Prava i ravan

$$l: \frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$$

$$\pi: Ax+By+Cz+D=0$$

$$l: \begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases}$$ uvrstimo u jednačinu ravni π.

$$A(x_0+mt) + B(y_0+nt) + C(z_0+pt) + D = 0$$

$$Ax_0 + By_0 + Cz_0 + D + (Am+Bn+Cp)t = 0 \quad (*)$$

Ako $(*)$ važi za svako $t \in \mathbb{R}$ tada prava l leži u ravni

1- tj. Prava l leži u ravni π akko

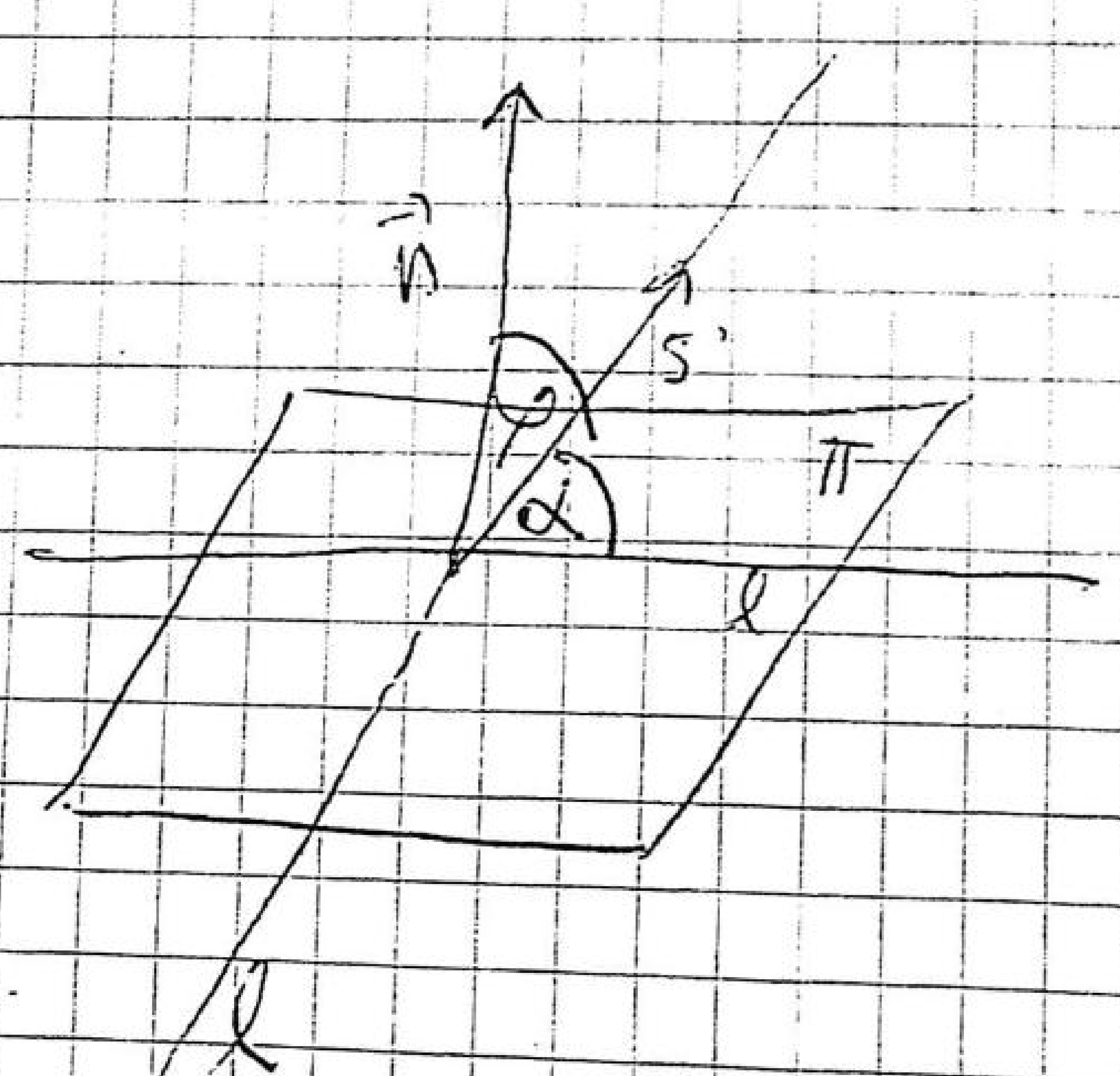
$$\begin{cases} Ax_0 + By_0 + Cz_0 + D = 0 & (1) \\ Am + Bn + Cp = 0 & (2) \end{cases}$$

$(M(x_0, y_0, z_0) \in \pi)$

$\cancel{\vec{s} \cdot \vec{n} = 0}$ $\quad \vec{s} \cdot \vec{n} = 0$

$\cancel{\vec{s} \perp \vec{n}}$ $\quad \vec{s} \perp \vec{n}$

Ako ne važi (1) a važi (2) tada je $l \parallel \pi$, ako ne važi (2) tada se prava l i ravan π sijeku.



