Vektorski proizvod

<u>Pr</u> Ako je $\vec{a} = (3, -1, 2)$ i $\vec{b} = (1, 2, -1)$ naći : a) $\vec{a} \times \vec{b}$, b) $(2\vec{a} + \vec{b}) \times \vec{b}$

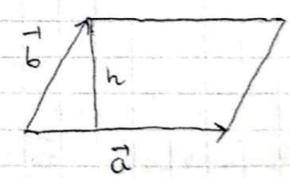
c) $(2\vec{a} - \vec{b}) \times (2\vec{a} + \vec{b})$

a) $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} \cdot \vec{j} + \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \vec{k} =$$

$$= (1 - 4)\vec{i} - (-3 - 2) \cdot \vec{j} + (6 - (-1))\vec{k} = -3\vec{i} + 5\vec{j} + 7\vec{k}$$

$$\vec{a} \times \vec{b} = (-3, 5, 7)$$

P - površina paralelograma nad $\vec{a}$ i $\vec{b}$



$$P = |\vec{a} \times \vec{b}|$$

$$P = \sqrt{9 + 25 + 49} = \sqrt{83}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \sphericalangle(\vec{a}, \vec{b})$$

$$\left.\begin{array}{l} P = |\vec{a} \times \vec{b}| \\ P = |\vec{a}| h \end{array}\right\} \quad h = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$$

$$h = \frac{\sqrt{83}}{\sqrt{9+1+4}} = \frac{\sqrt{83}}{\sqrt{14}}$$

b) $(2\vec{a}+\vec{b})\times\vec{b} = 2\vec{a}\times\vec{b} + \underbrace{\vec{b}\times\vec{b}}_{\vec{0}} = 2\vec{a}\times\vec{b} = 2(-3,5,7) = (-6,10,14)$

c) $(2\vec{a}-\vec{b})\times(2\vec{a}+\vec{b}) = 4\underbrace{\vec{a}\times\vec{a}}_{\vec{0}} + 2\vec{a}\times\vec{b} - \vec{b}\times(2\vec{a}) - \underbrace{\vec{b}\times\vec{b}}_{\vec{0}} = 2\vec{a}\times\vec{b} + 2\vec{a}\times\vec{b} =$

$= 4\vec{a}\times\vec{b} = 4(-3,5,7) = (-12,20,28)$

2. Za koju vrijednost parametra k su vektori $\vec{p} = k\vec{a} - 5\vec{b}$ i $\vec{q} = 3\vec{a} - \vec{b}$ kolinearni ako vektori $\vec{a}$ i $\vec{b}$ nisu kolinearni.

$\vec{p}$ i $\vec{q}$ kolinearni

$\vec{p}\times\vec{q} = \vec{0}$

$(k\vec{a}-5\vec{b})\times(3\vec{a}-\vec{b}) = \vec{0}$

$3k\underbrace{\vec{a}\times\vec{a}}_{\vec{0}} - k\vec{a}\times\vec{b} - 15\vec{b}\times\vec{a} + 5\underbrace{\vec{b}\times\vec{b}}_{\vec{0}} = \vec{0}$

$-k\vec{a}\times\vec{b} + 15\vec{a}\times\vec{b} = \vec{0}$

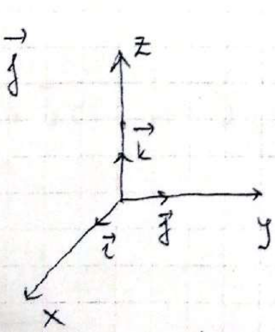
$(15-k)\underbrace{\vec{a}\times\vec{b}}_{\neq\vec{0}} = \vec{0} \Rightarrow \boxed{k = 15}$

