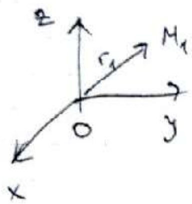
(9) 

$M_1(\vec{r}_1),\ M_2(\vec{r}_2),\ M(\vec{r})$



Tačka M dijeli duž M_1M_2 u omjeru $|\overrightarrow{M_1M}| : |\overrightarrow{MM_2}| = |\lambda|$, $\lambda \neq -1$

akko $\vec{r} = \frac{1}{1+\lambda}(\vec{r}_1 + \lambda\vec{r}_2)$

$M_1(x_1, y_1, z_1),\ M_2(x_2, y_2, z_2),\ M_3(x_3, y_3, z_3)$

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}$$

$$y = \frac{y_1 + \lambda y_2}{1+\lambda}$$

$$z = \frac{z_1 + \lambda z_2}{1+\lambda}$$

Ako je $\lambda = 1$, tačka M je središte duži

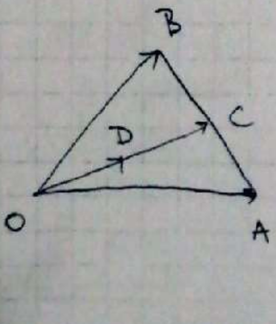
$$x = \frac{x_1+x_2}{2},\quad y = \frac{y_1+y_2}{2},\quad z = \frac{z_1+z_2}{2}$$

3. oktobar

1. Neka su $A(-1,-2,-4)$, $B(-4,-2,0)$, $O(3,-2,1)$ tjemena trougla. Tačka C dijeli duž AB u omjeru $3:4$, a tačka D dijeli duž OC u omjeru $2:5$.

a) Izračunati vektor $\overrightarrow{OD}$ preko vektora $\overrightarrow{OA}$ i $\overrightarrow{OB}$

b) Naći koordinate tačke C



$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + \frac{3}{7}\cdot\overrightarrow{AB} = \overrightarrow{OA} + \frac{3}{7}(\overrightarrow{OB} - \overrightarrow{OA}) =$$

$$= \frac{4}{7}\overrightarrow{OA} + \frac{3}{7}\overrightarrow{OB}$$

$$\overrightarrow{OD} = \frac{2}{7}\cdot\overrightarrow{OC} = \frac{2}{7}\left(\frac{4}{7}\overrightarrow{OA} + \frac{3}{7}\overrightarrow{OB}\right) = \frac{8}{49}\overrightarrow{OA} + \frac{6}{49}\overrightarrow{OB}$$

$A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

$M(x, y, z)$ - središte duži AB

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}$$

$$|AC| = |CB| = \frac{3}{4}$$

$C(x, y, z) \qquad x = -\frac{16}{7}, \ y = -2, \ z = -\frac{16}{7} \qquad C\left(-\frac{16}{7}, -2, -\frac{16}{7}\right)$

Skalarni proizvod vektora

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cdot \cos \measuredangle(\vec{a}, \vec{b})$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \, pr_{\vec{a}} \vec{b}$$

$$pr_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \qquad pr_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$\vec{a} = (x_1, y_1, z_1)$

$\vec{b} = (x_2, y_2, z_2)$

$$\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Pr Ako je $|\vec{a}| = 3$, $|\vec{b}| = 4$, $\measuredangle(\vec{a}, \vec{b}) = \frac{2\pi}{3}$, naći

a) $\vec{a} \cdot \vec{b}$

b) $(3\vec{a} - 2\vec{b})(\vec{a} + 2\vec{b})$

c) $|\vec{a} + \vec{b}|^2$

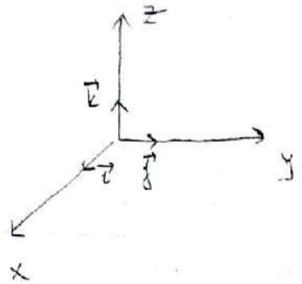
a) $\vec{a} \cdot \vec{b} = |\vec{a}| \, |\vec{b}| \cos \frac{2\pi}{3} = 3 \cdot 4 \cdot \left(-\frac{1}{2}\right) = -6$

b) $(3\vec{a} - 2\vec{b})(\vec{a} + 2\vec{b}) = 3\vec{a} \cdot \vec{a} + 6\vec{a} \cdot \vec{b} - 2\vec{b} \cdot \vec{a} - 4\vec{b} \cdot \vec{b} =$