$$\measuredangle(l, \pi) = \measuredangle(l, l')$$

l' – ortogonalna projekcija prave l na ravan π

$$\varphi + \alpha = \frac{\pi}{2}$$

$$\varphi = \measuredangle(\vec{n}, \vec{s})$$

$$\cos\varphi = \frac{\vec{n} \cdot \vec{s}}{|\vec{n}||\vec{s}|}$$

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$$\cos\left(\frac{\pi}{2}-\varphi\right) = \frac{\vec{n}\cdot\vec{s}}{|\vec{n}||\vec{s}|}$$

$$\sin\alpha = \frac{\vec{n}\cdot\vec{s}}{|\vec{n}||\vec{s}|}$$

1) Primjer: Ispitati uzajamni položaj prave i ravni

$$p: \frac{x}{-2} = \frac{y-1}{-1} = \frac{z-2}{1}$$

$$\alpha: x+y+3z-7=0$$

R|| $M(0,1,2) \in p$

$\vec{s} = (-2,-1,1)$

$\vec{n} = (1,1,3)$

2) $0+1+3\cdot 2-7=0$ $\;\;0=0$

$\underline{M\in\alpha}$

1) $\vec{s}\cdot\vec{n} = -2\cdot 1 - 1\cdot 1 + 1\cdot 3 = 0 \Rightarrow \underline{\vec{s}\perp\vec{n}}$. (prava ili lezi u ravni ili je paralelna s njom)

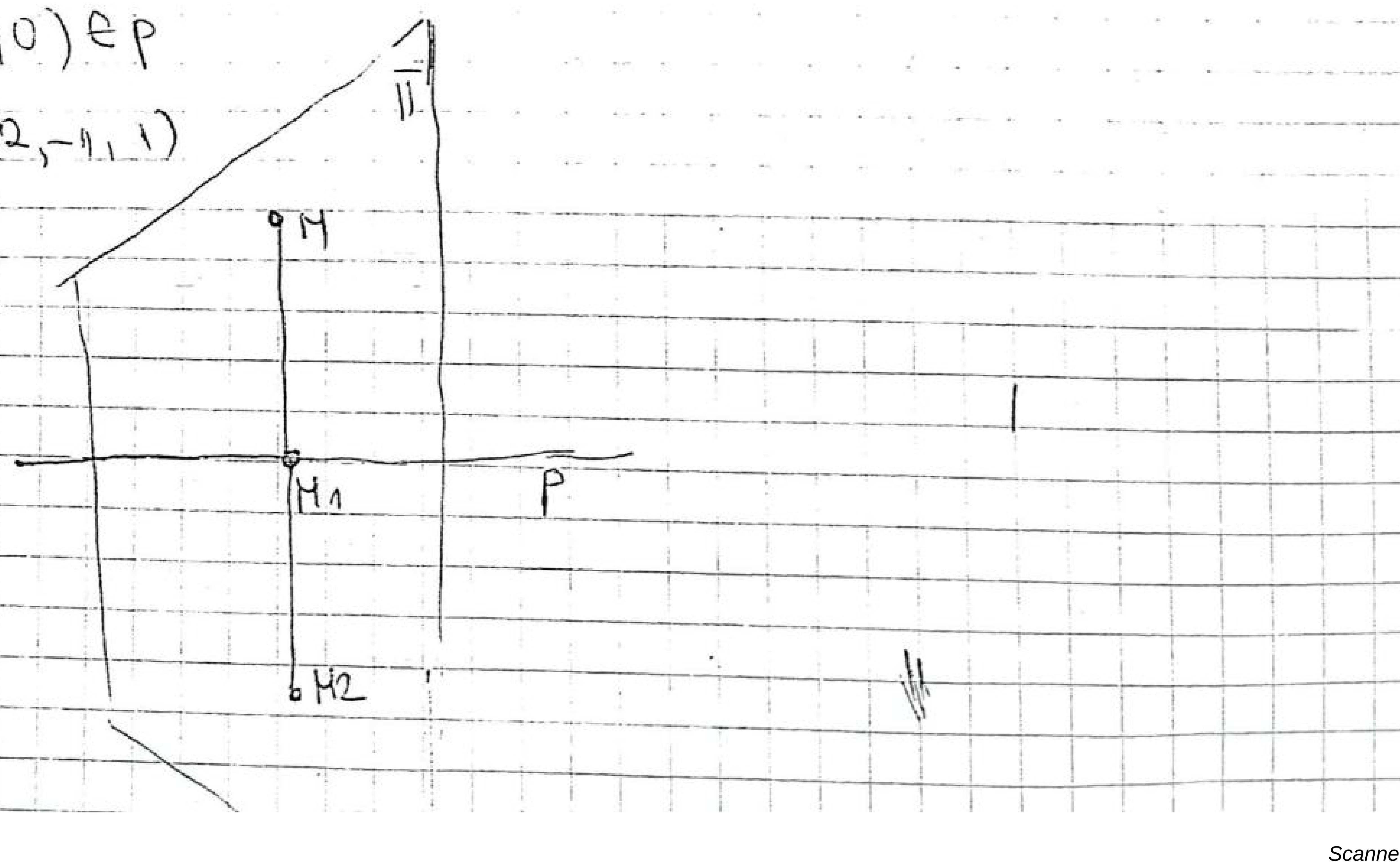
Iz 1) i 2) $\Rightarrow$ prava p leži u ravni α.

2) Naći tačku simetričnu tački $M(1,2,8)$ u odnosu na pravu

$p: \frac{x-1}{2} = \frac{y}{-1} = \frac{z}{1}$ i normalu iz M na pravu p.

