

Prava i ravan

$$l: \frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$$

$$\Pi: Ax + By + Cz + D = 0$$

$$l: \begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases}$$

Uvешtamo u jedinačnu ravnine Π .

$$A(x_0 + mt) + B(y_0 + nt) + C(z_0 + pt) + D = 0$$

$$Ax_0 + By_0 + Cz_0 + D + (Am + Bn + Cp)t = 0 \quad (*)$$

Ako (*) važi za svako $t \in \mathbb{R}$ tada prava l leži u ravnini

1- tj. Prava l leži u ravnini Π ako

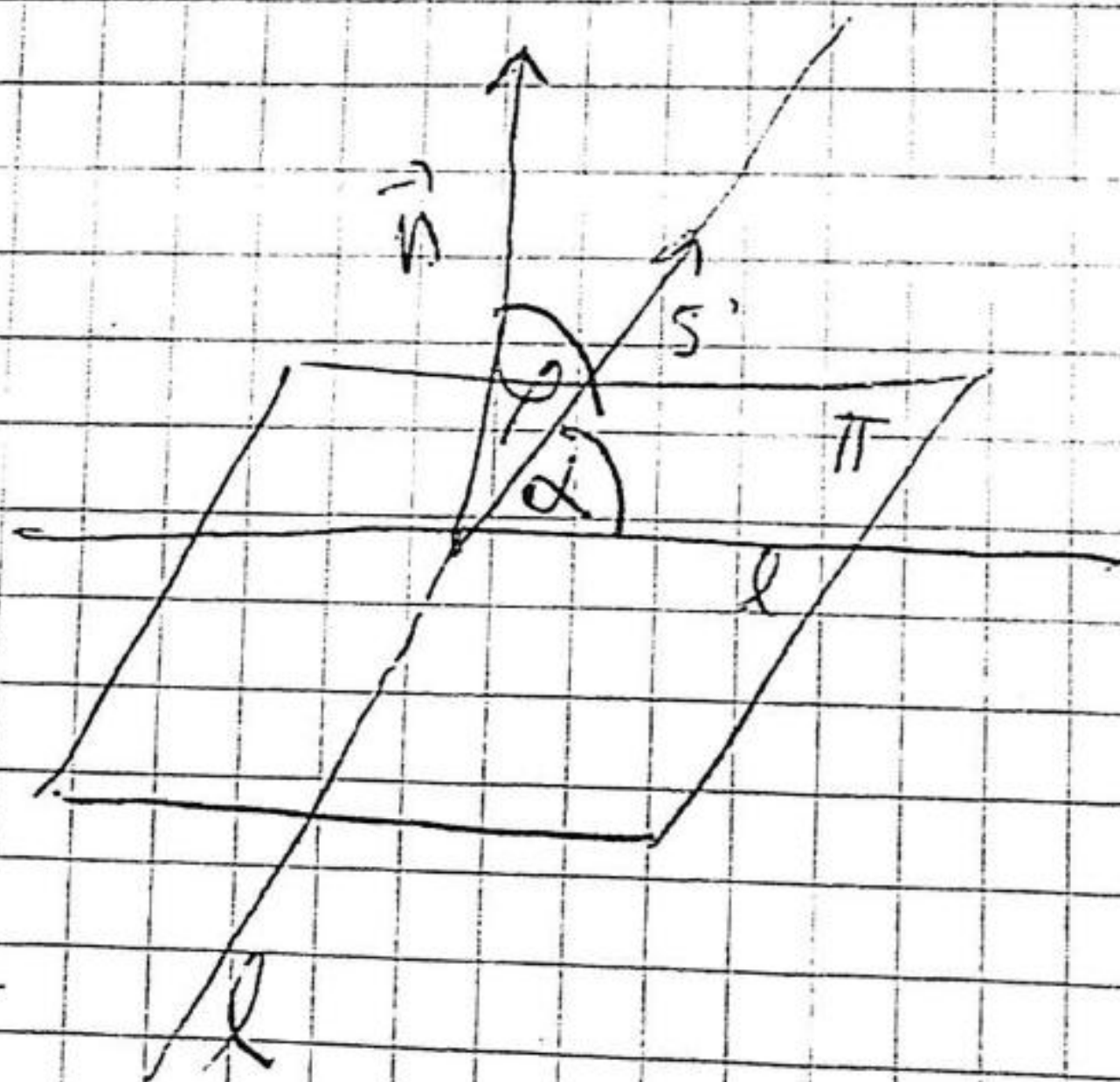
$$\begin{cases} Ax_0 + By_0 + Cz_0 + D = 0 & (1) & (M(x_0, y_0, z_0) \in \Pi) \\ Am + Bn + Cp = 0 & (2) \end{cases}$$



$$\begin{aligned} \vec{s}' \cdot \vec{n} &= 0 \\ \vec{s}' &\perp \vec{n} \end{aligned}$$

Ako ne važi (1) a važi (2) tada je $l \parallel \Pi$, ako ne važi

2) tada se prava l i ravan Π sijeku.