$= 3|\vec{a}|^2 + 4\vec{a} \cdot \vec{b} - 4|\vec{b}|^2 = 3 \cdot 9 + 4 \cdot (-6) - 4 \cdot 16 = 27 - 24 - 64 = -61$

c) $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})(\vec{a} + \vec{b}) = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 9 + 2 \cdot (-6) + 16 = 13$

Pr. Naći skalarni proizvod vektora $\vec{a}=(3,4,7)$, $\vec{b}=(2,-5,2)$



$$\vec{a}\cdot\vec{b}=3\cdot 2+4\cdot(-5)+7\cdot 2=0$$

Primijetimo da je $\vec{a}\perp\vec{b}$

1. Naći vektor $\vec{c}$ koji je kolinearan sa vektorom $\vec{a}+\vec{b}$ ako su ispunjeni uslovi $\vec{a}\cdot\vec{b}=5$, $\vec{c}\cdot\vec{b}=18$, $|\vec{b}|=2$

$\vec{c}=k(\vec{a}+\vec{b})$

I $\quad k(\vec{a}+\vec{b})\cdot\vec{b}=18$

II $\quad \vec{c}=k(\vec{a}+\vec{b}) \quad /\cdot\vec{b}$

$\vec{c}\cdot\vec{b}=k\vec{a}\cdot\vec{b}+k\cdot\vec{b}\cdot\vec{b}$

I $\quad k\vec{a}\cdot\vec{b}+k|\vec{b}|^2=18$

$5k+4k=18$

$9k=18$

$k=2$

$\vec{c}=2(\vec{a}+\vec{b})$

2. Naći ugao između vektora $\vec{a}$ i $\vec{b}$ ako je $|\vec{a}|=2|\vec{b}|$ i vektor $2\vec{a}+\vec{b}$ je normalan na vektor $\vec{a}-3\vec{b}$

$(2\vec{a}+\vec{b})\perp(\vec{a}-3\vec{b})$

$(2\vec{a}+\vec{b})(\vec{a}-3\vec{b})=0$

$2|\vec{a}|^2-5\vec{a}\cdot\vec{b}-3|\vec{b}|^2=0$

$8|\vec{b}|^2-5\vec{a}\cdot\vec{b}-3|\vec{b}|^2=0$

$|\vec{b}|^2-\vec{a}\cdot\vec{b}=0$

$\vec{a}\cdot\vec{b}=|\vec{b}|^2$

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\varphi$

$$\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

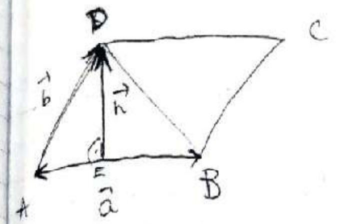
$$\cos\varphi=\frac{\cancel{|\vec{b}|^2}}{2\cancel{|\vec{b}|}\cancel{|\vec{b}|}}$$

$$\cos\varphi=\frac{1}{2}$$

$$\boxed{\varphi=\frac{\pi}{3}}$$

$\vec{a}, \vec{b}$

3. Dokazati da je vektor $\vec{h}$ visine paralelograma nad vektorima $\vec{a}, \vec{b}$ dat sa $\vec{h} = \vec{b} - \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\vec{a}_0$ gdje je $\vec{a}_0$ jedinični vektor vektora $\vec{a}$