R|| $A(1,0,0) \in p$

$\vec{s}_p = (2,-1,1)$



Postavljamo ravan Π koja sadrži tačku M i ortogonalan je na pravu p:

$\vec{n}_\Pi$ – njen vektor normale

$\Pi \cap p = \{M_1\}$

M_1 – ortogonalna projekcija tačke M na pravu p.

M_2 – tačka koja simetrična tački M u odnosu na pravu p

M_1 je središte duži MM_2.

$p \perp \Pi \Rightarrow \vec{n}_\Pi = \lambda \vec{s}_p$ ✓

Za $\lambda = 1$, $\vec{n}_\Pi = (2,-1,1)$

Π: $M(1,2,8)$, $\vec{n}_\Pi = (2,-1,1)$

$2(x-1) - 1(y-2) + 1 \cdot (z-8) = 0$

$\boxed{\Pi: 2x - y + z - 8 = 0}$

$p: \begin{cases} x = 1+2t \\ y = -t \\ z = t \end{cases}$ uvestimo u jednačinu ravni Π

$2(1+2t) - (-t) + t - 8 = 0$

$6t = 6$

$t = 1$

$x = 1 + 2 \cdot 1 = 3$

$y = -1$ $\quad M_1(3,-1,1)$

$z = 1$

$M(1,2,8)$ $\quad M_1(3,-1,1)$ $\quad M_2(x,y,z)$

$3 = \frac{1+x}{2} \Rightarrow x = 5$

$-1 = \frac{2+y}{2} \Rightarrow y = -4$

$1 = \frac{8+z}{2} \Rightarrow z = -6$

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$M_2(5,-4,-6)$

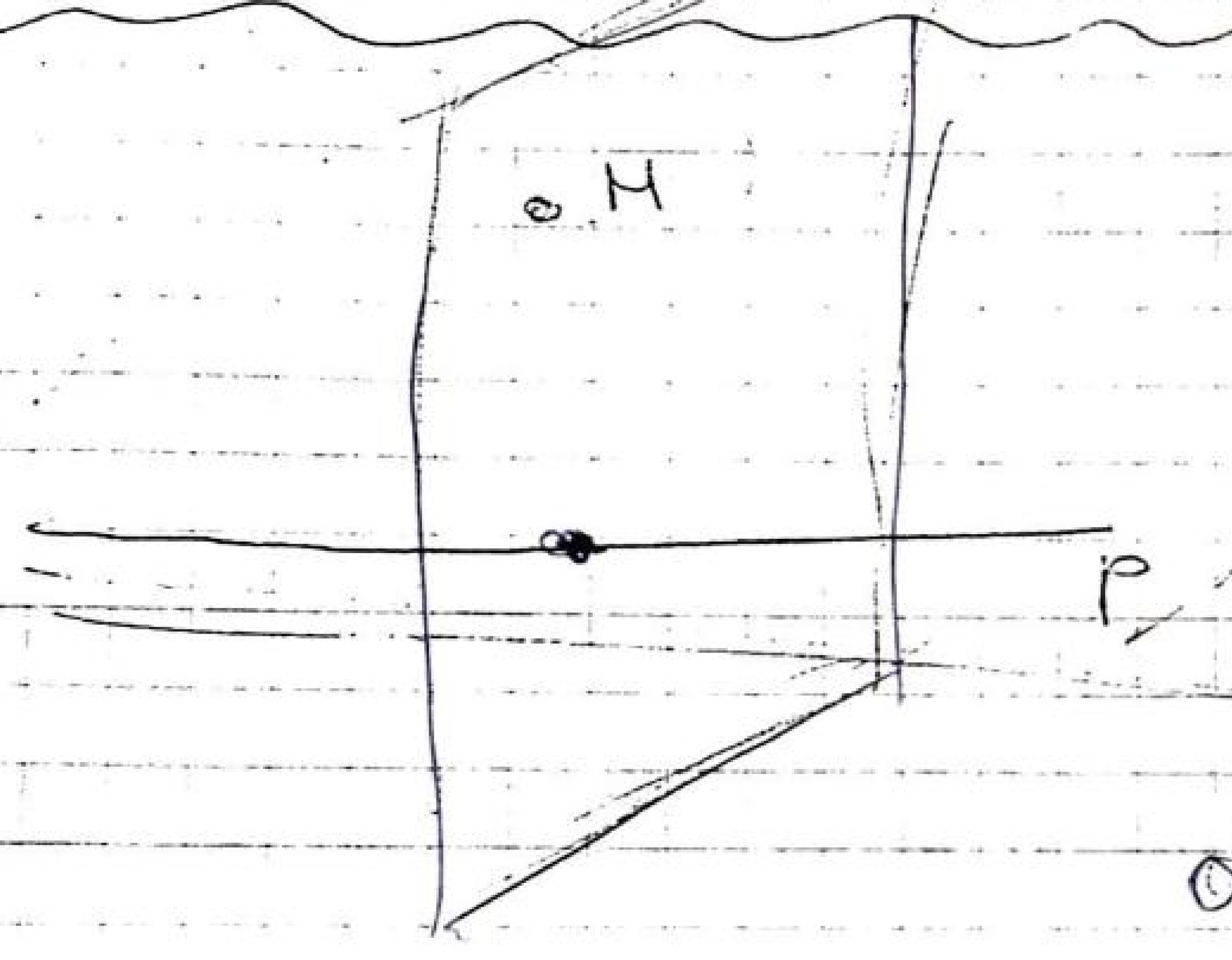
l - normala iz tacke M na pravu p

$l(M, M_1)$

$$l: \frac{x-1}{3-1} = \frac{y-2}{-1-2} = \frac{z-8}{1-8}$$

$$l: \frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-8}{-7}$$

II način za normalu iz tacke na pravu



$$\alpha \cap \beta = \{l\}$$

α - ravan koja sadrži tacku i pravu p

β - ravan koja sadrži tacku M i ortogonalna na pravu p.

$$\alpha \cap \beta = \{l\}$$

l - normala prave p iz tacke M na pravu p.