$$\angle(l, \Pi) = \angle(l, l')$$

l - ortogonalna projekcija

prave l na ravan Π

$$\varphi + \psi = \frac{\pi}{2}$$

$$\varphi = \angle(\vec{n}, \vec{s}')$$

$$\cos \varphi = \frac{|\vec{n} \cdot \vec{s}'|}{|\vec{n}| |\vec{s}'|}$$

$$\cos\left(\frac{\pi}{2} - \varphi\right) = \frac{\vec{n} \cdot \vec{s}'}{|\vec{n}'| |\vec{s}'|}$$

$$\sin \alpha = \frac{|\vec{n} \cdot \vec{s}'|}{|\vec{n}'| |\vec{s}'|}$$

Przykład: Ispytaj wzajemni polożaj proste i równi

$$p: \frac{x}{-2} = \frac{y-1}{-1} = \frac{z-2}{1}$$

$$\alpha: x + y + 3z - 7 = 0$$

$$R|| M(0, 1, 2) \in p$$

$$\vec{s}' = (-2, -1, 1)$$

$$\vec{n} = (1, 1, 3)$$

$$1) 0 + 1 + 3 \cdot 2 - 7 = 0 \quad \checkmark \quad 0 = 0$$

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$$2) \vec{s}' \cdot \vec{n} = -2 \cdot 1 - 1 \cdot 1 + 1 \cdot 3 = 0 \Rightarrow \underline{\vec{s}' \perp \vec{n}}$$

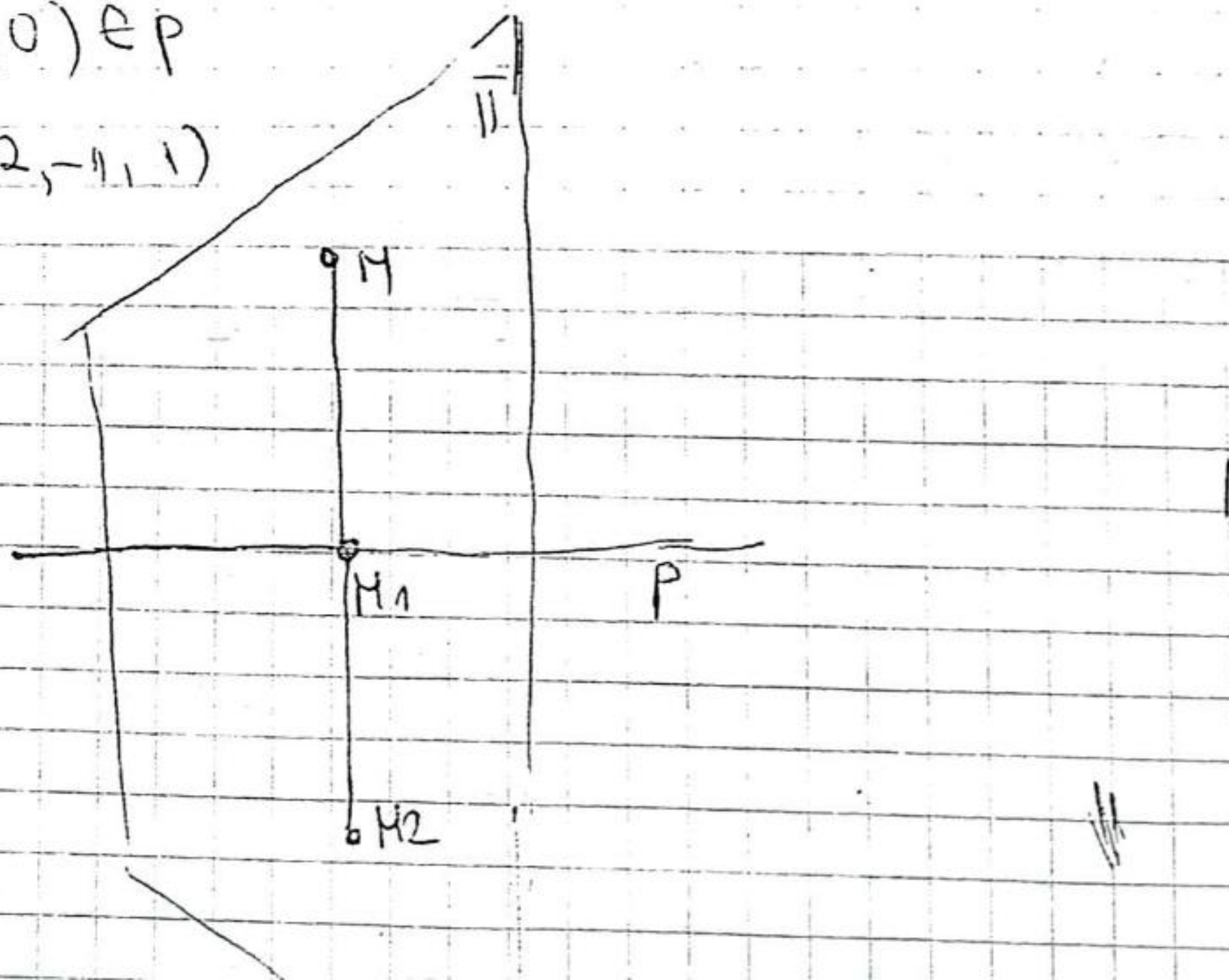
Iz 1) i 2) \Rightarrow prosta p leży w równi α .

2) Naci: taczki symetrycznej taczki $M(1, 2, 8)$ w odnosu na proste

$$p: \frac{x-1}{2} = \frac{y}{-1} = \frac{z}{1} \quad \text{normalu } \vec{n} \text{ na proste } p.$$

$$R|| A(1, 0, 0) \in p$$

$$\vec{s}_p = (2, -1, 1)$$



Postavljamo ravan Π koja sadrži tačku M i ortogonalna je na

pravu p .

\vec{n}_{Π} - njen vektor normale

$$\Pi \cap p = \{M\}$$

M_1 - ortogonalna projekcija tačke M na pravu p .

