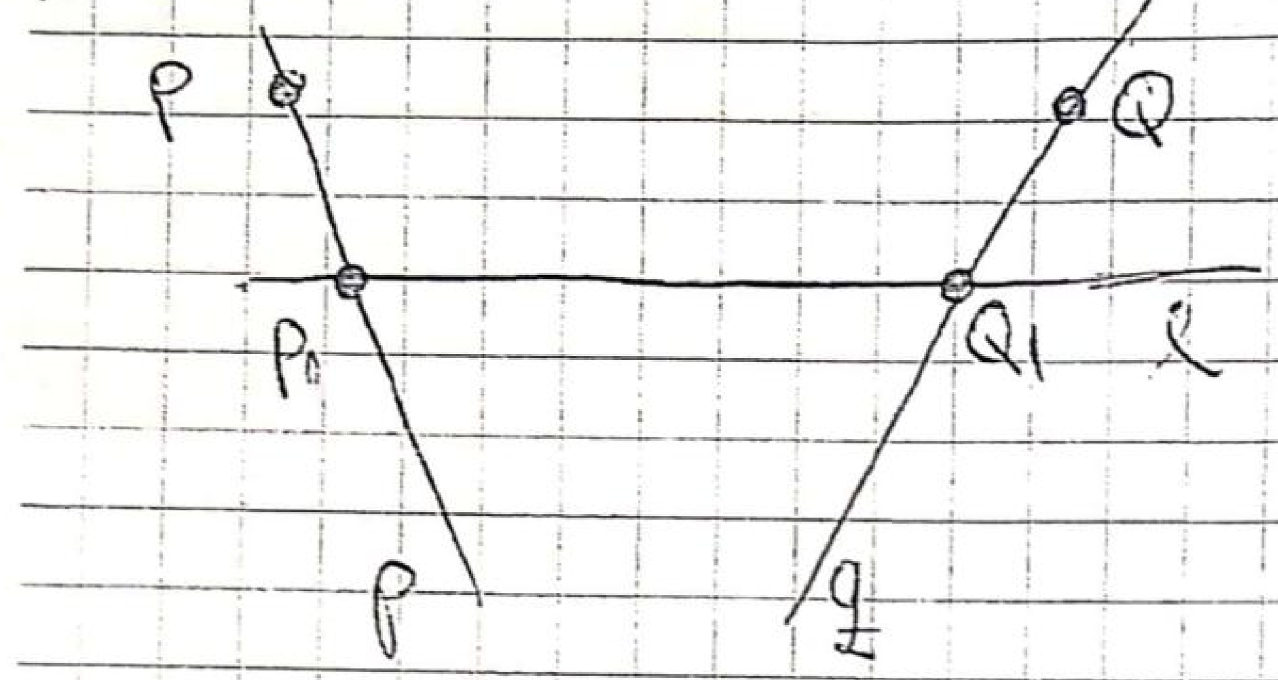
8) Naći zajedničku normalu i kanonske jednačine pravih p i q je su paralelne ravni $\alpha: 2x-y+z=0$, sijeku pravu $l: \frac{x+1}{2}=\frac{y}{-2}=\frac{z}{1}$ prolaze kroz tačke $P(-2,3,5)$ i $Q(2,0,1)$ respektivno.



$l \cap p = \{P_1\}$

$l \cap q = \{Q_1\}$

$p \parallel \alpha \Rightarrow p$ pripada ravni Π_1 koja je paralelna ravni α

$\Pi_1: 2x-y+z+D=0$

$P \in p \subset \Pi_1 \Rightarrow 2(-2)-3+5+D=0$

$\Rightarrow -2+D=0$

$\Rightarrow \underline{D=2}$

$\Pi_1: 2x-y+z+2=0$

$$\left.\begin{array}{l} P_1 \in l \\ P_1 \in p \subset \Pi_1 \end{array}\right\} \Rightarrow l \cap \Pi_1 = \{P_1\}$$

$$l: \begin{cases} x=-1+2t \\ y=-2t \\ z=t \end{cases} \qquad \begin{array}{l} x=-1 \\ y=0 \\ z=0 \end{array}$$

$\cancel{-2}+4t+2t+t\cancel{+2}=0$

$7t=0$

$\boxed{t=0}$

$\Rightarrow P_1(-1,0,0)$

$p: P_1(-1,0,0),\ P(-2,3,5)$

$$\frac{x+1}{-1} = \frac{y}{3} = \frac{z}{5}$$

$q \parallel \alpha \Rightarrow$ q pripada ravni Π_2 paralelnoj ravni α

$\Pi_2: 2x - y + z + D = 0$

$Q \in q \subset \Pi_2 \Rightarrow 2 \cdot 2 - 0 + 1 + D = 0$

$5 + D = 0$

$\boxed{D = -5}$

$\Pi_2: 2x - y + z - 5 = 0$

$$\left.\begin{array}{l} Q_1 \in l \\ Q_1 \in q \subset \Pi_2 \end{array}\right\} \Rightarrow l \cap \Pi_2 = \{Q_1\}$$

