Mješoviti proizvod

$(\vec{a} \times \vec{b}) \cdot \vec{c}$

Odrediti parametar t iz uslova da su vektori $\vec{a}, \vec{b}$ i $\vec{c}$ komplanarni ako je $\vec{a} = (\ln(t-2), -2, 6)$, $\vec{b} = (t, -2, 5)$ i $\vec{c} = (0, -1, 3)$

$$\begin{vmatrix} \ln(t-2) & -2 & 6 \\ t & -2 & 5 \\ 0 & -1 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} -2 & 5 \\ -1 & 3 \end{vmatrix} \ln(t-2) - \begin{vmatrix} t & 5 \\ 0 & 3 \end{vmatrix} (-2) + \begin{vmatrix} t & -2 \\ 0 & -1 \end{vmatrix} \cdot 6 = 0$$

$$-\ln(t-2) + \cancel{6t} - \cancel{6t} = 0$$

$$-\ln(t-2) = 0$$

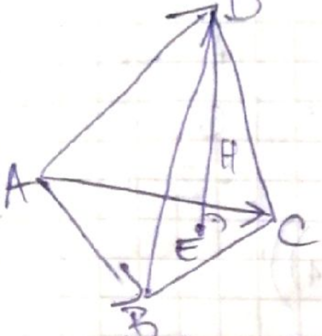
$$\ln(t-2) = 0$$

$$t-2 = 1$$

$$t = 3$$

MJEŠOVITI I VEKTORSKO-VEKTORSKI PROIZVODI

1\. TJEMENA TETRAEDRA SU $A(1,1,1)$ $B(0,2,1)$ $C(-2,-2,3)$ $D(3,4,-3)$. ODREDI DUŽINU VISINE IZ TJEMENA D I VEKTOR VISINE.



$$V_t = \frac{1}{6}\left|(\vec{AB}\times\vec{AC})\cdot\vec{AD}\right|$$

$$V_t = \frac{1}{3}\cdot B\cdot H = \frac{1}{3}\cdot\frac{1}{2}\left|\vec{AB}\times\vec{AC}\right|\cdot H$$

$$V_t = \frac{1}{6}\left|\vec{AB}\times\vec{AC}\right|\cdot H$$

$$H = \frac{\left|(\vec{AB}\times\vec{AC})\cdot\vec{AD}\right|}{\left|\vec{AB}\times\vec{AC}\right|}$$

$\vec{AB} = (-1,1,0)$

$\vec{AC} = (-3,-3,2)$

$\vec{AD} = (2,3,-4)$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -3 & -3 & 2 \end{vmatrix} = (2,2,6)$$

$$(\vec{AB}\times\vec{AC})\cdot\vec{AD} = \begin{vmatrix} -1 & 1 & 0 \\ -3 & -3 & 2 \\ 2 & 3 & -4 \end{vmatrix} = -14$$

$$H = \frac{|-14|}{\sqrt{4+4+36}} = \frac{7\sqrt{11}}{11}$$

$\vec{ED}$ - VEKTOR VISINE

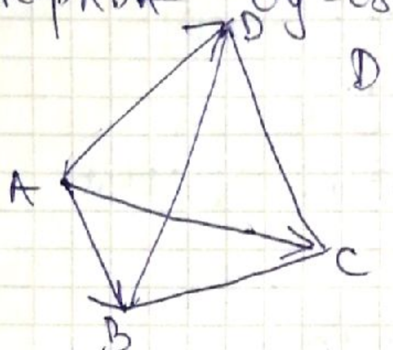
$\vec{ED} = |\vec{ED}|\cdot\vec{e_0}$ $\quad$ $\vec{e_0}$, JEDINIČNI VEKTOR, VEKTORA $\vec{ED}$

$$\left.\begin{array}{l} \vec{ED}\perp\vec{AB} \\ \vec{ED}\perp\vec{AC} \end{array}\right\} \Rightarrow \vec{ED} = \lambda(\vec{AB}\times\vec{AC})$$

$$\vec{e_0} = \frac{1}{\left|\vec{AB}\times\vec{AC}\right|}\cdot\vec{AB}\times\vec{AC} = \frac{1}{2\sqrt{11}}(2,2,6) = \frac{\sqrt{11}}{11}(1,1,3) =$$

$$\vec{ED} = \frac{7\sqrt{11}}{11} \cdot \frac{\sqrt{11}}{11}(1,1,3) = \left(\frac{7}{11}, \frac{7}{11}, \frac{21}{11}\right)$$