3) Naći jednačine normalne projekcije prave p: $\frac{x}{5} = \frac{y+1}{-2} = \frac{z+1}{-3}$

na ravan α: $2x - 3y + z - 4 = 0$.

R) β - ravan koja sadrži pravu p i ortogonalna na ravan α.

$\alpha \cap \beta = \{p_1\}$ p_1 ortogonalna projekcija prave p na ravan α.

$M(0,-1,-1) \in p \subset \beta$

$$\left.\begin{array}{l} p \subset \beta \Rightarrow \vec{n}_\beta \perp \vec{S}_p \\ \beta \perp \alpha \Rightarrow \vec{n}_\beta \perp \vec{n}_\alpha \end{array}\right\} \vec{n}_\beta = \lambda(\vec{S}_p \times \vec{n}_\alpha)$$

$$\vec{s_p} \times \vec{n_\gamma} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -2 & -3 \\ 2 & -3 & 1 \end{vmatrix} = -11\vec{i} - 11\vec{j} - 11\vec{k} \qquad \vec{s_p} \times \vec{n_\gamma} = (-11, -11, -11)$$

za $\lambda = \frac{1}{-11}$ $\vec{n_\beta} = (1, 1, 1)$

β: $M(0, -1, -1)$ $\vec{n_\beta} = (1, 1, 1)$

$$1(x-0) + 1(y+1) + 1(z+1) = 0$$

$$x + y + 1 + z + 1 = 0$$

β: $x + y + z + 2 = 0$

$$p_1 = \begin{cases} 2x - 3y + z - 4 = 0 \\ x + y + z + 2 = 0 \end{cases}$$ (Naći kanonski oblik jednačine)

(4) Odrediti ugao između prave $p: \frac{x-1}{2} = \frac{y+2}{-1} = \frac{z-3}{4}$ i ravni α određene tačkama $A(0,0,1)$ $B(3,1,0)$ $C(3,2,2)$

R// $M(x,y,z)$ proizvoljna tačka ravni α

$\vec{AM}, \vec{AB}, \vec{AC} \Rightarrow$ komplanarni

$\vec{AM} = (x, y, z-1)$

$\vec{AB} = (3, 1, -1)$

$\vec{AC} = (3, 2, 1)$

$$(\vec{AM} \times \vec{AB}) \cdot \vec{AC} = \begin{vmatrix} x & y & z-1 \\ 3 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 3x - 6y + 3(z-1) = 0$$

α: $x - 2y + z - 1 = 0$

$\vec{n_\alpha} = (1, -2, 1)$

$\vec{s_p} = (2, -1, 4)$

$\varphi = \measuredangle(\alpha, p)$

$$\sin\varphi = \frac{|\vec{n_\alpha} \cdot \vec{s_p}|}{|\vec{n_\alpha}||\vec{s_p}|}$$

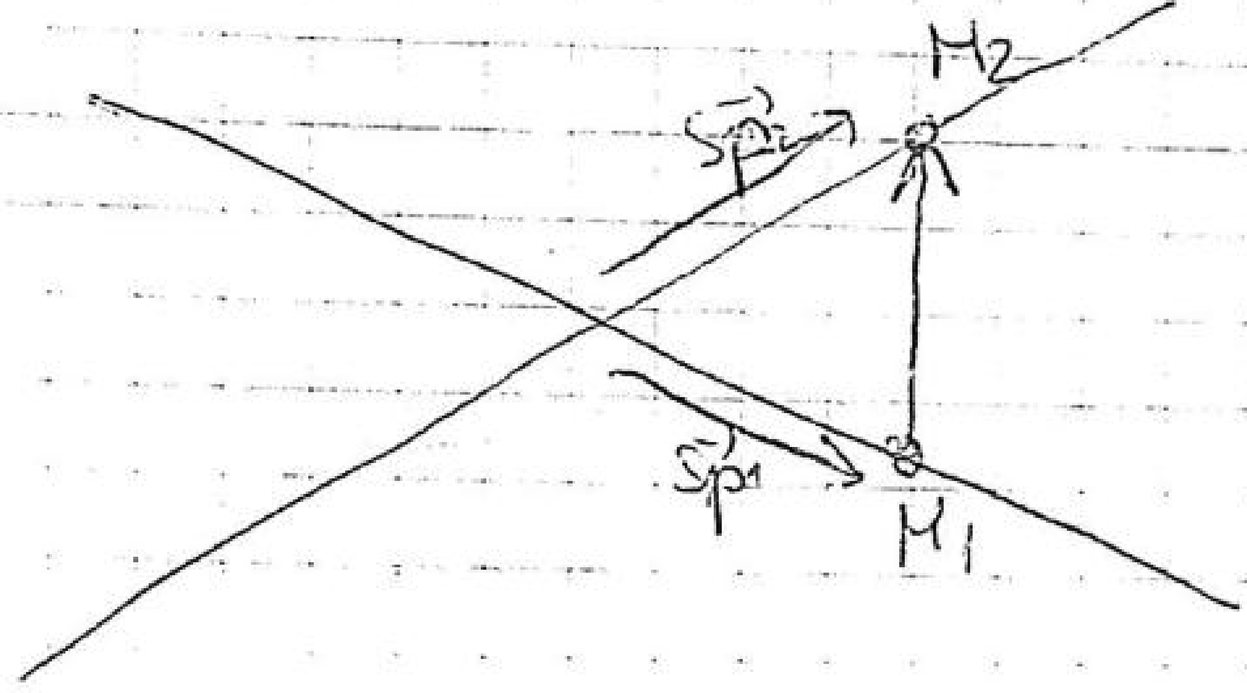
$$\sin\varphi = \frac{8}{\sqrt{6}\sqrt{21}} = \frac{8}{3\sqrt{14}} = \frac{4\sqrt{14}}{21}$$

