Метод смене

1. $$\int (5-2x)^9 dx = \begin{bmatrix} 5-2x=t \\ -2dx=dt \\ dx=-\frac{dt}{2} \end{bmatrix} = \int t^9\left(-\frac{dt}{2}\right) = -\frac{1}{2}\int t^9 dt = -\frac{1}{2}\frac{t^{10}}{10}+c = -\frac{1}{2}\frac{(5-2x)^{10}}{10}+c$$

2. $$\int \frac{x\,dx}{\sqrt{1-x^2}} = \begin{bmatrix} 1-x^2=t \\ -2x\,dx=dt \\ x\,dx=-\frac{dt}{2} \end{bmatrix} = \int \frac{-\frac{dt}{2}}{\sqrt{t}} = -\frac{1}{2}\int\frac{dt}{\sqrt{t}} = -\frac{1}{2}\int t^{-\frac{1}{2}}dt =$$

$$= -\frac{1}{2}\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c = -\frac{1}{\cancel{2}}\frac{t^{\frac{1}{2}}}{\frac{1}{\cancel{2}}}+c = -\sqrt{t}+c = -\sqrt{1-x^2}+c$$

3. $$\int \frac{\ln x}{x}dx = \begin{bmatrix} \ln x=t \\ \frac{dx}{x}=dt \end{bmatrix} = \int t\,dt = \frac{t^2}{2}+c = \frac{\ln^2 x}{2}+c$$

4. $$\int\sqrt{\frac{1-x}{1+x}}\,dx = \int\sqrt{\frac{1-x}{1+x}\cdot\frac{1-x}{1-x}}\,dx = \int\sqrt{\frac{(1-x)^2}{1-x^2}}\,dx = \int\frac{1-x}{\sqrt{1-x^2}}\,dx =$$

$$= \int\frac{dx}{\sqrt{1-x^2}} - \int\frac{x}{\sqrt{1-x^2}}\,dx = \arcsin x + \sqrt{1-x^2}+c$$

зад. 2.

5. $$\int\frac{dx}{x^2+3} = \int\frac{dx}{3\left(\frac{x^2}{3}+1\right)} = \frac{1}{3}\int\frac{dx}{1+\left(\frac{x}{\sqrt{3}}\right)^2} = \begin{bmatrix} \frac{x}{\sqrt{3}}=t \\ \frac{dx}{\sqrt{3}}=dt \\ dx=\sqrt{3}\,dt \end{bmatrix} = \frac{1}{3}\int\frac{\sqrt{3}\,dt}{1+t^2} =$$

$$= \frac{\sqrt{3}}{3}\int\frac{dt}{1+t^2} = \frac{\sqrt{3}}{3}\operatorname{arctg} t+c = \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{x}{\sqrt{3}}+c$$

6. $$\int\frac{dx}{2+3x^2} = \int\frac{dx}{2\left(1+\frac{3x^2}{2}\right)} = \frac{1}{2}\int\frac{dx}{1+\left(\frac{\sqrt{3}x}{\sqrt{2}}\right)^2} = \begin{bmatrix} \frac{\sqrt{3}x}{\sqrt{2}}=t \\ \frac{\sqrt{3}}{\sqrt{2}}dx=dt \\ dx=\sqrt{\frac{2}{3}}\,dt \end{bmatrix} = \frac{1}{2}\int\frac{\sqrt{\frac{2}{3}}\,dt}{1+t^2} =$$

$$= \frac{1}{2}\sqrt{\frac{2}{3}}\int\frac{dt}{1+t^2} = \frac{1}{\sqrt{6}}\operatorname{arctg} t+c = \frac{1}{\sqrt{6}}\operatorname{arctg}\sqrt{\frac{3}{2}}x+c = \frac{\sqrt{6}}{6}\operatorname{arctg}\frac{\sqrt{6}}{2}x+c$$