$\vec{AB} = \vec{a} \quad \vec{AD} = \vec{b}$

E – podnožje visine iz tjemena D na osnovicu AB

$\vec{h} = \vec{ED} = \vec{EA} + \vec{AD}$

$\vec{EA} = |\vec{EA}|\cdot\vec{e}_0$, jedinični vektor vektora $\vec{EA}$

$\vec{h} = -\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\vec{a}_0 + \vec{b}$

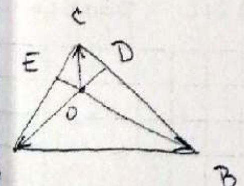
Vektori $\vec{EA}$ i $\vec{AB}$ su kolinearni, ali suprotnog smjera pa su njihovi jedinični vektori suprotni $\vec{e}_0 = -\vec{a}_0$

$$\boxed{\vec{h} = \vec{b} - \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\vec{a}_0}$$

Uočimo da je $|\vec{EA}| = |\vec{AE}| = pr_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$\vec{EA} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot(-\vec{a}_0)$

4. Dokazati da se u svakom trouglu visine sijeku u jednoj tački.



$\vec{AD}$ - visina iz A

BE - visina iz B

O - presjek visina AD i BE

Dokažimo da je $CO \perp AB$ i time ćemo pokazati da visina iz tjemena C prolazi kroz tačku O.

$\vec{OA} \perp \vec{BC} \qquad \vec{OB} \perp \vec{CA}$

$\vec{OA}\cdot(\vec{OC} - \vec{OB}) = 0 \qquad \vec{OB}\cdot(\vec{OA} - \vec{OC}) = 0$

$\vec{OA}\cdot\vec{OC} - \vec{OA}\cdot\vec{OB} = 0 \quad (1) \qquad \vec{OB}\cdot\vec{OA} - \vec{OB}\cdot\vec{OC} = 0 \quad (2)$

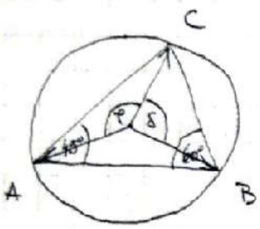
$(1) + (2): \quad \vec{OA}\cdot\vec{OC} - \vec{OB}\cdot\vec{OC} = 0 \qquad \vec{BA}\cdot\vec{OC} = 0 \Rightarrow \vec{BA} \perp \vec{OC}$

$\underbrace{(\vec{OA} - \vec{OB})}_{\vec{BA}}\cdot\vec{OC} = 0$

(*) Naći ugao koji obrazuju vektori $\vec{a}$ i $\vec{b}$ ako je $(\vec{a}-2\vec{b}) \perp (\vec{a}+3\vec{b})$?
i $(\vec{a}+\vec{b}) \perp (2\vec{a}+\vec{b})$

5. U $\triangle ABC$ dati su uglovi $\alpha = 45°$ i $\beta = 60°$. Oko trougla opisan je krug poluprečnika $r = 1$ sa centrom u tački O. Odrediti:

a) $\vec{OB} \cdot \vec{OC}$, b) $\vec{OC} \cdot \vec{OA}$



$|\vec{OA}| = |\vec{OB}| = |\vec{OC}| = r = 1$

$\measuredangle(\vec{OB}, \vec{OC}) = 2 \cdot 45° = 90°$

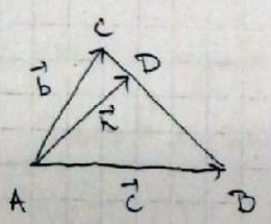
Centralni ugao dvostruko veći od periferijskog nad istim luk.

$\vec{OB} \cdot \vec{OC} = |\vec{OB}||\vec{OC}| \cos 90° = 0$

b) $\measuredangle(\vec{OA}, \vec{OC}) = 2 \cdot 60° = 120°$

$\vec{OA} \cdot \vec{OC} = |\vec{OA}||\vec{OC}| \cos 120° = 1 \cdot 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$

6. ΔABC zadat je vektorima $\vec{AB} = \vec{c}$ i $\vec{AC} = \vec{b}$. Izraziti pomoću vektora $\vec{c}$ i $\vec{b}$ vektor visine iz tjemena A.