M_2 - tačka koja simetrična tački

M u odnosu na pravu p .

M_1 je središte duži MM_2 .

$$p \perp \Pi \Rightarrow \vec{n}_{\Pi} = \lambda \vec{sp} \quad \checkmark$$

$$\text{Za } \lambda=1, \vec{n}_{\Pi} = (2, -1, 1)$$

$$\Pi: M(1, 2, 8), \vec{n}_{\Pi} = (2, -1, 1)$$

$$2(x-1) - 1(y-2) + 1(z-8) = 0$$

$$\boxed{\Pi: 2x - y + z - 8 = 0}$$

$$p: \begin{cases} x = 1 + 2t \\ y = -t \\ z = t \end{cases}$$

Uvestimo u jednačinu ravni Π

$$2(1+2t) - (-t) + t - 8 = 0$$

$$6t = 6$$

$$t = 1$$

$$x = 1 + 2 \cdot 1 = 3$$

$$y = -1$$

$$z = 1$$

$$M_1(3, -1, 1)$$

$$M(1, 2, 8)$$

$$M_1(3, -1, 1)$$

$$M_2(x, y, z)$$

$$3 = \frac{1+x}{2} \Rightarrow x = 5$$

$$-1 = \frac{2+y}{2} \Rightarrow y = -4$$

$$1 = \frac{8+z}{2} \Rightarrow z = -6$$

$$M_2(5, -4, -6)$$

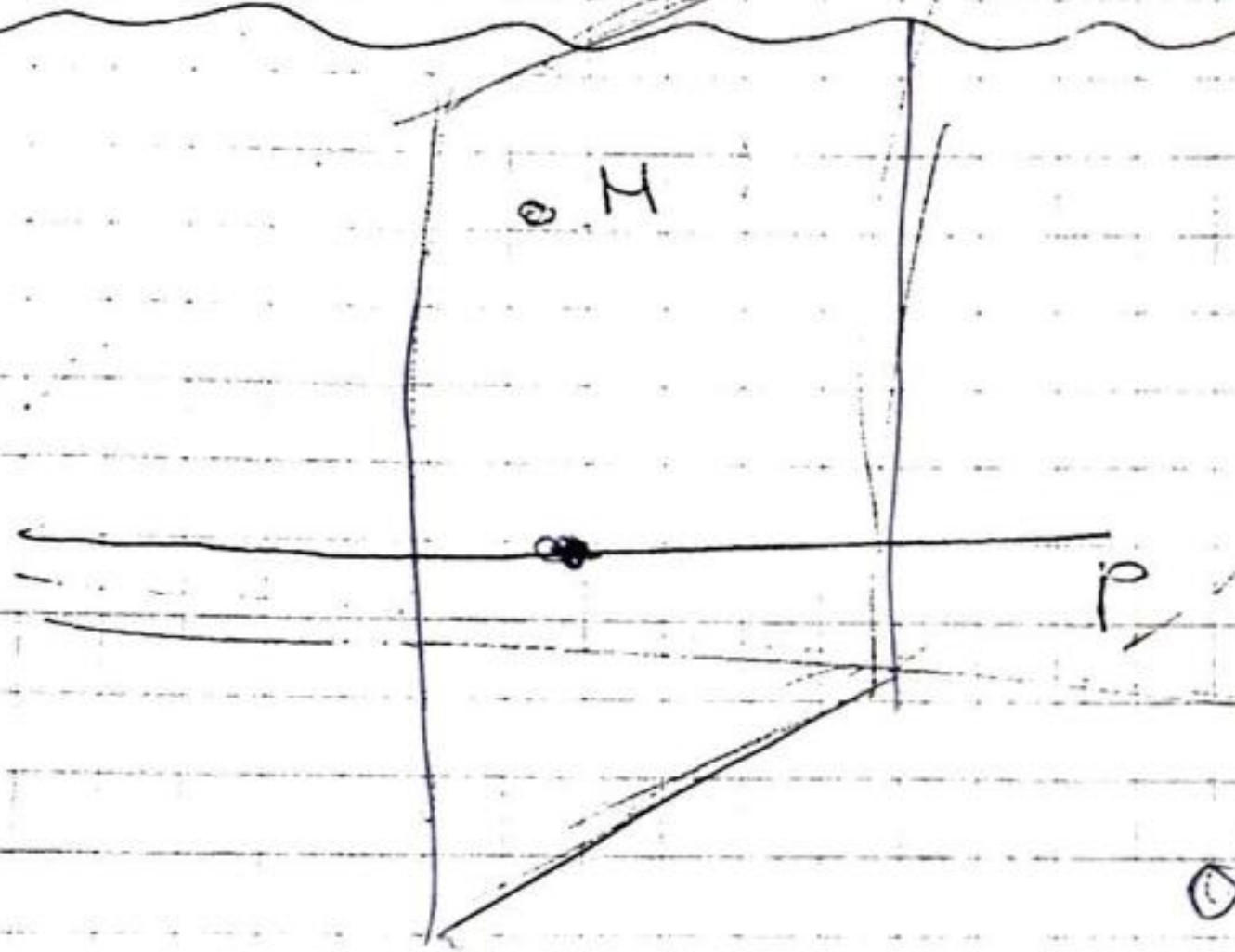
l - normala iz tacke H na pravu p

$$l(H, M_1)$$

$$l: \frac{x-1}{3-1} = \frac{y-2}{-1-2} = \frac{z-8}{1-8}$$

$$l: \frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-8}{-4}$$

II način za normalu iz tacke na pravu



$$\alpha \cap \beta = l$$

α - ravan koja sadrži tacku i pravu p

β - ravan koja sadrži tacku M i ortogonalna na pravu p.

$$\alpha \cap \beta = l$$

l - normala pravu p iz tacke M na pravu p.