7. $$\int e^{\cos x}\sin x\,dx = \begin{vmatrix} \cos x=t \\ -\sin x\,dx=dt \\ \sin x\,dx=-dt \end{vmatrix} = \int e^t(-dt) = -\int e^t dt = -e^t+c = -e^{\cos x}+c$$

8. $$\int\frac{e^{2x}dx}{1-3e^{2x}} = \begin{bmatrix} 1-3e^{2x}=t \\ -6e^{2x}dx=dt \\ e^{2x}dx=\frac{-dt}{6} \end{bmatrix} = \int\frac{-\frac{dt}{6}}{t} = -\frac{1}{6}\int\frac{dt}{t} = -\frac{1}{6}\ln|t|+c =$$

$$= -\frac{1}{6}\ln|1-3e^{2x}|+c$$

9. $\int \frac{\sin x\,dx}{\sqrt{1+2\cos x}} = \left[\begin{array}{l} 1+2\cos x = t^2 \\ -2\sin x\,dx = 2t\,dt \\ \sin x\,dx = -t\,dt \end{array}\right] = \int \frac{-t\,dt}{\sqrt{t^2}} = -\int \frac{\cancel{t}\,dt}{\cancel{t}} =$

$= -t + c = -\sqrt{1+2\cos x} + c$

10. $\int \frac{\sqrt{1+\ln x}}{x}\,dx = \left[\begin{array}{l} 1+\ln x = t \\ \frac{dx}{x} = dt \end{array}\right] = \int \sqrt{t}\,dt = \int t^{\frac{1}{2}}dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(1+\ln x)^{\frac{3}{2}} + c$

11. $\int \frac{dx}{x^2-x+1} = \int \frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}+1} = \int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} = \int \frac{dx}{\frac{3}{4}\left(\frac{4}{3}(x-\frac{1}{2})^2+1\right)} =$

$= \frac{1}{\frac{3}{4}} \int \frac{dx}{1+\left(\frac{2}{\sqrt{3}}(x-\frac{1}{2})\right)^2} = \left[\begin{array}{l} \frac{2}{\sqrt{3}}(x-\frac{1}{2}) = t \\ \frac{2}{\sqrt{3}}dx = dt \\ dx = \frac{\sqrt{3}}{2}dt \end{array}\right] = \frac{4}{3} \int \frac{\frac{\sqrt{3}}{2}dt}{1+t^2} = \frac{4}{3}\cdot\frac{\sqrt{3}}{2} \int \frac{dt}{1+t^2} =$

$= \frac{2}{\sqrt{3}} \operatorname{arctg} t + c = \frac{2\sqrt{3}}{3} \operatorname{arctg}\left(\frac{2\sqrt{3}}{3}(x-\frac{1}{2})\right) + c$

12. $\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{2} \int \frac{dx}{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} \cdot \cos^2\frac{x}{2}} = \frac{1}{2} \int \frac{dx}{\operatorname{tg}\frac{x}{2}\cos^2\frac{x}{2}} =$

$= \left[\begin{array}{l} \operatorname{tg}\frac{x}{2} = t \\ \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}dx = dt \\ \frac{dx}{\cos^2\frac{x}{2}} = 2dt \end{array}\right] = \frac{1}{2} \int \frac{2dt}{t} = \frac{\cancel{2}}{\cancel{2}} \int \frac{dt}{t} = \ln|t| + c = \ln\left|\operatorname{tg}\frac{x}{2}\right| + c$

13. $\int \sqrt{1-x^2}\,dx = \left[\begin{array}{l} x = \sin t \\ dx = \cos t\,dt \end{array}\right] = \int \sqrt{1-\sin^2 t}\,\cancel{d}\cos t\,dt = \int \sqrt{\cos^2 t}\,\cos t\,dt =$

$= \int \cos t \cdot \cos t\,dt = \int \cos^2 t\,dt = \int \frac{1+\cos 2t}{2}dt = \frac{1}{2}\int dt + \frac{1}{2}\int \cos 2t\,dt =$