5) Date su prave $p_1: \frac{x+2}{2} = \frac{y+2}{a} = \frac{z+2}{2}$, $p_2: \frac{x-1}{a} = \frac{y-1}{1} = \frac{z-7}{3}$

Odrediti a tako da se prave sijeku. Za veću vrijednost parametra a napisati jednačinu ravni α koja sadrži prave.

R|| $M_1(-2,-2,-2) \in p_1$

$M_2(1,1,7) \in p_2$



$\vec{sp_1} = (2,a,2)$

$\vec{sp_2} = (a,1,3)$

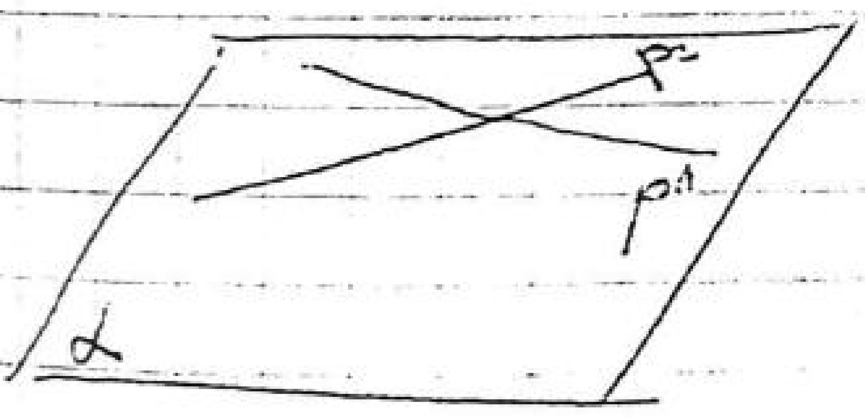
$\vec{sp_1}, \vec{sp_2}, \vec{M_1M_2}$ su komplanarni pa je:

$\vec{M_1M_2} = (3,3,9)$

$$(\vec{sp_1} \times \vec{sp_2}) \cdot \vec{M_1M_2} = \begin{vmatrix} 2 & a & 2 \\ a & 1 & 3 \\ 3 & 3 & 9 \end{vmatrix} = -9a^2 + 15a - 6 = 0$$

$$-3a^2 + 5a - 2 = 0$$

$$a_{1,2} = \frac{-5 \pm \sqrt{25-24}}{-6} = \frac{-5 \pm 1}{-6} \to a_1 = 1, \quad a_2 = \frac{2}{3}$$



$p_1 \subset \alpha \Rightarrow \vec{n_\alpha} \perp \vec{sp_1}$

$p_2 \subset \alpha \Rightarrow \vec{n_\alpha} \perp \vec{sp_2}$

$\Big\} \Rightarrow \vec{n_\alpha} = \lambda(\vec{sp_1} \times \vec{sp_2})$

$\vec{sp_1} = (2,1,2)$

$\vec{sp_2} = (1,1,3)$

$$\vec{sp_1} \times \vec{sp_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix} = \vec{i} - 4\vec{j} + 1\vec{k} \Rightarrow (1,-4,1)$$

za $\lambda = 1$ $\vec{n_\alpha} = (1,-4,1)$

α: $M_1(-2,-2,-2)$

$\vec{n_\alpha} = (1,-4,1)$

$1(x+2) - 4(y+2) + 1(z+2) = 0$

$x - 4y + z - 4 = 0$

(7)

1. Naći jednačinu ravni koja sadrži pravu $p: \frac{x}{2} = \frac{y-1}{-1} = \frac{z+1}{2}$ i čije je rastojanje od tačke $A(1,2,-1)$ iznosi $\frac{4}{3}$.

$\boxed{\text{! Svaka prava određuje pramen !}}$

Odredimo dvije ravni koje sadrže pravu p.

$$\frac{x}{2} = \frac{y-1}{-1} \qquad \frac{x}{2} = \frac{z+1}{2}$$

$$\alpha: -x-2y+2=0 \qquad \beta: x-z-1=0$$

Ravni α i β određuju pramen ravni a tražena ravan Π pripada tom pramenu.