3) Koji jednacine normalne projekcije pravu p: $\frac{x}{5} = \frac{y+1}{-2} = \frac{z+1}{-3}$ na ravan $\alpha: 2x - 3y + z - 4 = 0$.

R1 $\beta \rightarrow$ ravan koja sadrži pravu p i ortogonalna na ravan α .

$\alpha \cap \beta = p$ i ortogonalna projekcija pravu p na ravan α .

$$M(0, -1, -4) \in p \subset \beta$$

$$p \subset \beta \Rightarrow \vec{n}_\beta \perp \vec{s}_p \quad \left\{ \begin{array}{l} \vec{n}_\beta = \lambda (\vec{s}_p \times \vec{n}_\alpha) \end{array} \right.$$

$$\beta \perp \alpha \Rightarrow \vec{n}_\beta \perp \vec{n}_\alpha$$

$$\vec{s}_p \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -2 & -3 \\ 2 & -3 & 1 \end{vmatrix} = -11\vec{i} - 11\vec{j} - 11\vec{k} \quad \vec{s}_p \times \vec{n}_\beta = (-11, -11, -11)$$

za $\lambda = \frac{1}{-11}$ $\Rightarrow \vec{n}_\beta = (1, 1, 1)$

β : $M(0, -1, -1)$ $\vec{n}_\beta = (1, 1, 1)$

$$1(x-0) + 1(y+1) + 1(z+1) = 0$$

$$x + y + z + 2 = 0$$

β : $x + y + z + 2 = 0$

$p_1 = \begin{cases} 2x - 3y + z - 4 = 0 \\ x + y + z + 2 = 0 \end{cases}$ (Naci kanonski oblik jednačine)

④ Odrediti ugao između prave $p: \frac{x-1}{2} = \frac{y+2}{-1} = \frac{z-3}{4}$

i ravni α određene tačkama $A(0, 0, 1)$ $B(3, 1, 0)$ $C(3, 2, 1)$

$R // M(x, y, z)$ proizvoljna tačka ravni α

$\vec{AM}, \vec{AB}, \vec{AC} \Rightarrow$ komplanarni

$$\vec{AM} = (x, y, z-1)$$

$$\vec{AB} = (3, 1, -1)$$

$$\vec{AC} = (3, 2, 1)$$

$$(\vec{AM} \times \vec{AB}) \cdot \vec{AC} = \begin{vmatrix} x & y & z-1 \\ 3 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 3x - 6y + 3(z-1) = 0$$

$$2 \cdot x - 2y + z - 1 = 0$$

$$\vec{n}_\alpha = (1, -2, 1)$$

$$\vec{s}_p = (2, -1, 4)$$

$$\varphi = \angle(\alpha, p)$$

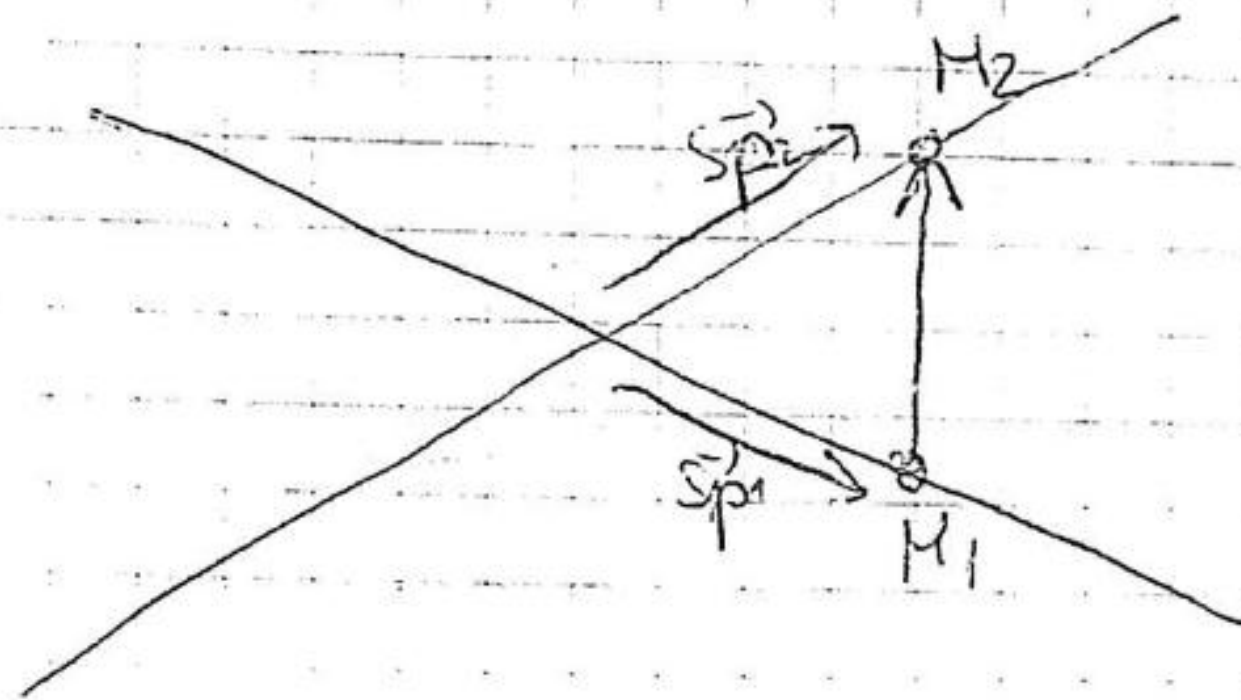
$$\sin \varphi = \frac{|\vec{n}_\alpha \cdot \vec{s}_p|}{|\vec{n}_\alpha| |\vec{s}_p|}$$

$$\sin \varphi = \frac{8}{\sqrt{6} \sqrt{21}} = \frac{8}{8\sqrt{14}} = \frac{4\sqrt{14}}{21}$$