$\vec{h} = \vec{AD}$

$\vec{h} = \vec{AB} + \vec{BD}$

$\vec{h} = \vec{c} + k \cdot \vec{BC}$

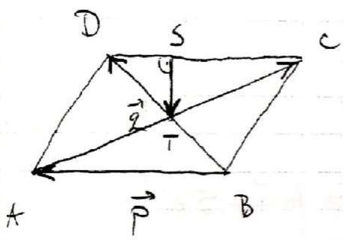
$\vec{h} = \vec{c} + k(\vec{b} - \vec{c}) \ / \cdot \quad \vec{BC} = \vec{b} - \vec{c}$

$0 = \vec{c}(\vec{b} - \vec{c}) + k|\vec{b} - \vec{c}|^2$

$k = -\dfrac{\vec{c}(\vec{b} - \vec{c})}{|\vec{b} - \vec{c}|^2}$

$\vec{h} = \vec{c} - \dfrac{\vec{c}(\vec{b} - \vec{c})}{|\vec{b} - \vec{c}|^2} \cdot (\vec{b} - \vec{c})$

7. Neka je T presječna tačka dijagonala paralelograma $ABCD$ i neka je S podnožje visine $\triangle TCD$ iz tjemena T. Izraziti vektor $\vec{ST}$ u bazi $\vec{BA} = \vec{p}$ i $\vec{BD} = \vec{q}$



$\vec{ST} = -\vec{TS}$

$$\vec{ST} = \vec{SD} + \vec{DT}$$

$$\vec{ST} = k \cdot \vec{CD} + \frac{1}{2}\vec{DB}$$

$$\vec{ST} = k \cdot (+\vec{p}) + \frac{1}{2} \cdot (-\vec{q}) \quad / \cdot \vec{p}$$

$$0 = k \cdot |\vec{p}|^2 - \frac{1}{2}\vec{p} \cdot \vec{q}$$

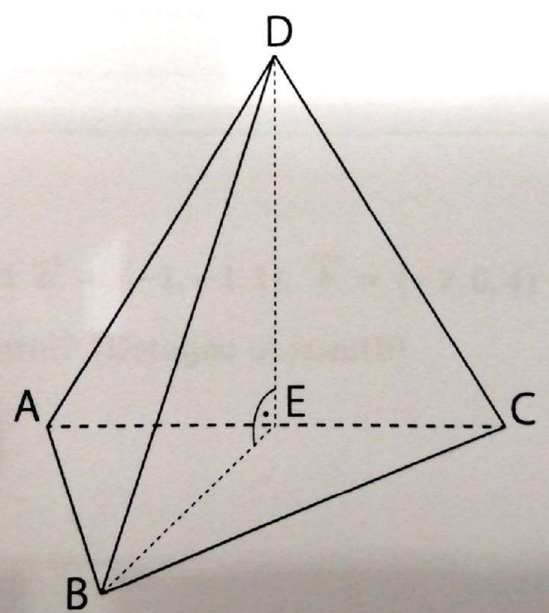
$$k = \frac{1}{2} \cdot \frac{\vec{p} \cdot \vec{q}}{|\vec{p}|^2}$$

$$\vec{ST} = \frac{1}{2} \cdot \frac{\vec{p} \cdot \vec{q}}{|\vec{p}|^2} \cdot \vec{p} - \frac{1}{2}\vec{q}$$

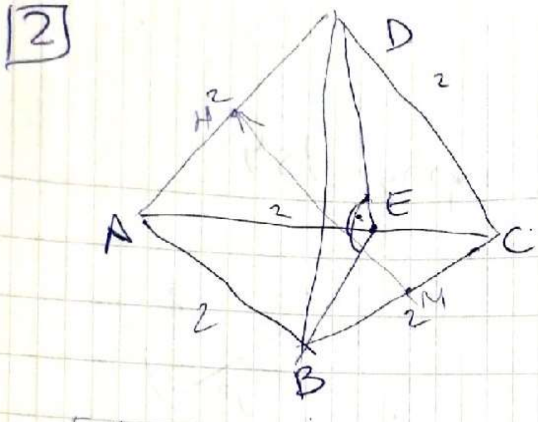
2. Tačka E je središte ivice AC datog tetredra $ABCD$. Strane ABC i ACD su jednakostranični trouglovi stranice 2, a ugao $\measuredangle BED$ je prav.