2. ZAPREMINA TETRAEDRA JE 5. TRI NJEGOVA TJEMENA SU TAČKE $A(2,1,-1)$, $B(3,0,1)$ i $C(2,-1,3)$. NAĆI KOORDINATE ČETVRTOG TJEMENA D AKO ONO PRIPADA Oy-OSI.



$D(0,b,0)$

$$V = \frac{1}{6}\left|(\vec{AB} \times \vec{AC}) \cdot \vec{AD}\right|$$

$V = 5$

$$|\vec{AB} \times \vec{AC}| \cdot AD = 30$$

$\vec{AB} = (1,-1,2)$

$\vec{AC} = (0,-2,4)$

$\vec{AD} = (-2, b-1, 1)$

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & b-1 & 1 \end{vmatrix} = -2+8-(8+4(b-1)) = 2-4b$$

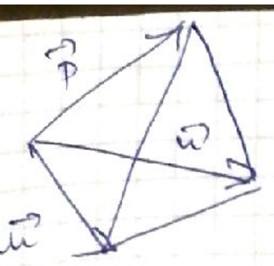
$|2-4b| = 30$

$|1-2b| = 15$

1° $1-2b = 15$, $2b = -14$, $b = -7$, $D(0,-7,0)$

2° $1-2b = -15$, $b = 8$, $D(0,8,0)$

3. DATI SU VEKTORI $\vec{m} = (1,1,1)$ i $\vec{n} = (2,1,1)$. VEKTOR $\vec{p}$ SA OSOM Ox ZAKLAPA UGAO $\pi/4$ SA OSOM Oy UGAO $\pi/3$, A SA OSOM Oz TUP UGAO. ZAPREMINA TETRAEDRA OBRAZOVANA NAD VEKTORIMA $\vec{m}, \vec{n}$ i $\vec{p}$ JE 2. ODREDITI $\vec{p}$ KOORDINAT



$\alpha = \sphericalangle(\vec{p}, Ox) = \pi/4$

$\beta = \sphericalangle(\vec{p}, Oy) = \pi/3$

$\gamma = \sphericalangle(\vec{p}, Oz)$ - TUP UGAO $\Rightarrow \cos\gamma < 0$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$$

$$\cos\gamma = \pm\frac{1}{2} \qquad \cos\gamma = -\frac{1}{2} \qquad \gamma = \frac{2\pi}{3}$$

$\vec{p_0}$ JEDINIČNI VEKTOR VEKTORA $\vec{p}$

$$\vec{p_0} = (\cos\alpha, \cos\beta, \cos\gamma)$$

$$\vec{p_0} = \left(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$\vec{p} = |\vec{p}| \cdot \vec{p_0}$$

$$V = 2$$

$$V = \frac{1}{6}\left|(\vec{m}\times\vec{n})\cdot\vec{p}\right|$$

$$|\vec{p}| = \lambda > 0$$

$$\vec{p} = \left(\frac{\sqrt{2}}{2}\lambda, \frac{\lambda}{2}, -\frac{\lambda}{2}\right) \qquad \left|(\vec{m}\times\vec{n})\cdot\vec{p}\right| = 12$$