$= \frac{t}{2} + \frac{1}{2}\int \cos 2t\,dt = \left[\begin{array}{l} 2t = u \\ 2dt = du \\ dt = \frac{du}{2} \end{array}\right] = \frac{t}{2} + \frac{1}{2}\int \cos u \cdot \frac{du}{2} = \frac{t}{2} + \frac{1}{4}\int \cos u\,du =$

$= \frac{t}{2} + \frac{\sin u}{4} + c = \frac{t}{2} + \frac{\sin 2t}{4} + c = \frac{t}{2} + \frac{2\sin t\cos t}{4} + c = \frac{\arcsin t}{2} + \frac{x\sqrt{1-x^2}}{2} + c$

[Коментар: $x = \sin t \Rightarrow t = \arcsin x$
$\cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}$]

14. $$\int \frac{dx}{\sqrt{3-x-x^2}} = \int \frac{dx}{\sqrt{3-(x^2+x)}} = \int \frac{dx}{\sqrt{3-\left(x^2+x+\frac{1}{4}-\frac{1}{4}\right)}} = \int \frac{dx}{\sqrt{3-\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}} =$$

$$= \int \frac{dx}{\sqrt{\frac{13}{4}-\left(x+\frac{1}{2}\right)^2}} = \int \frac{dx}{\sqrt{\frac{13}{4}}\cdot\sqrt{1-\frac{4}{13}\left(x+\frac{1}{2}\right)^2}} = \sqrt{\frac{4}{13}} \int \frac{dx}{\sqrt{1-\left(\frac{2}{\sqrt{13}}\left(x+\frac{1}{2}\right)\right)^2}} =$$

$$= \left[\begin{array}{l} \frac{2}{\sqrt{13}}\left(x+\frac{1}{2}\right) = t \\ \frac{2}{\sqrt{13}}\,dx = dt \\ dx = \frac{\sqrt{13}}{2}\,dt \end{array}\right] = \frac{2}{\sqrt{13}} \int \frac{\frac{\sqrt{13}}{2}\,dt}{\sqrt{1-t^2}} = \frac{\cancel{2}}{\cancel{\sqrt{13}}}\cdot\frac{\cancel{\sqrt{13}}}{\cancel{2}}\cdot\int \frac{dt}{\sqrt{1-t^2}} =$$

$$= \arcsin t + c = \arcsin\left(\frac{2\sqrt{13}}{3}\left(x+\frac{1}{2}\right)\right) + c$$

Интеграција рационалних функција

Рационална функција је функција облика $\frac{P(x)}{Q(x)}$, где су P, Q полиноми.
Свака рационална функција $\frac{P(x)}{Q(x)}$ се може записати у облику

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{\prod_{i=1}^{k}(x-x_i)^{n_i}\prod_{j=1}^{s}(a_j x^2+b_j x+g_j)^{m_j}}$$

где полиноми $a_j x^2+b_j x+g_j$ немају реалне нуле. Оваква функција се може раставити на

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{k}\left(\frac{A_{n_i}^{(i)}}{(x-x_i)^{n_i}}+\frac{A_{n_i-1}^{(i)}}{(x-x_i)^{n_i-1}}+\dots+\frac{A_1^{(i)}}{x-x_i}\right)+$$

$$+\sum_{j=1}^{s}\left(\frac{B_{m_j}^{(j)}x+C_{m_j}^{(j)}}{(a_j x^2+b_j x+g_j)^{m_j}}+\frac{B_{m_j-1}^{(j)}x+C_{m_j-1}^{(j)}}{(a_j x^2+b_j x+g_j)^{m_j-1}}+\dots+\frac{B_1^{(j)}x+C_1^{(j)}}{a_j x^2+b_j x+g_j}\right).$$

Коефицијенте $A_k^{(i)}, B_e^{(j)}, C_e^{(j)}$ налазимо методом неодређених коефицијената.
На крају, интеграл $\int\frac{P(x)}{Q(x)}dx$ рачунамо тако што израчунамо интеграле свих сабирака са десне стране једнакости, и саберемо их.