$-2 + 4t + 2t + t - 5 = 0$

$7t - 7 = 0$

$\boxed{t = 1}$

$x = 1$

$y = -2$

$z = 1$

$\Rightarrow Q_1(1,-2,1)$

$q: Q_1(1,-2,1)\ Q(2,0,1)$

$$q: \frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{0} \checkmark$$

$d \parallel p$

$d \parallel q$

$\beta \rightarrow$ ravan koja sadrži pravu q i ortogonalna je na α.

$$\left.\begin{array}{l} \vec{n_\beta} \perp \vec{s_q} \\ \vec{n_\beta} \perp \vec{n_\alpha} \end{array}\right\} \Rightarrow \vec{n_\beta} = \lambda(\vec{s_q} \times \vec{n_\alpha})$$

$$\vec{s_q} \times \vec{n_\gamma} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 2 & -1 & 1 \end{vmatrix} = 2\vec{i} - \vec{j} + (-5)\vec{k} \quad , \vec{n_\beta} = (2, -1, -5)$$

β: $Q(2,0,1)$ $\vec{n_\beta} = (2,-1,5)$

$2(x-2) - 1(y-0) - 5(z-1) = 0$

β: $2x - y - 5z + 1 = 0$

$p \cap \beta = \{A\}$ $A \in n$, $n \to$ normala pentru p și q

$$p: \begin{cases} x = -1-t \\ y = 3t \\ z = 5t \end{cases} \qquad \begin{aligned} -2-2t-3t-25t+1 &= 0 \\ -30t - 1 &= 0 \\ t &= -\frac{1}{30} \end{aligned}$$

$x = -\frac{29}{30}$

$y = -\frac{1}{10} \Rightarrow A\left(-\frac{29}{30}, -\frac{1}{10}, -\frac{1}{6}\right)$

$z = -\frac{1}{6}$

$\vec{s}$ - vector pentru normale ⌐ Normala γ

$\left.\begin{aligned} \vec{s} \perp \vec{s_p} \\ \vec{s} \perp \vec{s_q} \end{aligned}\right\} \Rightarrow \vec{s} = \lambda(\vec{s_p} \times \vec{s_q})$

$\vec{s} = \lambda \cdot \vec{n_\alpha}$

$\vec{s} = (2, 1, -1)$

9) Naći jednačinu prave koje sadrži tačku $M(3,-2,-4)$, paralelna je sa ravni $\alpha: 3x-2y-3z-7=0$, a siječe pravu $l: \frac{x-2}{3}=\frac{y+4}{-2}=\frac{z-1}{2}$

