forall x \in L$

1. Naći svoj. vrij. i svoj vektore operatora zadatog matricom

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$p(\lambda) = \det(A_e - \lambda E) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)((1-\lambda)^2 - 1) = (1-\lambda)(\lambda^2 - 2\lambda) = -\lambda(\lambda-1)(\lambda-2)$$

$p(\lambda) = -\lambda(\lambda-1)(\lambda-2)$

$p(\lambda) = 0 \Rightarrow \lambda = 0 \vee \lambda = 1 \vee \lambda = 2$

Svojst. vrij. su $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$ i njihove alg vrijedn. su $V_1 = 1$, $V_2 = 1$, $V_3 = 1$

$\lambda_1 = 0$

$$(A_e - 0 \cdot E)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot(-1) \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 0 \qquad (-x_3, 0, x_3) \quad x_3 \in R \setminus \{0\}$

$x_2 = 0$

$x_1 = -x_3$

$L_{\lambda_1} = \{(-x_3, 0, x_3) \mid x_3 \in R\} = \mathcal{L}in\{f_1\} \qquad f_1 = (-1, 0, 1)$

$g(\lambda_1) = 1$

$\lambda_2 = 1$

$$(A_e - E)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$L_{\lambda_2} = \{x_3, -x_3, x_3 \mid x_3 \in R\}$

$L\lambda_2 = \mathcal{L}in\{f_2\} \quad f_2 = (1,-1,1)$

$g(\lambda_2) = \dim L\lambda_2 = 1$

$\lambda_3 = 2$

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$L\lambda_3 = \{(x_3, 0, x_3) \mid x_3 \in \mathbb{R}\} = \mathcal{L}in\{f_3\} \quad f_3 = (1,0,1)$

$g(\lambda_3) = \dim L\lambda_3 = 1$

$g(\lambda) = (-1)^3 = -1$

$v_j = g(\lambda_j) \quad j = \overline{1,3}$

Postoji baza prostora V sastavljena od svoj. vektora operatora $\mathcal{A}$ $\{f_1, f_2, f_3\}$

$\mathcal{A}f_1 = \lambda_1 \cdot f_1 = 0 = 0 \cdot f_1 = 0 \cdot f_1 + 0 \cdot f_2 + 0 \cdot f_3$

$\mathcal{A}f_2 = 1 \cdot f_2$

$\mathcal{A}f_3 = 2 \cdot f_3$

$$A_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Na dijagonali su svojstvene vrijednosti

A je proste strukture jer baza sv. vrijednosti postoji

2. Ispitati da li je operator $\mathcal{A}: P_{\leq 2} \to P_{\leq 2}$ zadat sa $\mathcal{A}p = 2p - p''$ proste strukture?

$\{e_1(x) = 1,\ e_2(x) = x,\ e_3(x) = x^2\}$ stand baza u $P_{\leq 2}$

Nađimo matricu op. $\mathcal{A}$ u bazi e

$\mathcal{A}e_1 = 2e_1 - e_1'' = 2e_1$

$\mathcal{A}e_2 = 2e_2 - e_2'' = 2e_2$

$\mathcal{A}e_3 = 2e_3 - 2 = -2 \cdot e_1 + 2 \cdot e_3$

$$Ae = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$p(\lambda) = \det(Ae - \lambda E) = \begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 = -(\lambda-2)^3$$

$p(\lambda) = -(\lambda-2)^3$

$p(\lambda) = 0 \Rightarrow \lambda = 2$

Svoj. vr. je $\lambda = 2$, a njena alg. viš. je $a(\lambda) = 3$ $V=3$

$\lambda = 2$

$$(Ae - 2E)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} -2x_3 = 0 \\ x_3 = 0 \end{matrix}$$

$L\lambda = \{(x_1, x_2, 0) \mid x_1, x_2 \in R\}$

$= \{x_1(1,0,0) + x_2(0,1,0) \mid x_1, x_2 \in R\} = \mathcal{L}(f_1, f_2)$

$f_1 = (1,0,0)$ $f_2 = (0,1,0)$ $f_1(x) = 1$ $f_2(x) = x$

Baza u $L\lambda$ je $\{f_1(x) = 1,\ f_2(x) = x\}$

$g(\lambda) = \dim L\lambda = 2$ $a(\lambda) \neq g(\lambda) \Rightarrow \mathcal{A}$ nije proste str.

3. Naći svoj. vrij. i vekt. op. diferenciranja nad prostorom $L = \mathcal{L}(f_1, f_2)$ gdje je $f_1(t) = \cos t$ $f_2(t) = \sin t$

$\mathcal{D}: L \to L$

$\mathcal{D}f_1 = f_1' = -\sin t = 0 \cdot f_1 + (-1) \cdot f_2$

$\mathcal{D}f_2 = f_2' = \cos t = 1 \cdot f_1 + 0 \cdot f_2$

Matrica op. $\mathcal{D}$ u bazi f je: $Df = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$p(\lambda) = \det(Df - \lambda E) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$p(\lambda)$ nema nula nad poljem $\mathbb{R}$ pa op. $\mathcal{A}$ nema svojstv. vrij.