$\vec{i}\quad \vec{j}\quad \vec{k}\quad \vec{i}\quad \vec{j}$

$\vec{i}\times\vec{i} = \vec{0}$

$\vec{j}\times\vec{j} = \vec{0}$

$\vec{k}\times\vec{k} = \vec{0}$



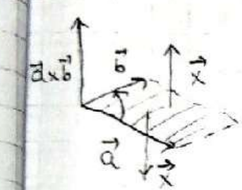
$\vec{i}\times\vec{j} = \vec{k}$

$\vec{j}\times\vec{k} = \vec{i}$

$\vec{k}\times\vec{i} = \vec{j}$

4. Vektor $\vec{x}$ je ortogonalan na vektore $\vec{a}=(4,-2,-3)$; $\vec{b}=(0,1,3)$, a sa Oy osom obrazuje tupi ugao. Naći njegove koordinate ako je $|\vec{x}|=26$

$$\left.\begin{array}{l}\vec{x}\perp\vec{a}\\ \vec{x}\perp\vec{b}\end{array}\right\} \Rightarrow \vec{x}=\lambda(\vec{a}\times\vec{b})$$



$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -2 & -3\\ 0 & 1 & 3\end{vmatrix}=\begin{vmatrix}-2 & -3\\ 1 & 3\end{vmatrix}\vec{i}-\begin{vmatrix}4 & -3\\ 0 & 3\end{vmatrix}\vec{j}+\begin{vmatrix}4 & -2\\ 0 & 1\end{vmatrix}\vec{k}=$$

$$=-3\vec{i}-12\vec{j}+4\vec{k}$$

$$\vec{a}\times\vec{b}=(-3,-12,4)$$

$$\vec{x}=(-3\lambda,-12\lambda,4\lambda)$$

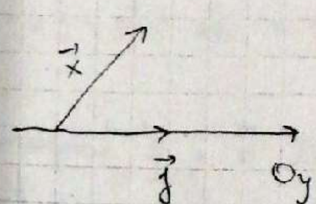
$$|\vec{x}|=26 \Rightarrow \sqrt{9\lambda^2+144\lambda^2+16\lambda^2}=26$$

$$\sqrt{169\lambda^2}=26$$

$$|13\lambda|=26$$

$$13|\lambda|=26$$

$$|\lambda|=2$$



$$\cos\alpha=\frac{\vec{x}\cdot\vec{j}}{|\vec{x}||\vec{j}|}$$

$\alpha=\measuredangle(\vec{x},O_y)$

$\alpha=\measuredangle(\vec{x},\vec{j})$

$\vec{j}=(0,1,0)$

$\vec{i}=(1,0,0)$

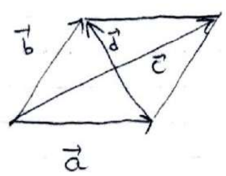
$$\left.\begin{array}{l}\cos\alpha=\dfrac{-12\lambda}{26\cdot 1}\\ \alpha\text{ - tup ugao}\\ \cos\alpha<0\end{array}\right\} \Rightarrow \underbrace{\lambda>0 \;\Rightarrow\; |\lambda|=2}$$

$$\Downarrow$$

$$\lambda=2$$

$$\boxed{\vec{x}=(-6,-24,8)}$$

5. Odrediti površinu paralelograma ako su vektori dijagonala $\vec{c} = 3$... i $\vec{d} = \vec{m} - \vec{n}$, $\measuredangle(\vec{m}, \vec{n}) = \frac{\pi}{6}$, a $\vec{m}$ i $\vec{n}$ su jedinični vektori



$$\vec{a} + \vec{b} = \vec{c}$$

$$\underline{\vec{b} - \vec{a} = \vec{d}}$$

$$2\vec{b} = \vec{c} + \vec{d}$$

$$\vec{b} = \frac{1}{2}(\vec{c} + \vec{d})$$

$$\vec{b} = \frac{1}{2}(3\vec{m} + 3\vec{n} + \vec{m} - \vec{n}) = \underline{\underline{2\vec{m} + \vec{n}}}$$