Коментар: Ако је степен полинома $P(x)$ већи или једнак од степена полинома $Q(x)$, тада постоје полиноми $q(x), r(x)$, $\deg r(x) < \deg Q(x)$, такви да је $P(x) = q(x)Q(x)+r(x)$.
Одавде је $\frac{P(x)}{Q(x)} = q(x)+\frac{r(x)}{Q(x)}$, па је $\int\frac{P(x)}{Q(x)}dx = \int q(x)dx+\int\frac{r(x)}{Q(x)}dx$

Израчунати:

1. $\int\frac{dx}{x^2-4} = *$

$x^2-4 = (x-2)(x+2)$

$$\frac{1}{x^2-4} = \frac{A_1}{x-2}+\frac{A_2}{x+2}$$

Нађимо A_1 и A_2.

$$\frac{1}{x^2-4} = \frac{A_1(x+2)+A_2(x-2)}{(x-2)(x+2)}$$

$$\frac{1}{x^2-4}=\frac{A_1x+2A_1+A_2x-2A_2}{x^2-4}$$

$$\frac{1}{x^2-4}=\frac{(A_1+A_2)x+2A_1-2A_2}{x^2-4}$$

$$1=(A_1+A_2)x+2A_1-2A_2 \Rightarrow \begin{cases} A_1+A_2=0 \Rightarrow A_2=-A_1 \\ 2A_1-2A_2=1 \end{cases}$$

$$A_1+A_2=0$$

$$4A_1=1,\ A_1=\frac{1}{4},\ A_2=-\frac{1}{4}$$

Дакле $\frac{1}{x^2-4}=\frac{1}{4}\frac{1}{x-2}-\frac{1}{4}\frac{1}{x+2}$, па је

$$*=\int\frac{dx}{x^2-4}=\frac{1}{4}\int\frac{dx}{x-2}-\frac{1}{4}\int\frac{dx}{x+2}=\frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+c=\ln\left|\frac{x-2}{x+2}\right|^{\frac{1}{4}}+c$$

2. $\int\frac{6x+10}{x^2+2x-3}dx=\sim$

$x^2+2x-3=0$, $x_{1,2}=\frac{-2\pm\sqrt{4+12}}{2}$, $x_{1,2}=\frac{-2\pm4}{2}$, $x_1=-3, x_2=1$

$x^2+2x-3=(x+3)(x-1)$

$$\frac{6x+10}{x^2+2x-3}=\frac{A_1}{x+3}+\frac{A_2}{x-1}$$

$$\frac{6x+10}{x^2+2x-3}=\frac{A_1(x-1)+A_2(x+3)}{(x+3)(x-1)}$$

$$\frac{6x+10}{x^2+2x-3}=\frac{A_1x-A_1+A_2x+3A_2}{x^2+2x-3}$$

$$\frac{6x+10}{x^2+2x-3}=\frac{(A_1+A_2)x-A_1+3A_2}{x^2+2x-3}$$

$$6x+10=(A_1+A_2)x-A_1+3A_2 \Rightarrow \begin{cases} A_1+A_2=6 \\ -A_1+3A_2=10 \end{cases} + \Rightarrow \begin{cases} A_1+A_2=6 \\ 4A_2=16 \end{cases} \Rightarrow \begin{matrix} A_1=2 \\ A_2=4 \end{matrix}$$

Дакле, $\frac{6x+10}{x^2+2x-3}=\frac{2}{x+3}+\frac{4}{x-1}$, па је

$$\sim=\int\frac{6x+10}{x^2+2x-3}dx=2\int\frac{dx}{x+3}+4\int\frac{dx}{x-1}=2\ln|x+3|+4\ln|x-1|+c=\ln(x+3)^2(x-1)^4+c$$

3. $\int \frac{dx}{x^3+1} = *$

$x^3+1 = (x+1)(x^2-x+1)$

$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$\frac{1}{x^3+1} = \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}$