Jednačina pramena je

$$-x-2y+2+\lambda(x-z-1)=0$$

$$(\lambda-1)x-2y-\lambda z+2-\lambda=0 \checkmark$$

$$d(A,\Pi)=\frac{4}{3} \Rightarrow \frac{|(\lambda-1)\cdot 1-2\cdot 2-\lambda(-1)+2-\lambda|}{\sqrt{(\lambda-1)^2+4+\lambda^2}} = \frac{4}{3} \checkmark$$

$A(1,2,-1)$

$$= \frac{|\lambda-3|}{\sqrt{2\lambda^2-2\lambda+5}} = \frac{4}{3} \Big/^2$$

$$= \frac{(\lambda-3)^2}{2\lambda^2-2\lambda+5} = \frac{16}{9}$$

$$= 9\lambda^2-54\lambda+81 = 32\lambda^2-32\lambda+80$$

$$\Rightarrow 23\lambda^2+22\lambda-1=0$$

$$\lambda_{1,2} = \frac{-22\pm\sqrt{484+92}}{46} = \frac{-22\pm\sqrt{576}}{46}$$

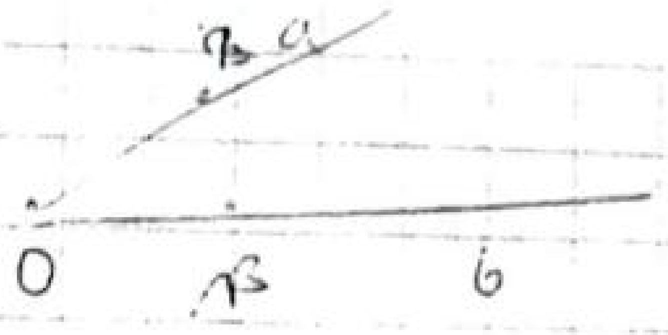
$$\Rightarrow \lambda_{1,2} = \frac{-22\pm 24}{46} \to \lambda_1=-1 \checkmark$$

$$\to \lambda_2 = \frac{1}{23} \checkmark$$

1° $-2x-2y+z+2+1=0$

$-2x-2y+z+3=0$

$2x+2y-z-3=0$



$f(0a.$

2° $\left(\frac{1}{23}-1\right)x-2y-\frac{1}{23}z+2-\frac{1}{23}=0$

$-22x-46y-z+45=0$

$22x+46y+z-45=0$ ✓

7) Naći rastojanje i zajedničku normalu mimoilaznih pravih:

$l_1: \frac{x+1}{1}=\frac{y}{1}=\frac{z-1}{2}$, $\quad l_2: \frac{x}{1}=\frac{y+1}{3}=\frac{z-2}{4}$