$$\vec{a} = \vec{c} - \vec{b}$$

$$\vec{a} = \vec{c} - \frac{1}{2}\vec{c} - \frac{1}{2}\vec{d} = \frac{1}{2}(\ldots$$

$$= \frac{1}{2}(3\vec{m} + 3\vec{n} - \vec{m} + \vec{n}) =$$

$$= \underline{\underline{\vec{m} + 2\vec{n}}}$$

$$P = |\vec{a} \times \vec{b}| = |(\vec{m} + 2\vec{n}) \times (2\vec{m} + \vec{n})| = |\vec{m} \times \vec{n} + 4\vec{n} \times \vec{m}| = |\vec{m} \times \vec{n} - 4\vec{m} \times \vec{n}| =$$

$$= |-3\vec{m} \times \vec{n}| = |-3| \cdot |\vec{m} \times \vec{n}| = 3|\vec{m}||\vec{n}| \sin\frac{\pi}{6} = 3 \cdot 1 \cdot 1 \cdot \frac{1}{2} = \frac{3}{2}$$

(*) Odrediti sinus ugla između dijagonala paralelograma konstrus... nad vektorima $\vec{a} = 2\vec{m} + \vec{n} - \vec{p}$, $\vec{b} = \vec{m} - 3\vec{n} + \vec{p}$ gdje je $\{\vec{m}, \vec{n}, \ldots$ ortonomirana baza

12 DATI SU VEKTORI. $\vec{a}=(-2,-6,1)$ $\vec{b}=(1,2,0)$
$\vec{c}=(2,-1,0)$

a) ODREDITI ONAJ OD JEDINIČNIH VEKTORA ORTOGONALNIH NA VEKTORE $\vec{a}$ i $\vec{b}$ KOJI SA VEKTOROM $\vec{c}$ OBRAZUJE OŠTAR UGAO.

b) U SMJERU TOG JEDINIČNOG VEKTORA ODREDITI VEKTOR $\vec{d}$ TAKO DA ZAPREMINA PARALELOPIPEDA NAD VEKTORIMA $\vec{a}, \vec{b}$ i $\vec{d}$ BUDE 18.

R a) $|\vec{x}|=1$ $\quad \vec{x}\perp\vec{a}$ $\quad \vec{x}\perp\vec{b}$

$$\vec{x}=\lambda(\vec{a}\times\vec{b})$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & -6 & 1\\ 1 & 2 & 0\end{vmatrix}=(-2)\vec{i}-(-1)\vec{j}+2\cdot\vec{k}$$