$\frac{1}{x^3+1} = \frac{Ax^2-Ax+A+Bx^2+Bx+Cx+C}{x^3+1}$

$\frac{1}{x^3+1} = \frac{(A+B)x^2+(-A+B+C)x+C+A}{x^3+1}$

$1 = (A+B)x^2+(-A+B+C)x+C+A \Rightarrow \begin{cases} A+B=0 \\ -A+B+C=0 \\ C+A=1 \end{cases} \Rightarrow \begin{cases} A+B=0 \\ 2B+C=0 \\ C-B=1 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} A=-B \\ C=-2B \\ -3B=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{3} \\ C=\frac{2}{3} \\ B=-\frac{1}{3} \end{cases}$

Дакле, $\frac{1}{x^3+1} = \frac{1}{3}\frac{1}{x+1} + \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1} = \frac{1}{3}\frac{1}{x+1} - \frac{1}{3}\frac{x-2}{x^2-x+1}$, па је

$* = \int\frac{dx}{x^3+1} = \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{3}\int\frac{x-2}{x^2-x+1}dx = \frac{1}{3}\ln|x+1| - \frac{1}{3}I_1 = **$

$I_1 = \int\frac{x-2}{x^2-x+1}dx = \int\frac{x-2}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}dx = \int\frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}dx = \int\frac{x-\frac{1}{2}-\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx =$

$= \int\frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx - \frac{3}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} = I_2 - \frac{3}{2}I_3 = \gamma$

$I_2 = \int\frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx = \left[\begin{array}{l} (x-\frac{1}{2})^2+\frac{3}{4}=t \\ 2(x-\frac{1}{2})dx=dt \\ (x-\frac{1}{2})dx=\frac{dt}{2} \end{array}\right] = \int\frac{\frac{dt}{2}}{t} = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\ln|t|+c =$

$= \frac{1}{2}\ln\left|(x-\frac{1}{2})^2+\frac{3}{4}\right|+c$

$I_3 = \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2\sqrt{3}}{3}(x-\frac{1}{2})\right)+c$ (зад. 11)

$\gamma = I_1 = \frac{1}{2}\ln\left|(x-\frac{1}{2})^2+\frac{3}{4}\right| + \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2\sqrt{3}}{3}(x-\frac{1}{2})\right)+c$

$** = \int\frac{dx}{x^3+1} = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left|(x-\frac{1}{2})^2+\frac{3}{4}\right| - \frac{2\sqrt{3}}{9}\operatorname{arctg}\left(\frac{2\sqrt{3}}{3}(x-\frac{1}{2})\right)+c$

4. $\displaystyle\int \frac{x\,dx}{(x-1)^2(x^2+2x+2)} = *$

$$\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{Bx+C}{x^2+2x+2}$$

$$\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{A_1(x-1)(x^2+2x+2)+A_2(x^2+2x+2)+(Bx+C)(x-1)^2}{(x-1)^2(x^2+2x+2)}$$

$$\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{A_1(x^3+2x^2+\cancel{2x}-x^2-\cancel{2x}-2)+A_2(x^2+2x+2)+(Bx+C)(x^2-2x+1)}{(x-1)^2(x^2+2x+2)}$$

$$\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{A_1x^3+A_1x^2-2A_1+A_2x^2+2A_2x+2A_2+Bx^3-2Bx^2+Bx+Cx^2-2Cx+C}{(x-1)^2(x^2+2x+2)}$$

$$\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{(A_1+B)x^3+(A_1+A_2-2B+C)x^2+(2A_2+B-2C)x+(-2A_1+2A_2+C)}{(x-1)^2(x^2+2x+2)}$$