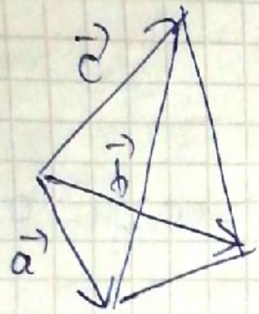
$$(\vec{m}\times\vec{n})\cdot\vec{p} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ \frac{\sqrt{2}\lambda}{2} & \frac{\lambda}{2} & -\frac{\lambda}{2} \end{vmatrix} = \lambda$$

$$|\lambda| = 12$$

$$\lambda = 12$$

$$\vec{p} = (6\sqrt{2}, 6, -6)$$

4. DATI SU VEKTORI $\vec{m}, \vec{n}, \vec{p}$ ZA KOJE VAŽI $|\vec{m}| = 1$, $|\vec{n}| = 1$, $|\vec{p}| = 2$, $\sphericalangle(\vec{m}, \vec{n}) = \pi/6$, $\sphericalangle(\vec{m}, \vec{p}) = \sphericalangle(\vec{n}, \vec{p}) = \pi/2$. AKO JE $\vec{a} = 2\vec{m} + \vec{n}$, $\vec{b} = 2\vec{n} - \vec{p}$ i $\vec{c} = 2\vec{p} + \vec{n}$, IZRAČUNATI ZAPREMINU TETRAEDRA NAD VEKTORIMA $\vec{a}, \vec{b}, \vec{c}$



$$V = \frac{1}{6}\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$$

$$\vec{a}\times\vec{b} = (2\vec{m}+\vec{n})\times(2\vec{n}-\vec{p})$$
$$= 4\vec{m}\times\vec{n} - 2\vec{m}\times\vec{p} + 2\underbrace{\vec{n}\times\vec{n}}_{\vec{0}} - \vec{n}\times\vec{p}$$
$$= 4\vec{m}\times\vec{n} - 2\vec{m}\times\vec{p} - \vec{n}\times\vec{p}$$
$$(\vec{a}\times\vec{b})\cdot\vec{c} = (4\vec{m}\times\vec{n} - 2\vec{m}\times\vec{p} - \vec{n}\times\vec{p})\cdot(2\vec{p}+\vec{n})$$
$$= 8(\vec{m}\times\vec{n})\cdot\vec{p} - 2(\vec{m}\times\vec{p})\vec{n}$$
$$= 8(\vec{m}\times\vec{n})\cdot\vec{p} - 2(\vec{p}\times\vec{n})\vec{m}$$
$$= 8(\vec{m}\times\vec{n})\cdot\vec{p} + 2(\vec{m}\times\vec{n})\vec{p}$$
$$= 10(\vec{m}\times\vec{n})\vec{p}$$

$$V = \frac{1}{6}\left|10(\vec{m}\times\vec{n})\vec{p}\right| = \frac{10}{6}\left|(\vec{m}\times\vec{n})\vec{p}\right|$$
$$= \frac{5}{3}\left|\,|\vec{m}\times\vec{n}|\cdot|\vec{p}|\cdot\cos\measuredangle(\vec{m}\times\vec{n},\vec{p})\right|$$
$$= \frac{5}{3}\left|\,|\vec{m}||\vec{n}|\cdot\sin\pi/6\right|\left|\cos\measuredangle(\vec{m}\times\vec{n},\vec{p})\right||\vec{p}|$$

$\left.\begin{array}{l}\vec{p}\perp\vec{m}\\ \vec{p}\perp\vec{n}\end{array}\right\} \Rightarrow \vec{p} = \lambda(\vec{m}\times\vec{n})$

$$= \frac{5}{3}\,1\cdot 1\cdot\frac{1}{\cancel{2}}\cdot\cancel{2}\cdot 1 = \frac{5}{3}$$

$\Rightarrow \measuredangle(\vec{m}\times\vec{n},\vec{p}) =$ [illegible]

$\Rightarrow \cos\measuredangle = 1 \lor -1$

5. NAĆI $\vec{x}$ AKO JE $\vec{x}\cdot\vec{a} = m$, $\vec{x}\cdot\vec{b} = n$ i $\vec{x}\cdot\vec{c} = p$ GDJE SU $\vec{a},\vec{b},\vec{c}$ DATI NEKOMPLANARNI VEKTORI, a m, n, i p DATI SKALARNI.

$$\vec{x}\cdot\vec{a} = m \quad /\cdot n$$
$$\vec{x}\cdot\vec{b} = n \quad /(-m)$$
$$n\,\vec{x}\cdot\vec{a} - m\,\vec{x}\cdot\vec{b} = 0$$
$$\vec{x}\cdot(n\vec{a} - m\vec{b}) = 0 \Rightarrow \vec{x}\perp n\vec{a} - m\vec{b} \quad (1)$$