4. Operator $\mathcal{A} \in \mathcal{L}(\mathbb{R}^{2\times 2})$ def. je sa $\mathcal{A}X = RX, \forall X \in \mathbb{R}^{2\times 2}$
$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Naći svojstv. vrij. i svojstv. vektore op. $\mathcal{A}$

$\left\{ E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ baza u $\mathbb{R}^{2\times 2}$

$$\mathcal{A}E_1 = R\cdot E_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0\cdot E_1 + 0\cdot E_2 + 1\cdot E_3 + 0\cdot E_4$$

$$\mathcal{A}E_2 = R\cdot E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0\cdot E_1 + 0\cdot E_2 + 0\cdot E_3 + 1\cdot E_4$$

$$\mathcal{A}E_3 = R\cdot E_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = -1\cdot E_1 + 0\cdot E_2 + 0\cdot E_3 + 0\cdot E_4$$

$$\mathcal{A}E_4 = R\cdot E_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = 0\cdot E_1 + 0\cdot E_2 + 0\cdot E_3 - 1\cdot E_4$$

$$A_E = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$p(\lambda) = \det(A_E - \lambda E) = \begin{vmatrix} -\lambda & 0 & -1 & 0 \\ 0 & -\lambda & 0 & -1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{vmatrix} = \lambda^4 + 2\lambda^2 + 1$$

$p(\lambda)$ nema nula u $\mathbb{R}$

5. Ako je rang $\mathcal{A} = 1$ koliko najviše svojstv. vrij. može imati: $\mathcal{A}: V \to V$ pri čemu je $\dim V = n$

$\operatorname{rang} \mathcal{A} = 1$
$\operatorname{rang}(\mathcal{A} - 0\cdot J) = 1$
$n_{\mathcal{A}-0J} + r_{\mathcal{A}-0J} = n$
$n_{\mathcal{A}-0J} = n-1$
$\dim \operatorname{Ker}(\mathcal{A} - 0J) = 1$
$\dim \{x \mid (\mathcal{A}-0J)x = 0\} = n-1$
$\dim L\lambda_0 = n-1, \quad \lambda_0 = n-1$
$g(\lambda_0) = n-1$
$g(\lambda_0) \leq a(\lambda_0)$
$a(\lambda_0) \geq n-1$

λ_0 je nula karakt. polinoma $p(\lambda)$ i njena višestrukost kao nule tog polinoma je $\geq n-1$. Kako p, kao polinom n-tog stepena ima najviše n nula to λ_0 ima vr. $n-1$ ili n pa $p(\lambda)$ ima dvije ili jednu nulu x2 6. Neka je $A_e = \begin{pmatrix} 16 & 0 & -12 \\ 0 & 25 & 0 \\ -12 & 0 & 9 \end{pmatrix}$ matrica lin. op. $\mathcal{A}: X \to X$ u odn. na bazu $\{e_1, e_2, e_3\}$ prostora X. Da li je op. $\mathcal{A}$ proste strukture? Ako jeste naći njegovu matr. u odn. na bazu svojstv. vektora.

$p(\lambda) = \det(A_e - \lambda E) = \begin{vmatrix} 16-\lambda & 0 & -12 \\ 0 & 25-\lambda & 0 \\ -12 & 0 & 9-\lambda \end{vmatrix} = -\lambda(\lambda-25)^2$

$\underline{\lambda_1 = 0}$

$(A_e - 0 \cdot E)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \left(\frac{3}{4}x_3, 0, x_3\right)$

$L\lambda_1 = \{(\frac{3}{4}x_3, 0, x_3) \mid x_3 \in R\} = \{x_3(\frac{3}{4}, 0, 1)\} = \{c \cdot (3, 0, 4), c \in R\}$

$= \mathcal{L}(f_1) \quad f_1 = (3, 0, 4) \quad g(\lambda_1) = \dim L\lambda_1 = 1$

$a(\lambda_1) = 1$

$a(\lambda_1) = g(\lambda_1)$

$\underline{\lambda_2 = 25}$

$(A_e - 25 \cdot E) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$L\lambda_2 = \{(-\frac{4}{3}x_3, x_2, x_3) \mid x_2, x_3 \in R\} = \{c_1(0,1,0) + c_2(4,0,-3) \mid c_1, c_2 \in R\}$

$= \mathcal{L}(f_2, f_3) \quad f_2 = (0,1,0) \quad f_3 = (4, 0, -3)$

$g(\lambda_2) = \dim L\lambda_2 = 2 \quad a(\lambda_2) = 2 \quad a(\lambda_2) = g(\lambda_2)$

$p(\lambda) = g(\lambda)(\lambda - 0)(\lambda - 25)^2$

$\left.\begin{array}{l} g(\lambda) = (-1)^3 = -1 \\ a(\lambda_1) = g(\lambda_1) \\ a(\lambda_2) = g(\lambda_2) \end{array}\right\} \Rightarrow$ op. $\mathcal{A}$ je proste strukture