5) Dane su prave $p_1: \frac{x+2}{2} = \frac{y+2}{a} = \frac{z+2}{2}$, $p_2: \frac{x-1}{a} = \frac{y-1}{1} = \frac{z-1}{3}$

Obračunaj a tako da se prave sijeku. Za veću vrijednost parametra a napiši jednadžbu ravnine α koja sadrži prave.

R1 $M_1(-2, -2, -2) \in p_1$
 $M_2(1, 1, 1) \in p_2$



$\vec{s}_{p_1} = (2, a, 2)$

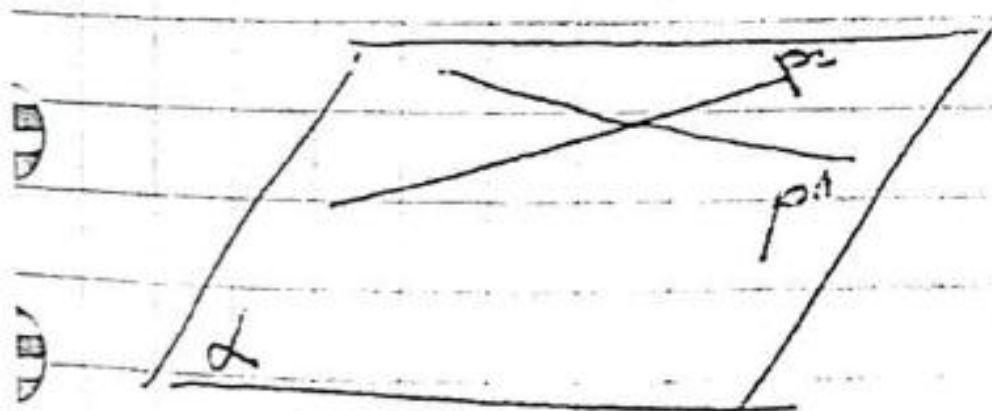
$\vec{s}_{p_2} = (a, 1, 3)$

$\vec{s}_{p_1}, \vec{s}_{p_2}, \vec{M_1M_2}$ su komplanarni pa je:

$\vec{M_1M_2} = (3, 3, 3)$

$(\vec{s}_{p_1} \times \vec{s}_{p_2}) \cdot \vec{M_1M_2} = \begin{vmatrix} 2 & a & 2 \\ a & 1 & 3 \\ 3 & 3 & 3 \end{vmatrix} = -9a^2 + 15a - 6 = 0$
 $-3a^2 + 5a - 2 = 0$ ✓

$a_{1/2} = \frac{-5 \pm \sqrt{25 - 24}}{-6} = \frac{-5 \pm 1}{-6} \rightarrow a_1 = 1$ ✓
 $\rightarrow a_2 = \frac{2}{3}$



$p_1 \subset \alpha \Rightarrow \vec{n}_\alpha \perp \vec{s}_{p_1}$
 $p_2 \subset \alpha \Rightarrow \vec{n}_\alpha \perp \vec{s}_{p_2} \Rightarrow \vec{n}_\alpha = \lambda(\vec{s}_{p_1} \times \vec{s}_{p_2})$

$\vec{s}_{p_1} = (2, 1, 2)$

$\vec{s}_{p_2} = (1, 1, 3)$

$\vec{s}_{p_1} \times \vec{s}_{p_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix} = \vec{i} - 4\vec{j} + \vec{k} \Rightarrow (1, -4, 1)$
 $2a\lambda = 1 \Rightarrow \vec{n}_\alpha = (1, -4, 1)$

$\alpha: M_1(-2, -2, -2)$

$\vec{n}_\alpha = (1, -4, 1)$

$1(x+2) - 4(y+2) + 1(z+2) = 0$

$x - 4y + z - 4 = 0$

*) Neki jednacina ravnih koja sadrži pravu $p: \frac{x}{2} = \frac{y-1}{-1} = \frac{z+1}{2}$ je je rastojanje od tačke $A(1, 2, -1)$ iznosi $\frac{4}{3}$.

|| **Svaka prava određuje pravu!**

Određimo duge ravni koje sadrže pravu p .

$$\frac{x}{2} = \frac{y-1}{-1}$$

$$\frac{x}{2} = \frac{z+1}{2}$$

$$\underline{-x - 2y + 2 = 0}$$

$$\underline{\beta: x - z - 1 = 0}$$

Ravni α i β određuju pravu ravnih a tražimo ravan π perpendikularnu pravini.