R|| p – tražena prava

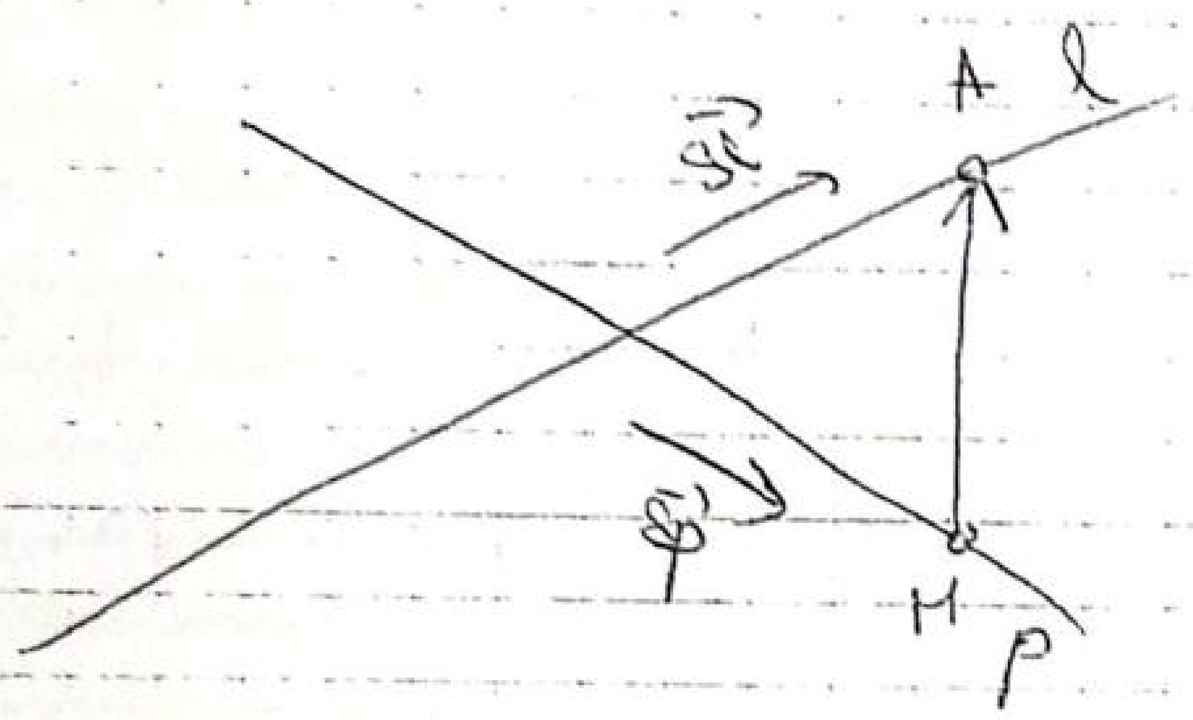
$\vec{s_p}=(m,n,k)$ → njen vektor prava

$\vec{s_l}=(3,-2,2)$

$A(2,-4,1)\in l$ $\qquad \vec{MA}=(-1,-2,5)$

$\vec{n_\alpha}=(3,-2,-3)$

$p \parallel \alpha \Rightarrow \vec{n_\alpha}\perp\vec{s_p} = \vec{n_\alpha}\cdot\vec{s_p}=0 \Rightarrow 3m-2n-3k=0 \quad (1)$



prave p i l se sijeku pa su vektori $\vec{s_p}, \vec{s_l}, \vec{MA}$ komplanarni i važi $(\vec{s_p}\times\vec{s_l})\cdot\vec{MA}=0$

$$\begin{vmatrix} m & n & k \\ 3 & -2 & 2 \\ -1 & -2 & 5 \end{vmatrix}=0$$

$-6m-17n-8k=0 \quad (2)$

$$\begin{cases} 3m-2n-3k=0 \quad /2 \\ -6m-17n-8k=0 \end{cases} \qquad \begin{cases} 3m-2n-3k=0 \\ -21n-14k=0 \end{cases}$$

$n=\frac{-2k}{3}$

$m=\frac{2n+3k}{3}$

$m=\frac{-\frac{4k}{3}+3k}{3}=\frac{5k}{9}$

$\vec{s_p}=\left(\frac{5}{9}k,-\frac{2}{3}k,k\right)$

$\vec{s_p}=k\left(\frac{5}{9},-\frac{2}{3},1\right)$

$\vec{s_p}=(5,-6,9)$ $\quad p: \frac{x-3}{5}=\frac{y+2}{-6}=\frac{z+4}{9}$ ✓

10) Date su prave $p: \frac{x+3}{l} = \frac{y-2}{m} = \frac{z-5}{4}$, $q: \frac{x-2}{1} = \frac{y+1}{2} = \frac{z-3}{-2}$

i $r: \frac{x-9}{3} = \frac{y-3}{1} = \frac{z-5}{2}$

Odrediti vrijednosti parametara l i m tako da prava p prolazi kroz presjek pravih q i r, a zatim naći ugao između prave i ravni koju određuju prave q i r.

I Odredimo $q \cap r$

$$q: \begin{cases} x = 2+t \\ y = -1+2t \\ z = 3-2t \end{cases}$$

$$\frac{2+t-9}{3} = \frac{-1+2t-3}{1} = \frac{3-2t-5}{2}$$

$$\frac{-7+t}{3} = \frac{-4+2t}{1} = \frac{-2-2t}{2}$$

$$t-7 = 6t-12$$

$$5t = 5$$

$$\boxed{t=1}$$

$A(3,1,1)$

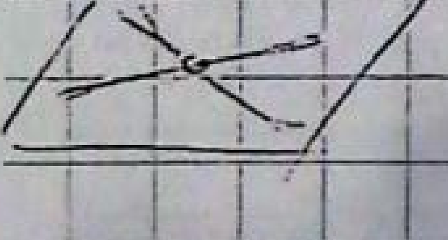
$q \cap r = \{A\}$

$$A(3,1,1) \in p \Rightarrow \frac{3+3}{l} = \frac{1-2}{m} = \frac{1-5}{4}, \quad \frac{6}{l} = \frac{-1}{m} = -1$$

$$\boxed{l=-6, \ m=1}$$

$$p: \frac{x-3}{-6} = \frac{y-2}{1} = \frac{z-5}{4}$$

α: → ravan koju određuju q i r



$$\left.\begin{array}{l} q \subset \alpha \Rightarrow \vec{n_\alpha} \perp \vec{s_q} \\ r \subset \alpha \Rightarrow \vec{n_\alpha} \perp \vec{s_r} \end{array}\right\} \vec{n_\alpha} = \lambda(\vec{s_q} \times \vec{s_r})$$