$\{f_1, f_2, f_3\}$ baza svojstv. vektora

$\mathcal{A}f_1 = 0 \cdot f_1$

$\mathcal{A}f_2 = 25 \cdot f_2 \qquad A_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{pmatrix}$

$\mathcal{A}f_3 = 25 \cdot f_3$

7. Neka su $\mathcal{A}$ i $\mathcal{B}$ komutativni op.
a) Dokazati da su slika i jezgro op. $\mathcal{B}$ invar. potpr. op. $\mathcal{A}$
b) Dokazati da je svaki svojstv. potpr. oper. $\mathcal{B}$ invar. u odn. na op. $\mathcal{A}$

a) N_B - jezgro

$x \in N_B \Rightarrow \mathcal{B}x = 0$

$\mathcal{B}(\mathcal{A}x) = (\mathcal{B}\mathcal{A})x = (\mathcal{A}\mathcal{B})x = \mathcal{A}(\mathcal{B}x) = \mathcal{A}0 = 0 \Rightarrow \mathcal{A}x \in N_B$

$(\mathcal{A}x \in N_B)\ \forall x \in N_B \Rightarrow N_B$ je invar. potpr. u odn. na op. $\mathcal{A}$

$y \in T_B \Rightarrow \exists x \in X\ \ y = \mathcal{B}x$

$\mathcal{A}y = \mathcal{A}(\mathcal{B}x) = (\mathcal{A}\mathcal{B})x = (\mathcal{B}\mathcal{A})x = \mathcal{B}(\mathcal{A}x) \in T_B$

$(\mathcal{A}y \in T_B)\ \forall y \in T_B \Rightarrow T_B$ je invar.

b) λ - svoj. vr. op. $\mathcal{B}$

$L_\lambda = \{x \mid \mathcal{B}x = \lambda x\} = \{x \mid (\mathcal{B} - \lambda \mathcal{I})x = 0\} = \mathcal{K}er(\mathcal{B} - \lambda \mathcal{I})$

$$\left.\begin{array}{l}(\mathcal{B} - \lambda \mathcal{I}) \cdot \mathcal{A} = \mathcal{B}\mathcal{A} - \lambda \mathcal{A} \\ \mathcal{A}(\mathcal{B} - \lambda \mathcal{I}) = \mathcal{A}\mathcal{B} - \lambda \mathcal{A} = \mathcal{B}\mathcal{A} - \lambda \mathcal{A}\end{array}\right\} \Rightarrow \mathcal{B} - \lambda \mathcal{I} \text{ i } \mathcal{A} \text{ su}$$

komutativni pa na osnovu a) $\mathcal{K}er(\mathcal{B} - \lambda \mathcal{I})$ je svojstv. pp. op. $\mathcal{A}$ tj. L_λ je svojst. pp. op. $\mathcal{A}$

8. Odrediti sve potpr. od R^2 koji su invar. u odn. na op. $\mathcal{A}$ čija je matrica $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$

R// Traženi pp-i mogu biti dimenzija 0, 1, 2.

$L = \{0\}$ je triv. invar. potpr. dimenzije 0

R^2 je invar. potpr. dimenzije 2

Ostaje da nađemo invar. potpr. dimenzije 1

Neka je S takav potpr.

$S = \mathcal{L}(x),\ x \neq 0$

S je generisan vektorom x

S invar. pa je $\mathcal{A}x \in S = \mathcal{L}(x)$

postoji λ, $A x = \lambda x$

$(A - \lambda J) x = 0$

$x \in Ker(A - \lambda J)$

$\alpha \cdot x \in Ker(A - \lambda J)$ jer je jezgro pp.

$S \subseteq Ker(A - \lambda J)$ (jer $(\forall \alpha \cdot x \in S)(\alpha x \in Ker(A - \lambda J))$)

$\dim S = 1$

$\dim Ker(A - \lambda J) \geq 1$

$(A - \lambda E) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Homogeni sistem mora imati netriv. rešenja

$\det(A - \lambda E) = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

$-3\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$3(\lambda - 1)(\lambda - 2) = 0 \quad \lambda = 1 \lor \lambda = 2$

$\underset{\sim}{\lambda = 1}$

$(A - 1 \cdot E)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -x_1 + x_2 = 0$

$x_1 = x_2$

$S_1 = L_{\lambda_1} = \{(x_2, x_2) \mid x_2 \in \mathbb{R}\} = \mathcal{L}(f_1) \quad f_1 = (1,1)$

$\underset{\sim}{\lambda = 2}$

$(A - 2E)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -2x_1 + x_2 = 0$

$x_2 = 2x_1$

$S_2 = L_{\lambda_2} = \{(x_1, 2x_1) \mid x_1 \in \mathbb{R}\} = \mathcal{L}(f_2) \quad f_2 = (1,2)$

S_1, S_2 - invar. potpr. dimenzije 1