Jednacina pravine je

$$-x - 2y + 2 + \lambda(x - z - 1) = 0$$

$$(\lambda - 1)x - 2y - \lambda z + 2 - \lambda = 0 \checkmark$$

$$d(A, \pi) = \frac{4}{3} = \frac{|(\lambda - 1) \cdot 1 - 2 \cdot 2 - \lambda(-1) + 2 - \lambda|}{\sqrt{(\lambda - 1)^2 + 4 + \lambda^2}} = \frac{4}{3} \checkmark$$

$A(1, 2, -1)$

$$= \frac{|\lambda - 3|}{\sqrt{2\lambda^2 - 2\lambda + 5}} = \frac{4}{3} \quad |^2$$

$$= \frac{(\lambda - 3)^2}{2\lambda^2 - 2\lambda + 5} = \frac{16}{9}$$

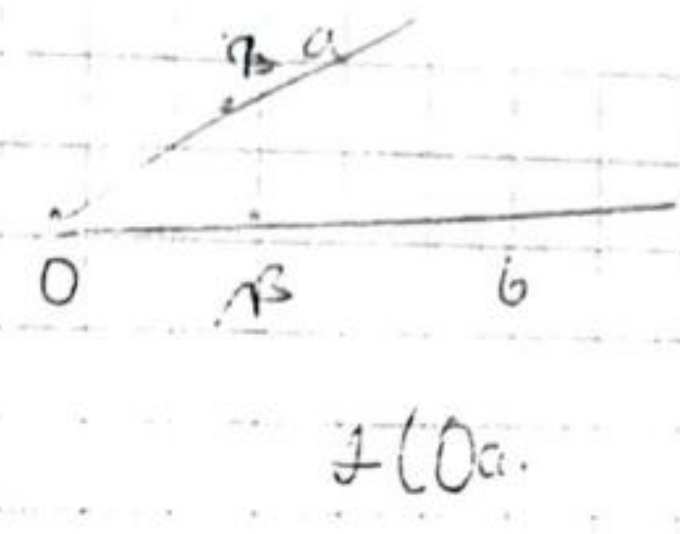
$$= 9\lambda^2 - 54\lambda + 81 = 32\lambda^2 - 32\lambda + 80$$

$$\Rightarrow 23\lambda^2 + 22\lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{-22 \pm \sqrt{484 + 92}}{46} = \frac{-22 \pm \sqrt{576}}{46}$$

$$\Rightarrow \lambda_{1,2} = \frac{-22 \pm 24}{46} \rightarrow \lambda_1 = -1 \checkmark$$

$$\rightarrow \lambda_2 = \frac{1}{23} \checkmark$$



$$1^{\circ} -2x - 2y + z + 2 + 1 = 0$$

$$-2x - 2y + z + 3 = 0$$

$$2x + 2y - z - 3 = 0$$

$$2^{\circ} \left(\frac{1}{23} - 1\right)x - 2y - \frac{1}{23}z + 2 - \frac{1}{23} = 0$$

$$-22x - 46y - z + 45 = 0$$

$$22x + 46y + z - 45 = 0 \quad \checkmark$$

7) Naci rastojanje i zajednicku normalu rimovanih pravih:

$$l_1: \frac{x+1}{1} = \frac{y}{1} = \frac{z-1}{2}, \quad l_2: \frac{x}{1} = \frac{y+1}{3} = \frac{z-2}{4}$$

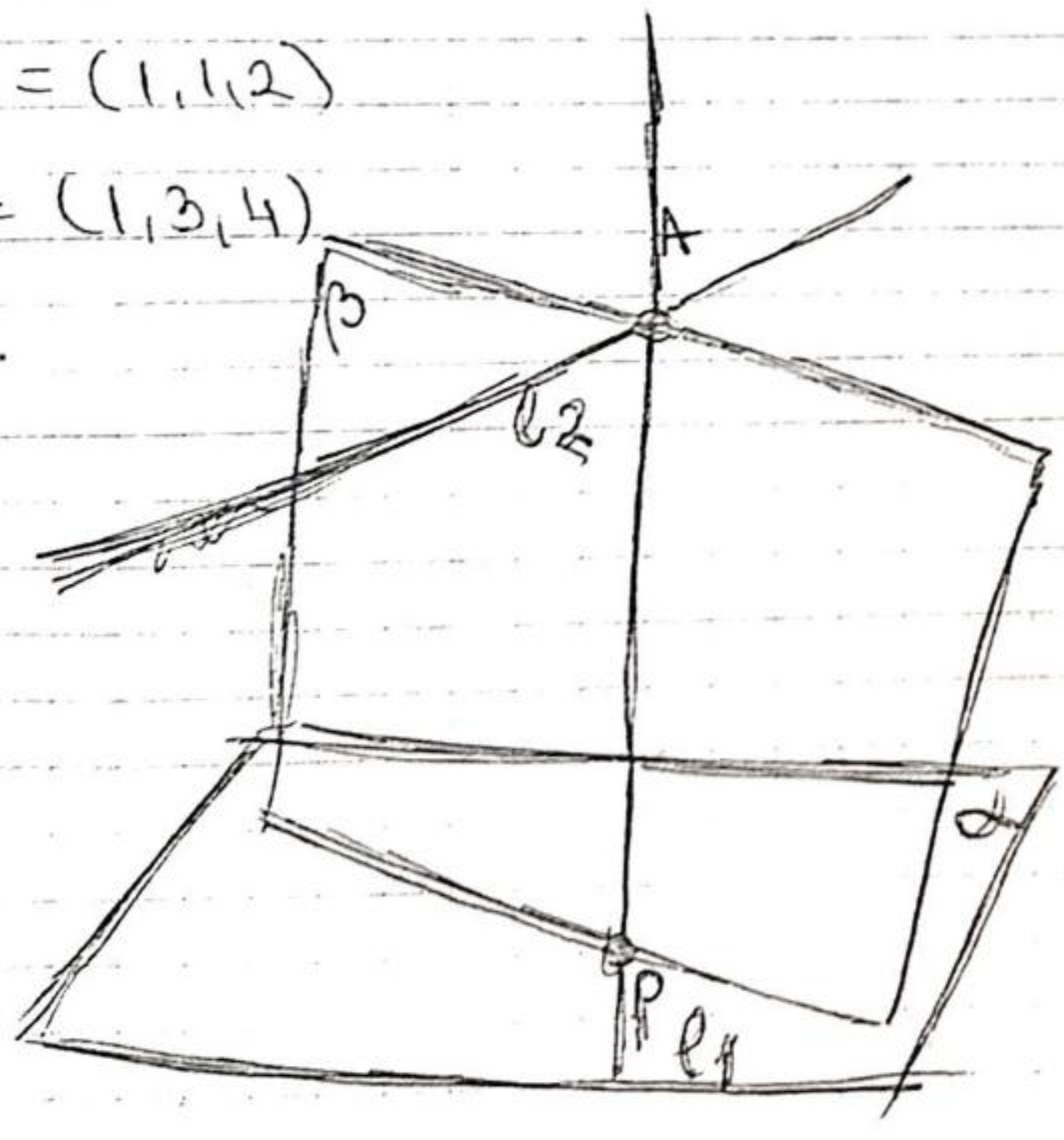
$$R_{l_1} \quad H_1 (-1, 0, 1) \in l_1$$