$\vec{x}\cdot\vec{b} = n \quad /\cdot p$

$\vec{x}\cdot\vec{c} = p \quad /(-n)$

$\vec{x}(p\vec{b} - n\vec{c}) = 0 \Rightarrow \vec{x} \perp (p\vec{b} - n\vec{c}) \quad (2)$

IZ (1) I (2) SLIJEDI

$\vec{x} = \lambda\left((n\vec{a} - m\vec{b}) \times (p\vec{b} - n\vec{c})\right)$

$\vec{x} = \lambda\left(np\, \vec{a}\times\vec{b} - n^2 \vec{a}\times\vec{c} + m\cdot n\, \vec{b}\times\vec{c}\right)$

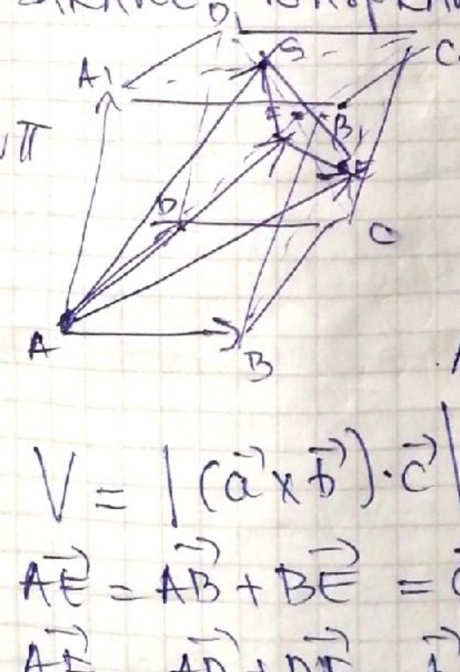
$\vec{x} = \lambda n\left(p\, \vec{a}\times\vec{b} - n\, \vec{a}\times\vec{c} + m\, \vec{b}\times\vec{c}\right) \quad /\cdot\vec{c}$

$p = \lambda\cdot n\left(p(\vec{a}\times\vec{b})\cdot\vec{c}\right)$

$\lambda = \frac{1}{n\underbrace{(\vec{a}\times\vec{b})\vec{c}}_{\neq 0}}$ JER SU $\vec{a}, \vec{b}$ I $\vec{c}$ NEKOMPLANARNI VEKTORI.

$\vec{x} = \frac{1}{(\vec{a}\times\vec{b})\vec{c}}\left(p\, \vec{a}\times\vec{b} - n\, \vec{a}\times\vec{c} + m\, \vec{b}\times\vec{c}\right)$

6. DAT SE PARALELOPIPED. ODREDITI ODNOS ZAPREMINE PARALELOPIDEA I ZAPREMINE TETRAEDRA ČIJA SU TJEMENA JEDNO TJEME PARALELOPIPEDA I CENTRI TRI STRANE NASPRAM TOG TJEMENA.



E, F, G CENTRI STRANE BCC_1B_1, DCC_1D_1 I $A_1B_1C_1D_1$

$\vec{AB} = \vec{a}$

$\vec{AD} = \vec{b}$

$\vec{AA_1} = \vec{c}$

$V = \left|(\vec{a}\times\vec{b})\cdot\vec{c}\right| \qquad V_t = \frac{1}{6}\left|(\vec{AE}\times\vec{AF})\vec{AG}\right|$

$\vec{AE} = \vec{AB} + \vec{BE} = \vec{a} + \frac{1}{2}(\vec{b}+\vec{c}) = \vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}$

$\vec{AF} = \vec{AD} + \vec{DF} = \vec{b} + \frac{1}{2}(\vec{a}+\vec{c}) = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{c} + \vec{b}$