$$x=(A_1+B)x^3+(A_1+A_2-2B+C)x^2+(2A_2+B-2C)x+(-2A_1+2A_2+C) \Rightarrow$$

$$\Rightarrow \begin{cases} A_1+B=0 \\ A_1+A_2-2B+C=0 \\ 2A_2+B-2C=1 \\ -2A_1+2A_2+C=0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 1 & -2 & 1 \\ -2 & 2 & 0 & 1 & 0 \end{array}\right) (-1)\ 2 \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 2 & 1 & -2 & 1 \\ 0 & 2 & 2 & 1 & 0 \end{array}\right) (-2)(-2) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 7 & -4 & 1 \\ 0 & 0 & 8 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & -1 & 8 & 0 \\ 0 & 0 & -4 & 7 & 1 \end{array}\right) (-4) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & -1 & 8 & 0 \\ 0 & 0 & 0 & -25 & 1 \end{array}\right)$$

$A_1+B=0 \Rightarrow A_1=\frac{1}{25}$

$A_2+C-3B=0 \Rightarrow A_2=\frac{1}{5}$

$-C+8B=0 \Rightarrow C=-\frac{8}{25}$

$-25B=1 \Rightarrow B=-\frac{1}{25}$

Дакле $\displaystyle\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{1}{25}\,\frac{1}{x-1}+\frac{1}{5}\,\frac{1}{(x-1)^2}+\frac{-\frac{1}{25}x-\frac{8}{25}}{x^2+2x+2} = \frac{1}{25}\,\frac{1}{x-1}+\frac{1}{5}\,\frac{1}{(x-1)^2}-\frac{1}{25}\,\frac{x+8}{x^2+2x+2}$

па је

$$* = \int\frac{x\,dx}{(x-1)^2(x^2+2x+2)} = \frac{1}{25}\int\frac{dx}{x-1}+\frac{1}{5}\int\frac{dx}{(x-1)^2}-\frac{1}{25}\int\frac{x+8}{x^2+2x+2}\,dx =$$

$$= \frac{1}{25}\ln|x-1| + \frac{1}{5}I_1 - \frac{1}{25}I_2 = **$$

$$I_1 = \frac{1}{5}\int\frac{dx}{(x-1)^2} = \begin{bmatrix} x-1=t \\ dx=dt \end{bmatrix} = \frac{1}{5}\int\frac{dt}{t^2} = \frac{1}{5}\int t^{-2}dt = \frac{1}{5}\,\frac{t^{-1}}{-1}+c = -\frac{1}{5}\,\frac{1}{x-1}+c$$

$$I_2 = \int\frac{x+8}{x^2+2x+2}\,dx = \int\frac{x+8}{(x+1)^2+1}\,dx = \int\frac{x+1+7}{(x+1)^2+1}\,dx = \int\frac{x+1}{(x+1)^2+1}\,dx + 7\int\frac{dx}{(x+1)^2+1} = I_3+7I_4$$

[redacted]

$$I_3 = \int \frac{x+1}{(x+1)^2+1}\,dx = \left[\begin{array}{l}(x+1)^2+1=t \\ 2(x+1)\,dx=dt \\ (x+1)\,dx=\frac{dt}{2}\end{array}\right] = \int \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\ln|t| + c =$$

$$= \frac{1}{2}\ln|(x+1)^2+1| + c$$

$$I_4 = \int \frac{dx}{(x+1)^2+1} = \left[\begin{array}{l}x+1=t \\ dx=dt\end{array}\right] = \int \frac{dt}{1+t^2} = \operatorname{arctg} t + c = \operatorname{arctg}(x+1) + c$$

$$** = \int \frac{x}{(x-1)^2(x^2+2x+2)}\,dx = \frac{1}{25}\ln|x-1| - \frac{1}{25}\,\frac{1}{x-1} - \frac{1}{50}\ln|x^2+2x+2| - \frac{7}{25}\operatorname{arctg}(x+1) + c$$

Интеграција ирационалних функција

Израчунати

1. $\displaystyle\int \frac{dx}{x(1-2\sqrt{x}+\sqrt[3]{x})} = \left| \begin{matrix} x=t^6 \\ dx=6t^5dt \end{matrix} \right| = \int \frac{6\cancel{t^5}\,dt}{t^{\cancel{6}}(1-2\sqrt{t^6}+\sqrt[3]{t^6})} =$