$$H_2 (0, -1, 2) \in l_2$$

$$\vec{s}_1 = (1, 1, 2)$$

$$\vec{s}_2 = (1, 3, 4)$$

I nacrt.



α - ravnina koja sadrži l_1
i paralelna je sa l_2

β - ravnina koja sadrži
pravu l_2 i ortogonalna
na ravnini α .

$l_2 \cap \beta = \{A\}$, A pripada zajednickoj normali p

\vec{s}_p - vektor pravca prave p

$$\left. \begin{array}{l} p \perp \alpha \Rightarrow \vec{s}_p \perp \vec{s}_1 \\ p \perp \beta \Rightarrow \vec{s}_p \perp \vec{s}_2 \end{array} \right\} \Rightarrow \vec{s}_p = \lambda (\vec{s}_1 \times \vec{s}_2).$$

$$d(p_1, p_2) = d(A, \alpha)$$

$$\vec{s}_1 \times \vec{s}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 2 \\ 1 & 3 & 4 \end{vmatrix} = -2\vec{i} - 2\vec{j} + 2\vec{k}$$

$$\exists a \lambda = -\frac{1}{2} \quad \vec{s}_p = (1, 1, -1)$$

$$\left. \begin{array}{l} l_1 \subset \alpha \Rightarrow \vec{n}_\alpha \perp \vec{s}_1 \\ l_2 \parallel \alpha \Rightarrow \vec{n}_\alpha \perp \vec{s}_2 \end{array} \right\} \Rightarrow \vec{n}_\alpha = \lambda (\vec{s}_1 \times \vec{s}_2)$$

$$\vec{n}_\alpha = (1, 1, -1)$$

$$\alpha: M_1(-1, 0, 1), \vec{n}_\alpha = (1, 1, -1)$$

$$1(x+1) + 1(y-0) - 1(z-1) = 0$$

$$\alpha: x + y - z + 2 = 0$$

$$\left. \begin{array}{l} \gamma \subset \beta \Rightarrow \vec{n}_\beta \perp \vec{s}_1 \\ \alpha \perp \beta \Rightarrow \vec{n}_\beta \perp \vec{n}_\alpha \end{array} \right\} \Rightarrow \vec{n}_\beta = \lambda (\vec{s}_1 \times \vec{n}_\alpha)$$

$$\vec{s}_1 \times \vec{n}_\alpha = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -3\vec{i} + 3\vec{j} + 0\vec{k} \Rightarrow (-3, 3, 0)$$