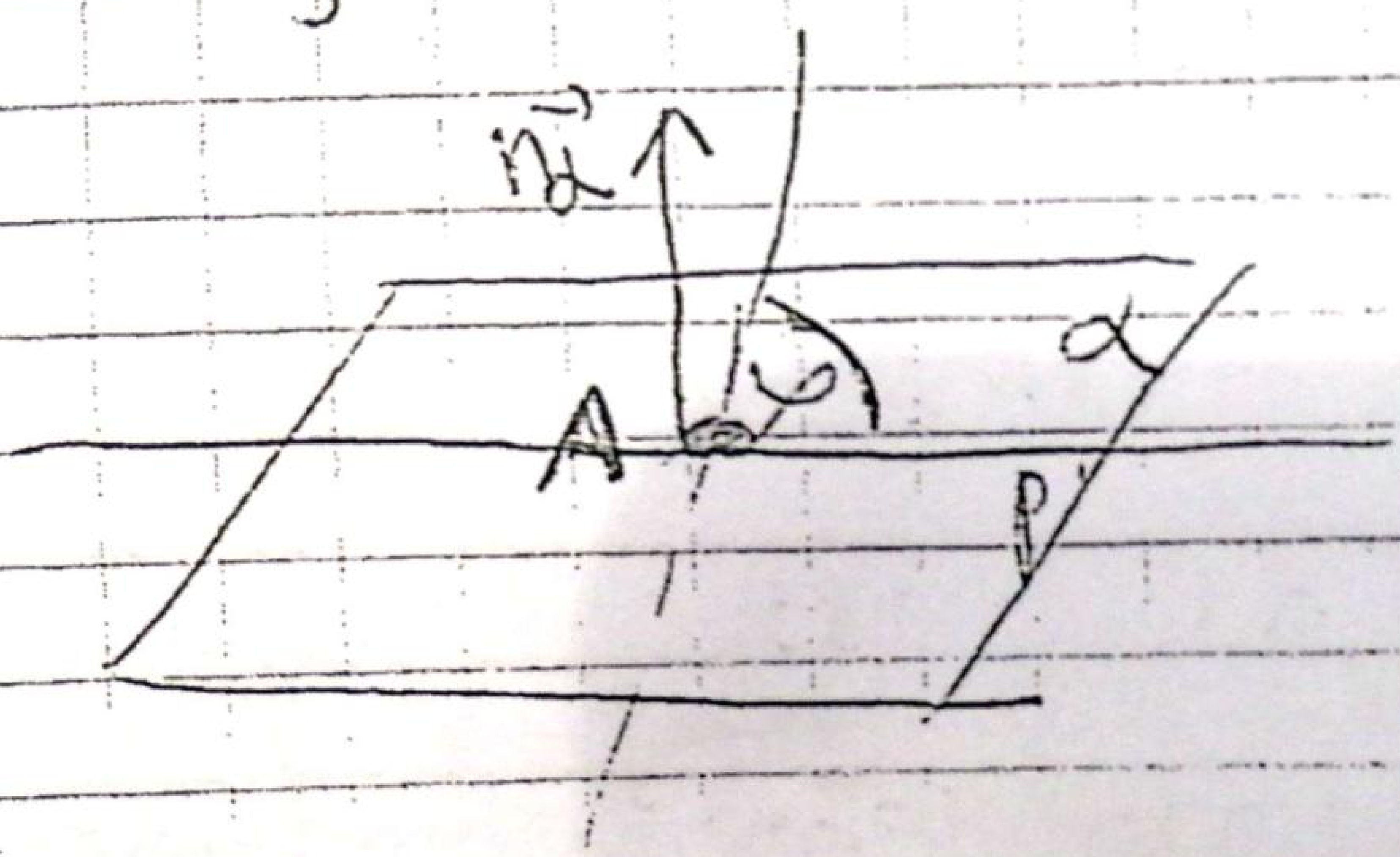
$$\vec{s_q} \times \vec{s_r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ 3 & 1 & 2 \end{vmatrix} = 6\vec{i} - 8\vec{j} + (-5)\vec{k}$$