$\displaystyle = 6\int \frac{dt}{t(1-2t^3+t^2)} = *$

Посматрајмо полином $R(x)=1-2x^3+x^2+1$

$R(1)=-2+1+1=0 \Rightarrow 1$ је нула полинома $R(x)$

$R(x):(x-1)=(-2x^3+x^2\quad+1):(x-1)=-2x^2-x-1$

$\mp 2x^3 \pm 2x^2$

$// \quad -x^2$

$\mp x^2 \pm x$

$// \quad -x+1$

$\mp x \pm 1$

$// \quad //$

$R(x)=(x-1)(-2x^2-x-1)$

$-2x^2-x-1=0 \Rightarrow x_{1,2}=\dfrac{1\pm\sqrt{1-8}}{-4} \notin \mathbb{R} \Rightarrow$ полином $(-2x^2-x-1)$ нема реалних нула $\Rightarrow$

$\displaystyle \Rightarrow * = 6\int \frac{dt}{t(t-1)(-2t^2-t-1)} = \ldots$ (сами)

2. $\displaystyle\int \frac{x^3}{\sqrt{1+2x-x^2}}\,dx = (Ax^2+Bx+C)\sqrt{1+2x-x^2} + D\int\frac{dx}{\sqrt{1+2x-x^2}}$

Диференцирајући овај израз добијамо

$$\frac{x^3}{\sqrt{1+2x-x^2}} = (2Ax+B)\sqrt{1+2x-x^2} + (Ax^2+Bx+C)\frac{1}{\cancel{2}\sqrt{1+2x-x^2}}\cdot\overset{\cancel{2}(1-x)}{(2-2x)} + D\cdot\frac{1}{\sqrt{1+2x-x^2}}$$

$$\frac{x^3}{\sqrt{1+2x-x^2}} = \frac{(2Ax+B)(1+2x-x^2)+(Ax^2+Bx+C)(1-x)+D}{\sqrt{1+2x-x^2}}$$

$x^3=(-2A-A)x^3+(4A-B+A-B)x^2+(2A+2B+B-C)x+B+C+D$

$-3A=1 \Rightarrow A=-\frac{1}{3}$

$5A-2B=0 \Rightarrow B=-\frac{5}{6}$

$2A+3B-C=0 \Rightarrow C=-\frac{19}{3}$

$B+C+D=0 \Rightarrow D=4$

Дакле, добијамо

$$\int \frac{x^3\,dx}{\sqrt{1+2x-x^2}} = \left(-\frac{1}{3}x^2 - \frac{5}{6}x - \frac{19}{3}\right)\sqrt{1+2x-x^2} + 4\overbrace{\int \frac{dx}{\sqrt{1+2x-x^2}}}^{I_1}$$

$$I_1 = \int \frac{dx}{\sqrt{1+2x-x^2}} = \int \frac{dx}{\sqrt{1-(x^2-2x)}} = \int \frac{dx}{\sqrt{1-(x^2-2x+1-1)}} = \int \frac{dx}{\sqrt{2-(x-1)^2}} =$$

$$= \int \frac{dx}{\sqrt{2}\sqrt{1-\left(\frac{x-1}{\sqrt{2}}\right)^2}} = \left[\begin{array}{l} \frac{x-1}{\sqrt{2}} = t \\ \frac{dx}{\sqrt{2}} = dt \end{array}\right] = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + c =$$

$$= \arcsin \frac{x-1}{\sqrt{2}} + c$$

па је

$$\int \frac{x^3\,dx}{\sqrt{1+2x-x^2}} = \left(-\frac{1}{3}x^2 - \frac{5}{6}x - \frac{19}{3}\right)\sqrt{1+2x-x^2} + 4\arcsin\frac{x-1}{\sqrt{2}} + c$$