$$\vec{a}\times\vec{b}=(-2,1,2)$$

$|\vec{x}|=1 \Rightarrow \sqrt{9\lambda^2}=1$

$|3\lambda|=1$

$3|\lambda|=1$

$|\lambda|=1/3$

$\alpha=\measuredangle(\vec{x},\vec{c})$

$$\cos\alpha=\frac{\vec{x}\cdot\vec{c}}{|\vec{x}||\vec{c}|}$$

$$\cos\alpha=\frac{-4\lambda-\lambda}{\sqrt{4+1+0}}$$

α OŠTAR UGAO

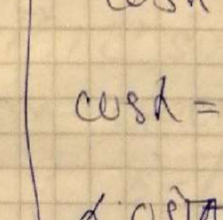
$\alpha>0 \quad \lambda<0$

$\lambda=-\frac{1}{3}$

$$\vec{x}=\left(\frac{2}{3},-\frac{1}{3},-\frac{2}{3}\right)$$

b) $\vec{d}=|\vec{d}|\cdot\vec{d}_0$

$\vec{d}=|\vec{d}|\cdot x$



$\vec{d}$ KOLINEARAN SA $\vec{x}$

$\vec{d}$ ORTOGONALAN NA $\vec{a}$ i $\vec{b}$

VISINA PARALELOPIPEDA je $H=|\vec{d}|$

$V = B \cdot H$

$V = |\vec{a} \times \vec{b}| \cdot |\vec{d}|$

$18 = 3 \cdot |\vec{d}|$

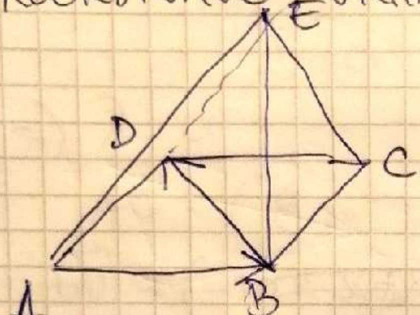
$6 = |\vec{d}|$

$\vec{d} = 6 \cdot \vec{x}$

$\vec{d} = 6 \cdot \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

$\vec{d} = (4, -2, -4)$

1. OSNOVA PIRAMIDE ABCDE ZAPREMINE $V=15$ JE PARALELOGRAM ABCD ČIJA JE DIJAGONALA $\vec{BD}=(1,1,0)$ I STRANICA $\vec{AB}=(3,2,1)$. PODNOŽJE VISINE PIRAMIDE SE POKLAPA SA TJEMENOM B, A TJEME D SA KOORDINATNIM POČETKOM. ODREDITI KOORDINATE VRHA E.



$\vec{BD}=(1,1,0)$ $\quad D=(0,0,0)$

$\vec{AB}=(3,2,1)$ $\quad A=(x,y,z)$

$\vec{AD}=\vec{AB}+\vec{BD}$ $\quad \vec{AD}=(-x,-y,z)$

$\vec{AD}=(4,3,1)$

$\left.\begin{array}{l} x=-4 \\ y=-3 \\ z=-1 \end{array}\right\} A=(-4,-3,-1)$

$V=\frac{1}{3}\cdot B\cdot H$ $\quad H=|\vec{BE}|,\ B=|\vec{AB}\times\vec{AD}|$

$V=\frac{1}{3}|\vec{AB}\times\vec{AD}|\cdot|\vec{BE}|$

$|\vec{BE}|=\frac{45}{|\vec{AB}\times\vec{AD}|}$

$\vec{AB}\times\vec{AD}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}=(-1)\vec{i}-(-1)\vec{j}+1\cdot\vec{k}$

$\vec{AB}\times\vec{AD}=(-1,1,1)$

$|\vec{BE}|=\frac{45}{\sqrt{3}}=15\sqrt{3}$

$\vec{BE}=|\vec{BE}|\cdot\vec{b_0}$, $\vec{b_0}$ JEDINIČNI VEKTOR VEKTORA $\vec{BE}$

$\left.\begin{array}{l} \vec{BE}\perp\vec{AB} \\ \vec{BE}\perp\vec{AD} \end{array}\right\} \Rightarrow \vec{BE}=\lambda(\vec{AB}\times\vec{AD})$

KAKO SU $\vec{BE}$ I $(\vec{AB}\times\vec{AD})$ ISTOG SMJERA, TO SU NJIHOVI JEDINIČNI VEKTORI JEDNAKI.

$\vec{b_0}=\frac{1}{|\vec{AB}\times\vec{AD}|}\cdot(\vec{AB}\times\vec{AD})=\frac{1}{\sqrt{3}}(-1,1,1)$

$\vec{BE}=15\cancel{\sqrt{3}}\cdot\frac{1}{\cancel{\sqrt{3}}}(-1,1,1)=(-15,15,15)$

$\vec{AB}=(3,2,1)$

$A(-4,-3,-1)$

$B(x,y,z)$

$\vec{AB}(x+4,y+3,z+$

$B(-1,-1,0)$ $\quad x=-1,\ y=-1,\ z=0$

$E(x, y, z)$

$\overrightarrow{BE} = (x+1, y+1, z)$

$x = -16$

$y = 14$

$z = 15$

$E(-16, 14, 15)$