$$\exists a \lambda = -\frac{1}{3} \Rightarrow (1, -1, 0)$$

$$\beta: M_1(-1, 0, 1), \vec{n}_\beta = (1, -1, 0)$$

$$1(x+1) - 1(y-0) + 0(z-1) = 0$$

$$\beta: x - y + 1 = 0$$

Wartung $l_2 \cap \beta$

$$l_2: \begin{cases} x = t \\ y = -1 + 3t \\ z = 2 + 4t \end{cases}$$

$$t - (-1 + 3t) + 1 = 0$$

$$-2t + 2 = 0$$

$$t = 1$$

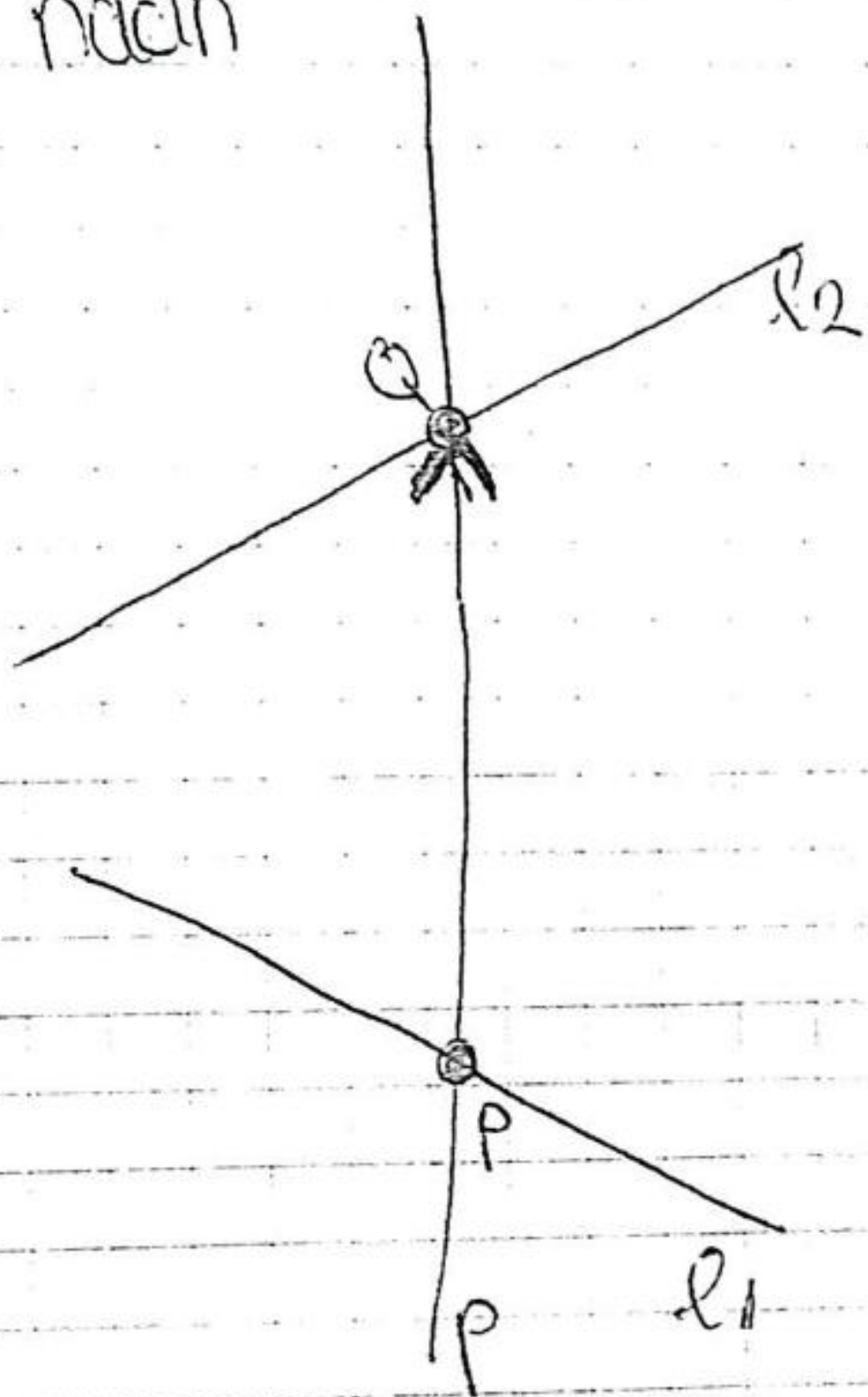
$$x = 1, y = 2, z = 6 \Rightarrow A(1, 2, 6)$$

$$p: A(1,2,6), \vec{s} = (1,1,-1)$$

$$p: \frac{x-1}{1} = \frac{y-2}{1} = \frac{z-6}{-1}$$

$$d(l_1, l_2) = d(A, \ell) = \frac{|1+2-6+2|}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

II nach



$$l_1: \begin{cases} x = -t+k \\ y = k \\ z = 1+2k \end{cases}$$

$$l_2: \begin{cases} x = t \\ y = -1+3t \\ z = 2+4t \end{cases}$$

$$l_1 \cap p = \{P\}$$

$$l_2 \cap p = \{Q\}$$

$$P(-1+k, k, 1+2k)$$

$$Q(t, -1+3t, 2+4t)$$

$$\vec{PQ} = (t+1-k, -1+3t-k, 1+4t-2k)$$

$$\vec{PQ} \perp \vec{s}_1 \Rightarrow \vec{PQ} \cdot \vec{s}_1 = 0$$

$$t+1-k-1+3t-k+2+8t-4k=0$$

$$\vec{PQ} \perp \vec{s}_2 \Rightarrow \vec{PQ} \cdot \vec{s}_2 = 0$$

$$t+1-k-3+9t-3k+4+16t-8k=0$$

$$12t - 6k + 2 = 0$$

$$26t - 12k + 2 = 0$$

$$6t - 3k + 1 = 0$$

$$13t - 6k + 1 = 0$$

$$l_1: \begin{cases} x = \frac{4}{3} \\ y = \frac{4}{3} \\ z = \frac{14}{3} \end{cases}$$

$$P\left(\frac{4}{3}, \frac{4}{3}, \frac{14}{3}\right)$$

$$\Rightarrow t=1, k=\frac{4}{3}$$

$$l_2: \begin{cases} x=1 \\ y=2 \\ z=6 \end{cases}$$

$$Q(1, 2, 6)$$

$$P: P\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right), Q(1, 2, 6)$$

$$P: \frac{x-1}{\frac{1}{3}-1} = \frac{y-2}{\frac{2}{3}-1} = \frac{z-6}{\frac{6}{3}-1}$$

$$\frac{x-1}{\frac{1}{3}} = \frac{y-2}{\frac{2}{3}} = \frac{z-6}{\frac{6}{3}}$$

$$P: \frac{x-1}{1} = \frac{y-2}{1} = \frac{z-6}{1}$$

$$d(Q_1, Q_2) = |PQ| \checkmark$$