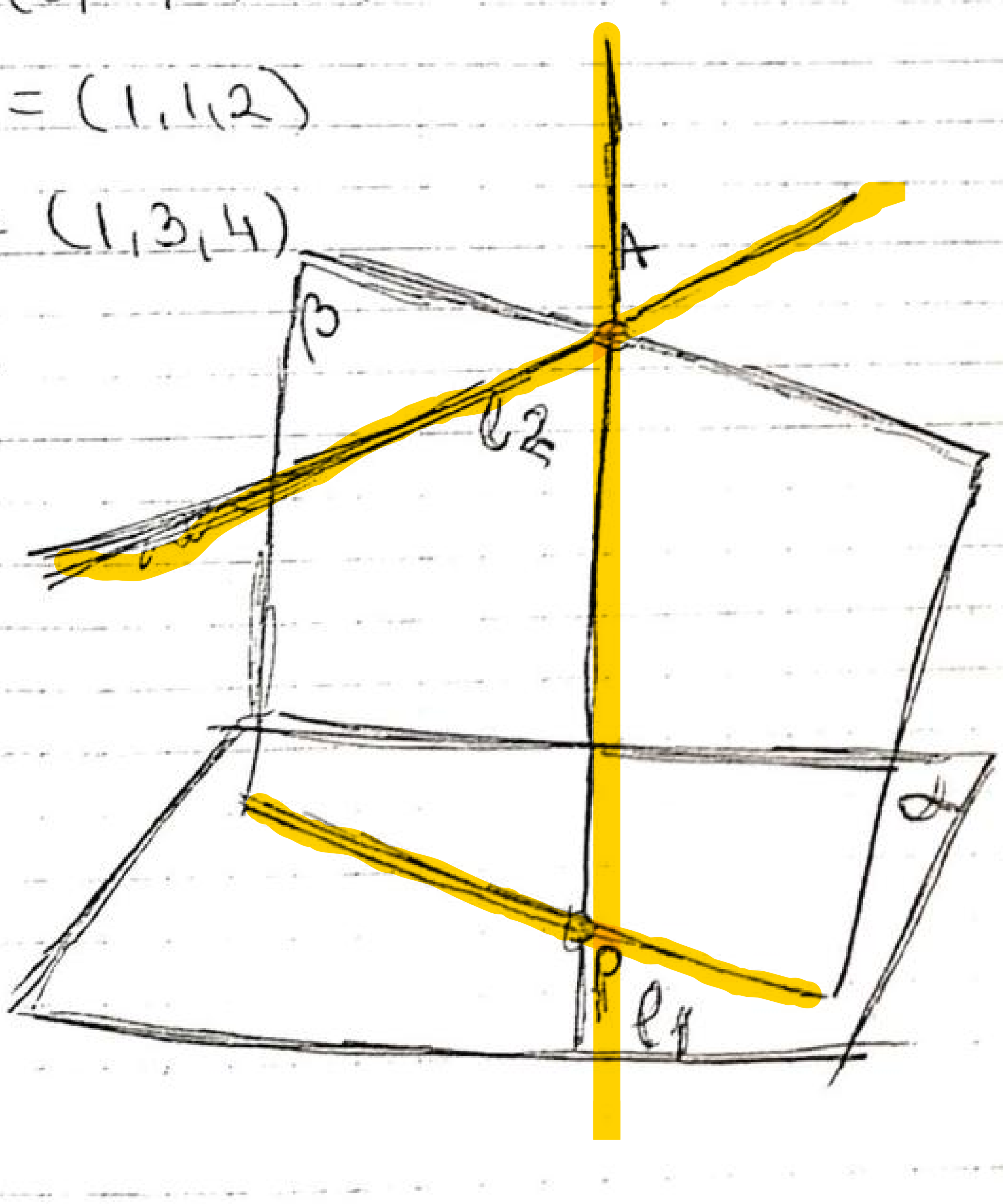
R|| $M_1(-1,0,1)\in l_1$

$M_2(0,-1,2)\in l_2$

$\vec{s_1}=(1,1,2)$

$\vec{s_2}=(1,3,4)$

I način.



α - ravan koja sadrži l_1 i paralelna je sa l_2

β - ravan koja sadrži pravu l_1 i ortogonalna na ravan α.

$l_2 \cap \beta = \{A\}$, A pripada zajedničkoj normali p

$\vec{s_p}$ – vektor pravca prave p

$$\left.\begin{array}{l} p\perp l_1 \Rightarrow \vec{s_p}\perp\vec{s_1} \\ p\perp l_2 \Rightarrow \vec{s_p}\perp\vec{s_2} \end{array}\right\} \Rightarrow \vec{s_p}=\lambda(\vec{s_1}\times\vec{s_2}).$$

$d(p_1,p_2)=d(A,\alpha)$

$$\vec{s_1} \times \vec{s_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 2 \\ 1 & 3 & 4 \end{vmatrix} = -\vec{2i} - \vec{2j} + \vec{2k}$$

$$\text{za } \lambda = -\frac{1}{2} \quad \vec{sp} = (1, 1, -1)$$

$$\left.\begin{array}{l} l_1 \subset \alpha \Rightarrow \vec{n_\alpha} \perp \vec{s_1} \\ l_2 \parallel \alpha \Rightarrow \vec{n_\alpha} \perp \vec{s_2} \end{array}\right\} \Rightarrow \vec{n_\alpha} = \lambda(\vec{s_1} \times \vec{s_2})$$

$\vec{n_\alpha} = (1, 1, -1)$

$\alpha: \ M_1(-1, 0, 1), \ \vec{n_\alpha} = (1, 1, -1)$

$1(x+1) + 1(y-0) - 1(z-1) = 0$

$\alpha: \ x + y - z + 2 = 0$

$$\left.\begin{array}{l} l_1 \subset \beta \Rightarrow \vec{n_\beta} \perp \vec{s_1} \\ \alpha \perp \beta \Rightarrow \vec{n_\beta} \perp \vec{n_\alpha} \end{array}\right\} \Rightarrow \vec{n_\beta} = \lambda(\vec{s_1} \times \vec{n_\alpha})$$

$$\vec{s_1} \times \vec{n_\alpha} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -\vec{3i} + \vec{3j} + 0\vec{k} \Rightarrow (-3, 3, 0)$$

$$\text{za } \lambda = -\frac{1}{3} \Rightarrow (1, -1, 0)$$

$\beta: \ M_1(-1, 0, 1), \ \vec{n_\beta} = (1, -1, 0)$

$1(x+1) - 1(y-0) + 0(z-1) = 0$

$\beta: \ x - y + 1 = 0$

Nađimo $l_2 \cap \beta$

$$l_2: \begin{cases} x = t \\ y = -1 + 3t \\ z = 2 + 4t \end{cases} \qquad \begin{array}{l} t - (-1 + 3t) + 1 = 0 \\ -2t + 2 = 0 \\ \boxed{t = 1} \end{array}$$