Za $\lambda = 1 \Rightarrow \vec{n_\alpha} = (6, -8, -5)$

α: $A(3,1,1)$ $\vec{n_\alpha} = (6,-8,-5)$

$$6(x-3) - 8(y-1) - 5(z-1) = 0$$

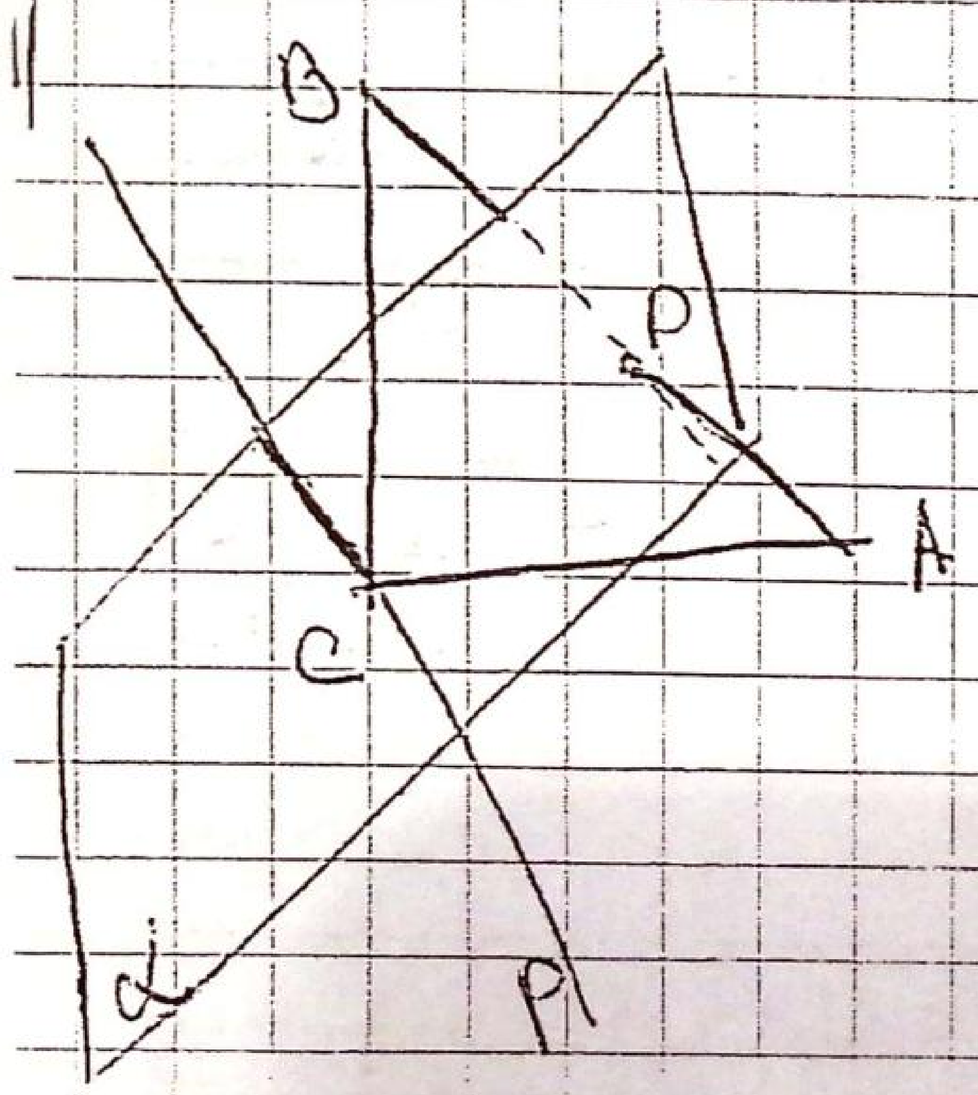
$6x - 8y - 5z - \cancel{10}\,5 = 0$



$$\sin\varphi = \frac{|\vec{n_\alpha} \cdot \vec{s_p}|}{|\vec{n_\alpha}||\vec{s_p}|}$$

$$\sin\varphi = \frac{|-36 - 8 - 20|}{\sqrt{36+64+25}\ \sqrt{36+1+16}} = \frac{64}{5\sqrt{265}}$$

12) Na pravoj $p: \frac{x-1}{-1} = \frac{y-2}{1} = \frac{z-1}{2}$ naći tačku C jednako udaljenu od tačaka $A(2,1,-2)$ i $B(4,-5,4)$.



D - središte duži AB

α - ravan koja sadrži tačku D i ortogonalna na duž AB.

$p \cap \alpha = \{C\}$.