$\vec{AG} = \vec{AA_1} + \vec{A_1G} = \vec{c} + \frac{1}{2}(\vec{a}+\vec{b}) = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} + \vec{c}$

$$\vec{AE} \times \vec{AF} = \left(\vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}\right) \times \left(\frac{1}{2}\vec{a} + \vec{b} + \frac{1}{2}\vec{c}\right)$$

$$= \vec{a} \times \vec{b} + \frac{1}{2}\vec{a} \times \vec{c} + \frac{1}{4}\vec{b} \times \vec{a} + \frac{1}{4}\vec{b} \times \vec{c} + \frac{1}{4}\vec{c} \times \vec{a} + \frac{1}{2}\vec{c} \times \vec{b}$$

$$= \frac{3}{4}\vec{a} \times \vec{b} + \frac{1}{4}\vec{a} \times \vec{c} - \frac{1}{4}\vec{b} \times \vec{c}$$

$$\left(\vec{AE} \times \vec{AF}\right) \cdot \vec{AD} = \left(\frac{3}{4}\vec{a} \times \vec{b} + \frac{1}{4}\vec{a} \times \vec{c} - \frac{1}{4}\vec{b} \times \vec{c}\right)\left(\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}\right.$$

$$= \frac{3}{4}(\vec{a} \times \vec{b}) \cdot \vec{c} + \frac{1}{8}(\vec{a} \times \vec{c}) \cdot \vec{b} - \frac{1}{8}(\vec{b} \times \vec{c})\vec{a}$$

$$= \frac{3}{4}(\vec{a} \times \vec{b})\vec{c} + \frac{1}{8}(\vec{c} \times \vec{b})\vec{a} - \frac{1}{8}(\vec{c} \times \vec{a})\vec{b}$$

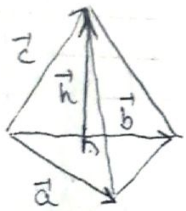
$$= \frac{5}{8}(\vec{a} \times \vec{b})\vec{c} - \frac{1}{8}(\vec{a} \times \vec{b})\vec{c} =$$

$$= \frac{1}{2}(\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$V = \frac{1}{6}\left|\frac{1}{2}(\vec{a} \times \vec{b}) \cdot \vec{c}\right|$$

$$V = \frac{1}{12}\left|(\vec{a} \times \vec{b}) \cdot \vec{c}\right|$$

1. Naći vektor visine tetraedra konstruisanog nad vektorima $\vec{a}, \vec{b}$ i $\vec{c}$.



$\vec{h} = |\vec{h}| \cdot \vec{h}_0$, $\vec{h}_0$ - jedinični vektor vektora $\vec{h}$

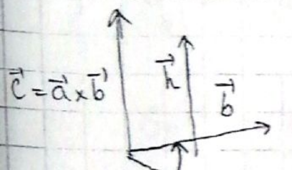
$$V_t = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

$$V_t = \frac{1}{3} B \cdot |\vec{h}| = \frac{1}{3} \cdot \frac{1}{2} |\vec{a} \times \vec{b}| \cdot |\vec{h}|$$

$$|\vec{h}| = \frac{|(\vec{a} \times \vec{b}) \cdot \vec{c}|}{|\vec{a} \times \vec{b}|}$$

$$\left.\begin{array}{l} \vec{h} \perp \vec{a} \\ \vec{h} \perp \vec{b} \end{array}\right\} \vec{h} = \lambda (\vec{a} \times \vec{b})$$

$$\vec{h}_0 = \frac{1}{|\vec{a} \times \vec{b}|} (\vec{a} \times \vec{b})$$



S obzirom na izabranu orijentaciju vektora $\vec{h}$, vektori $\vec{h}$ i $\vec{a} \times \vec{b}$ su istog smjera pa su im jedinični vektori jednaki

$$\vec{h} = \frac{|(\vec{a} \times \vec{b}) \cdot \vec{c}|}{|\vec{a} \times \vec{b}|} \cdot \frac{1}{|\vec{a} \times \vec{b}|} \vec{a} \times \vec{b}$$

$$\vec{h} = \frac{|(\vec{a} \times \vec{b}) \cdot \vec{c}|}{|\vec{a} \times \vec{b}|^2} \cdot \vec{a} \times \vec{b}$$