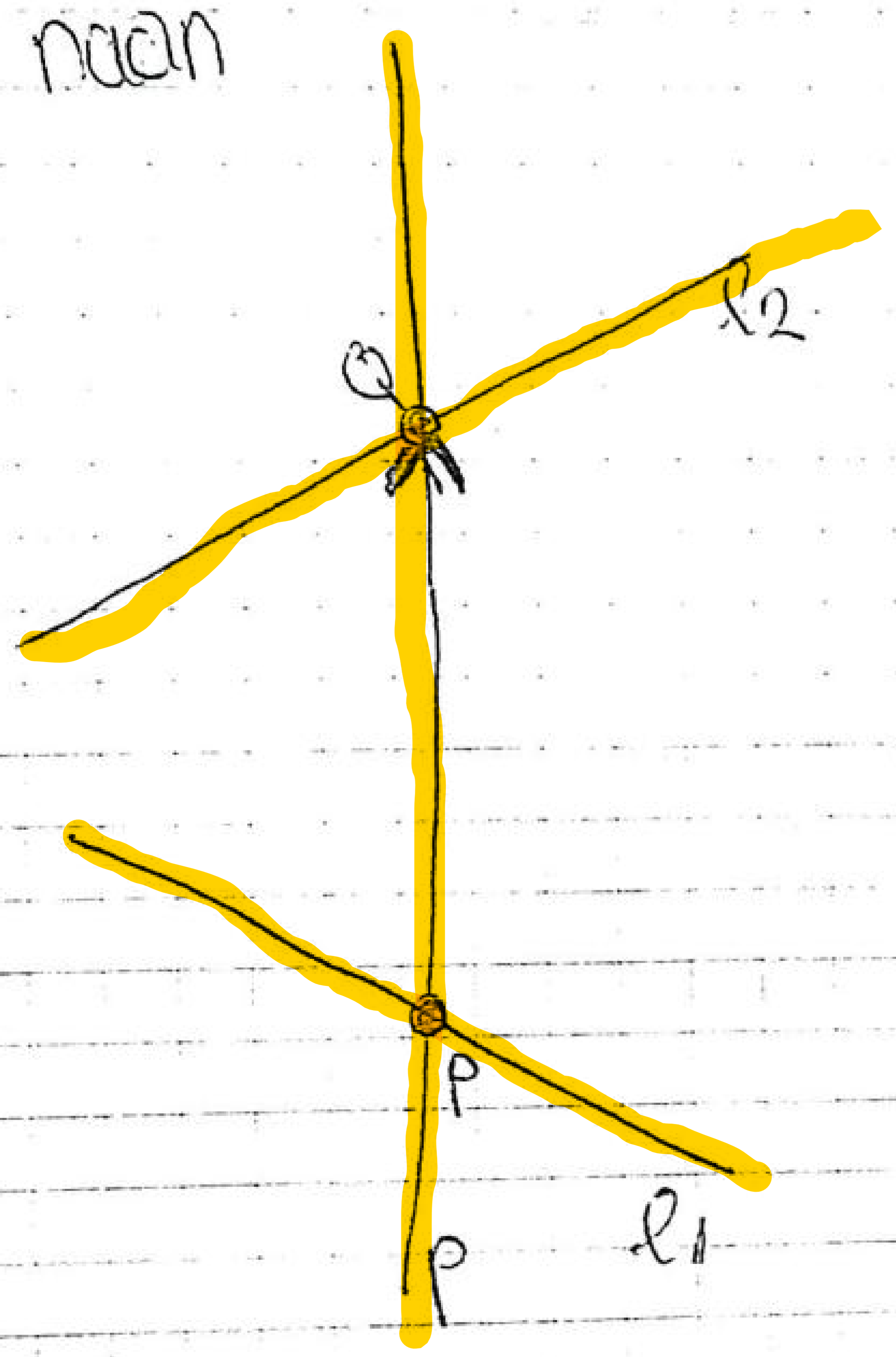
$x = 1, \ y = 2, \ z = 6 \Rightarrow A(1, 2, 6)$

$p: A(1,2,6)\ ,\ \vec{s_p} = (1,1,-1)$

$p: \frac{x-1}{1} = \frac{y-2}{1} = \frac{z-6}{-1}$

$d(l_1,l_2) = d(A,\alpha) = \frac{|1+2-6+2|}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

II naćin



$l_1: \begin{cases} x = -1+k \\ y = k \\ z = 1+2k \end{cases}$

$l_2: \begin{cases} x = t \\ y = -1+3t \\ z = 2+4t \end{cases}$

$l_1 \cap p = \{P\}$ $\qquad$ $l_2 \cap p = \{Q\}$

$P(-1+k, k, 1+2k)$ $\qquad$ $Q(t, -1+3t, 2+4t)$

$\vec{PQ} = (t+1-k, -1+3t-k, 1+4t-2k)$

$\vec{PQ} \perp \vec{s_1} \Rightarrow \vec{PQ} \cdot \vec{s_1} = 0 \implies t+\cancel{1}-k-\cancel{1}+3t-k+2+8t-4k=0$

$\vec{PQ} \perp \vec{s_2} \Rightarrow \vec{PQ} \cdot \vec{s_2} = 0 \implies t+1-k-3+9t-3k+4+16t-8k=0$

$12t-6k+2=0$

$26t-12k+2=0$

$6t-3k = -1$

$13t-6k = -1$

$\Rightarrow t=1\ ,\ k=\frac{7}{3}$

$l_1: \begin{cases} x = \frac{4}{3} \\ y = \frac{7}{3} \\ z = \frac{17}{3} \end{cases}$ $\qquad P\left(\frac{4}{3}; \frac{7}{3}; \frac{17}{3}\right)$

$l_2: \begin{cases} x = 1 \\ y = 2 \\ z = 6 \end{cases}$ $\qquad Q(1,2,6)$

$p: \; P\left(\frac{4}{3}, \frac{7}{3}, \frac{17}{3}\right), \quad Q(1,2,6)$

$$p: \; \frac{x-1}{\frac{4}{3}-1} = \frac{y-2}{\frac{7}{3}-1} = \frac{z-6}{\frac{17}{3}-6}$$

$$\frac{x-1}{\frac{1}{3}} = \frac{y-2}{\frac{1}{3}} = \frac{z-6}{-\frac{1}{3}}$$

$$p: \; \frac{x-1}{1} = \frac{y-2}{1} = \frac{z-6}{-1}$$

$d(l_1, l_2) = |\vec{PQ}|$ ✓