$D(x,y,z)$

$x = \frac{2+4}{2} = 3$

$y = \frac{1-5}{2} = -2$

$z = \frac{-2+4}{2} = 1$

$D(3,-2,1)$

$\vec{n_\alpha} = \lambda \vec{AB}$

$\vec{AB} = (2,-6,6)$

Za $\lambda = \frac{1}{2}$, $\vec{n_\alpha} = (1,-3,3)$

$\alpha: D(3,-2,1)$, $\vec{n_\alpha} = (1,-3,3)$

$\alpha: x-3y+3z-12=0$

$$p: \begin{cases} x = \cancel{1+t} \\ y = 2+t \\ z = 1+2t \end{cases}$$

$$1-t-3(2+t)+3(1+2t)-12=0$$
$$2t-14=0$$
$$t=7$$

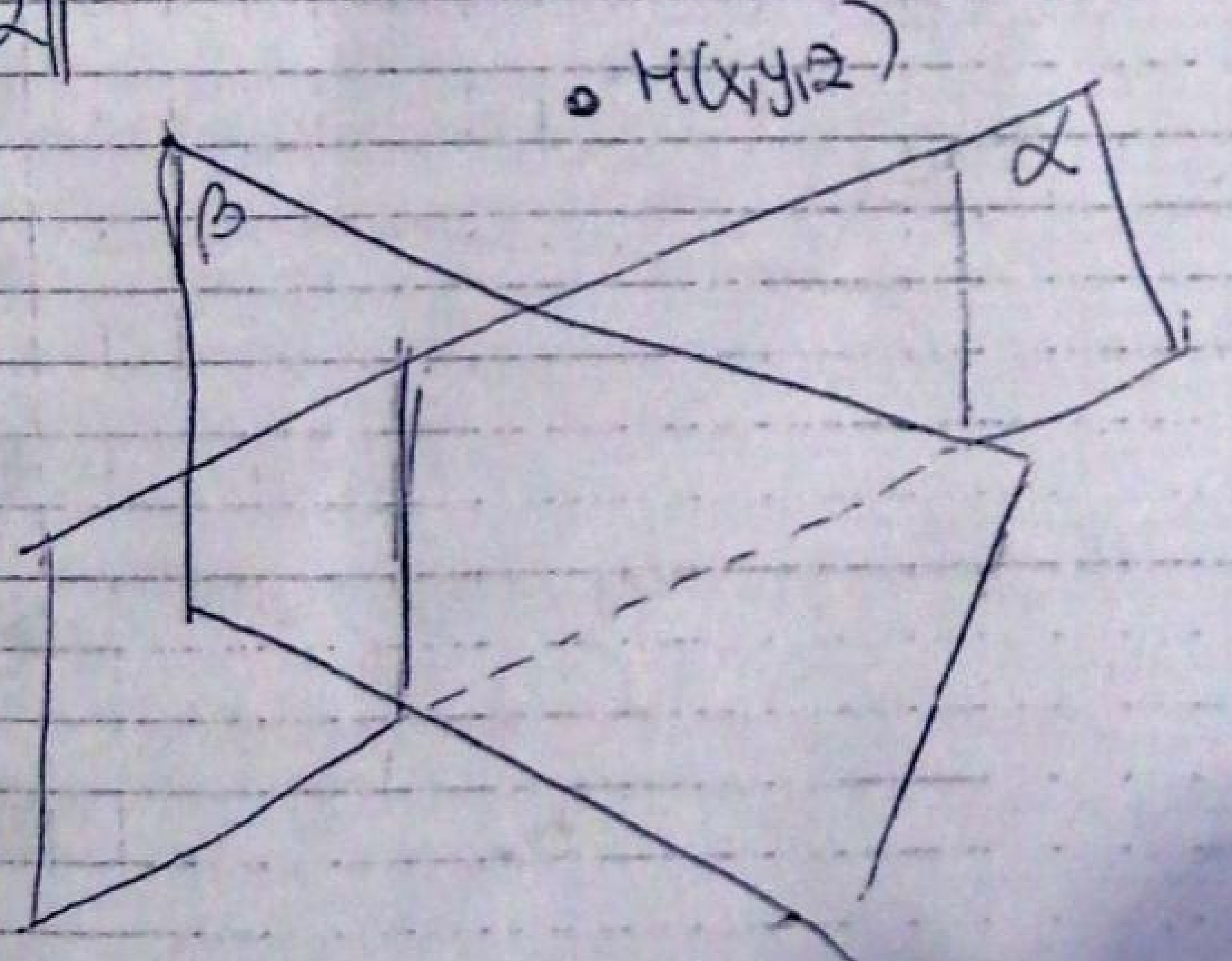
$C(-6,9,15)$

$$x=-6$$
$$y=9$$
$$z=15$$

13) Napisati jednačine ravni koje su simetrale uglova između ravni

$\alpha: x+2y+2z-4=0$ i $\beta: 6x-3y+2z+6=0$

R:



Neka je $M(x,y,z)$ proizvoljna tačka simetralne ravni ugla koje grade ravni α i β. Tada je $d(M,\alpha) = d(M,\beta)$.

$$\frac{|x+2y+2z-4|}{\sqrt{1+4+4}} = \frac{|6x-3y+2y+6|}{\sqrt{36+9+4}}$$

$$\frac{|x+2y+2z-4|}{3} = \frac{|6x-3y+2y+6|}{7}$$

$$7|x+2y+2z-4| = 3|6x-3y+2z+6|$$
$$7(x+2y+2z-4) = \pm 3(6x-3y+2z+6)$$

1° $7(x+2y+2z-4) = 3(6x-3y+2z+6)$

$\beta_1: 11x-23y-8z+46=0$

$7(x+2y+2z-4) = -3(6x-3y+2z+6)$

$\gamma_2: 25x+5y+20z-10=0$

4) Kroz tačku $M_0(3,-1,-1)$ postaviti pravu koja siječe pravu

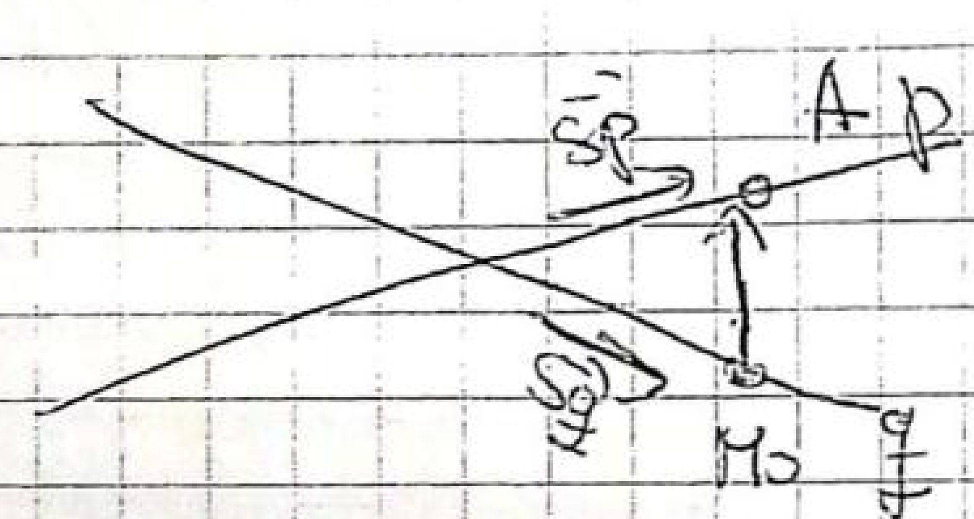
$p: \frac{x-3}{1} = \frac{y+2}{-1} = \frac{z}{2}$ pod uglom 60°.

// q - tražena prava

$\vec{s_q} = (m,n,k)$ - njen vektor pravca

$\vec{s_p} = (1,-1,2)$

$A(3,-2,0) \in p$



* Kako se prave p i q sijeku to su vektori $\vec{s_q}, \vec{s_p}, \overrightarrow{M_0A}$ komplanarni pa je

$(\vec{s_q} \times \vec{s_p}) \cdot \overrightarrow{M_0A} = 0 \quad (*)$

$\overrightarrow{M_0A} = (0,-1,1)$

$$(*) \Rightarrow \begin{vmatrix} m & n & k \\ 1 & -1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = 0$$

$m-n-k=0 \quad (1)$

$\alpha \rightarrow$ ugao između $\measuredangle(q,p) = \measuredangle(\vec{s_q}, \vec{s_p})$

$$\cos 60^\circ = \frac{|\vec{s_q} \cdot \vec{s_p}|}{|\vec{s_q}||\vec{s_p}|} = \frac{|m-n+2k|}{\sqrt{m^2+n^2+k^2}\sqrt{1+1+4}}$$

$$= \frac{|m-n-2k|}{\sqrt{m^2+n^2+k^2}\sqrt{6}}$$

$\frac{1}{2}$

$\sqrt{6}\sqrt{m^2+n^2+k^2} = 2|m-n+2k| \quad (2)$

Dobili smo sistem

$$\begin{cases} m-n-k=0 \\ \sqrt{6}\ \sqrt{m^2+n^2+k^2} = 2|m-n+2k| \end{cases}$$

za $k=1$

$$\begin{cases} m-n-1=0 \\ \sqrt{6}\ \sqrt{m^2+n^2+1} = 2|m-n+2| \quad /^2 \end{cases}$$