   (a) Izračunati $\overrightarrow{AB} \cdot \overrightarrow{AD}$.

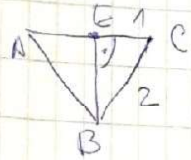
   (b) Ako je M središte ivice BC, a N središte ivice AD izračunati $|\overrightarrow{MN}|$.



$\boxed{2}$



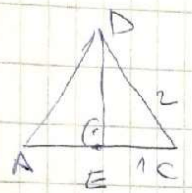
$\vec{AB} \cdot \vec{AD}$



Visina u jednakostr. trouglu pa je $\sphericalangle(\vec{EA}, \vec{EB})$ prav.

$|\vec{EB}| = \sqrt{4-1} = \sqrt{3}$

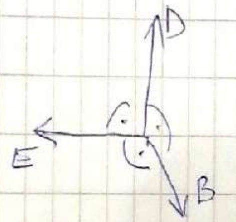
$|\vec{EA}| = 1$



Visina u jednakostr. trouglu pa je $\sphericalangle(\vec{EA}, \vec{ED})$ - prav.

$|\vec{ED}| = \sqrt{4-1} = \sqrt{3}$

$|\vec{EA}| = 1$



Tri nekomplanarna ortogonalna vektora

Izrazićemo sve preko njih.

a) $\vec{AB} = \vec{EB} - \vec{EA}$

$\vec{AD} = \vec{ED} - \vec{EA}$

$\vec{AB} \cdot \vec{AD} = (\vec{EB} - \vec{EA}) \cdot (\vec{ED} - \vec{EA}) =$

$= \underbrace{\vec{EB} \cdot \vec{ED}}_{0} - \underbrace{\vec{EB} \cdot \vec{EA}}_{0} - \underbrace{\vec{EA} \cdot \vec{ED}}_{0} - |\vec{EA}|^2 = -1$

b) $\vec{MN} = \vec{MB} + \vec{BA} + \vec{AN}$

$\vec{MB} = \frac{1}{2}\vec{CB} = \frac{1}{2}(\vec{EB} - \vec{EC}) = \frac{1}{2}(\vec{EB} + \vec{EA})$

$\vec{BA} = \vec{EA} - \vec{EB}$

$\vec{AN} = \frac{1}{2}\vec{AD} = \frac{1}{2} \cdot (\vec{ED} - \vec{EA})$

$\vec{MN} = \frac{1}{2}\vec{EB} + \cancel{\frac{1}{2}\vec{EA}} + \vec{EA} - \vec{EB} + \frac{1}{2}\vec{ED} - \cancel{\frac{1}{2}\vec{EA}} =$

$= \vec{EA} - \frac{1}{2}\vec{EB} + \frac{1}{2}\vec{ED}$

$|\vec{MN}|^2 = \vec{MN} \cdot \vec{MN} = (\vec{EA} - \frac{1}{2}\vec{EB} + \frac{1}{2}\vec{ED}) \cdot (\vec{EA} - \frac{1}{2}\vec{EB} + \frac{1}{2}\vec{ED}) =$

$= |\vec{EA}|^2 + \frac{1}{4}|\vec{EB}|^2 + \frac{1}{4}|\vec{ED}|^2 = 1 + \frac{1}{4} \cdot 3 + \frac{1}{4} \cdot 3 = 1 + \frac{3}{2} = \frac{5}{2}$

$|\vec{MN}| = \sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{2}$