$m=n+1$

$6\cdot[(n+1)^2+n^2+1] = 4(\cancel{n}+1-\cancel{n}+2)^2$

$3(2n^2+2n+2) = 2\cdot 9$

$n^2+n+1=3$

$n^2+n-2=0$

$n_{1/2} = \frac{-1\pm\sqrt{1+8}}{2}$

$n_{1/2} = \frac{-1\pm 3}{2}$

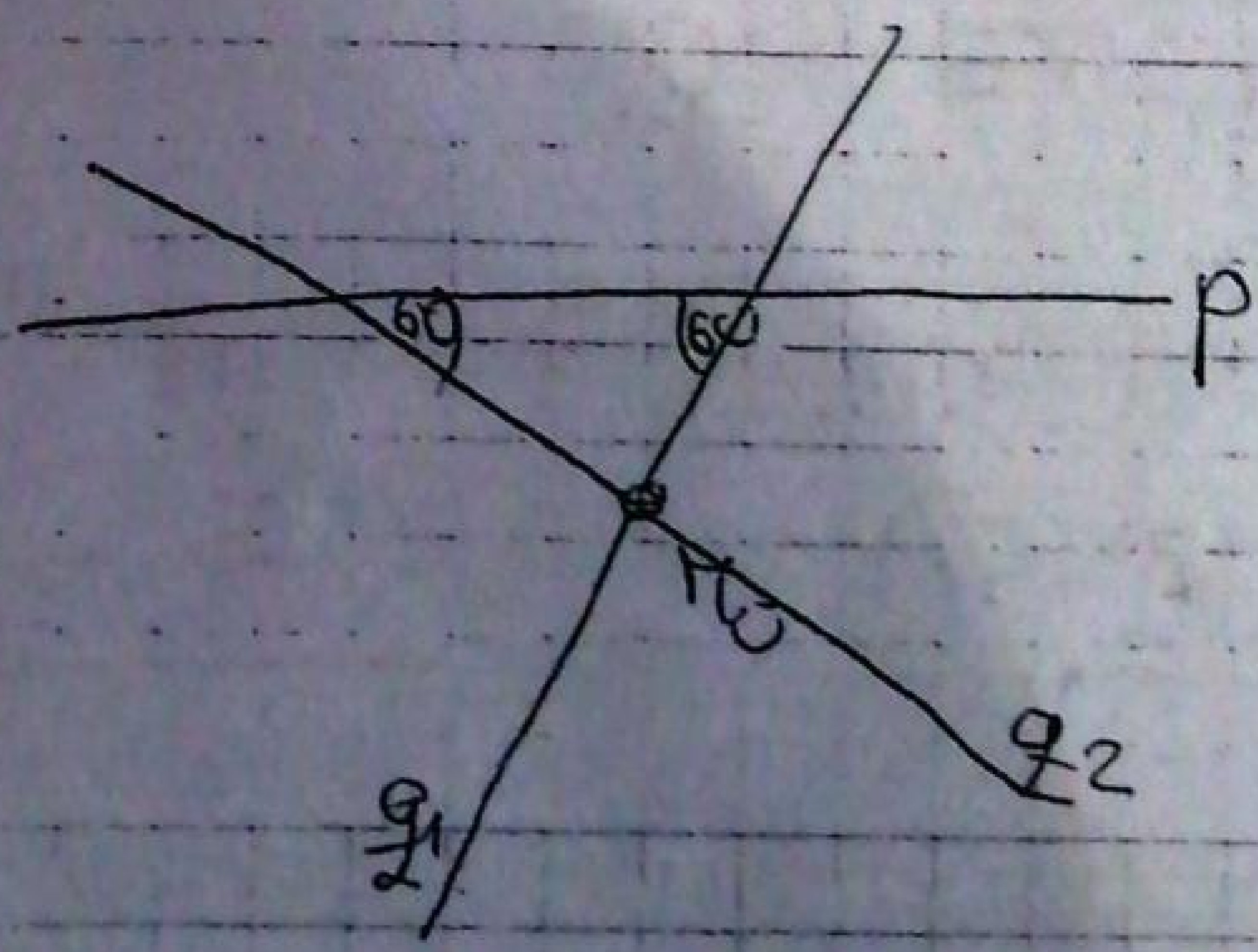
$\boxed{n_1=1,\ n_2=-2}$

1° $m_1=2$

$s\vec{q_1}=(2,1,1)$

q: $M_0(3,-1,-1)$ $s\vec{q_1}=(2,1,1)$

$q: \frac{x-3}{2} = \frac{y+1}{1} = \frac{z+1}{1}$



2° $m_2=-2+1=-1$

$s\vec{q_2}=(-1,-2,1)$

$q: \frac{x-3}{-1} = \frac{y+1}{-2} = \frac{z+1}{1}$