30.10.18.

PREDAVANJA 6

data jna i oblast egzistencije → napisati barem jedan Košijev zadatak npr.

Nehomogena lin. dif. jna

$L(y) = y^{(n)} + a_1(x) \cdot y^{(n-1)} + \dots + a_n(x) y = b(x)$

→ posmatramo jnu $L(y) = b(x)$

$b(x) \neq 0$

T: Neka je $\varphi_H(x)$ opšte rješenje odgovarajuće homogene dif. jednačine. $L(y) = 0$, i neka je $\varphi_p(x)$ neko partikularno rješenje jne $L(y) = b(x)$

Tada je opšte rješenje j-ne $L(y) = b(x)$: $\varphi_H + \varphi_p$

u homogenom rješenju imamo n const.

→ Sad je pitanje kako naći rješenje

METOD VARIJACIJE KONSTANTI

→ prvo tražimo homogeno rješenje:

$L(y) = 0$

$y_H = c_1 \varphi_1(x) + \dots + c_n \varphi_n(x)$

→ rješavamo $L(y) = b(x)$

→ pretpostavljamo da c nisu konstante već da zavise od x:

$y_p = c_1(x) \cdot \varphi_1 + \dots + c_n(x) \cdot \varphi_n$

→ ovo rješenje ubacujemo u jnu $L(y) = b(x)$

$$y' = \boxed{\underbrace{c_1'(x)\varphi_1(x) + \dots + c_n'(x)\varphi_n(x)}} +$$

$$+ \; c_1(x)\varphi_1'(x) + \dots + c_n(x)\varphi_n'(x)$$

→ pretpostavimo $=0$

$$y'' = \overbrace{c_1'(x)\cdot\varphi_1'(x) + \dots + c_n'(x)\varphi_n'(x)}^{=0} +$$

$$+ \; c_1(x)\cdot\varphi_1''(x) + \dots + c_n(x)\varphi''(x)$$

→ pretpostavljamo da je ovo gore $=0$

$\vdots$

$$y^{(n-1)} = \overbrace{c_1'(x)\cdot\varphi_1^{(n-2)}(x) + \dots + c_n'(x)\varphi_n^{(n-2)}(x)}^{=0} +$$

$$+ \; c_1(x)\varphi_1^{(n-1)}(x) + \dots + c_n(x)\varphi_n^{(n-1)}(x)$$

$$y^{(n)} = c_1'(x)\varphi_1^{(n-1)}(x) + \dots + c_n'(x)\cdot\varphi_n^{(n-1)}(x) +$$

$$+ \; c_1(x)\cdot\varphi_1^{(n)}(x) + \dots + c_1(x)\varphi_n^{(n)}(x)$$

→ ovdje ne smijemo pretpostaviti da je ovo gore $=0$

→ vratimo se sa ovim u 1-ju

$$L(y) = y^{(n)} + a_1(x)\cdot y^{(n-1)} + \dots + a_n(x)\cdot y = b(x)$$

→ $L(y) = b(x)$:

$$\underbrace{c_1'(x)\cdot\varphi_1^{(n-1)}(x) + \dots + c_n'(x)\cdot\varphi_n^{(n-1)}(x) + c_1(x)\cdot\varphi_1^{(n)}(x) + \dots + c_n(x)\varphi_n^{(n)}(x)}_{y^{(n)}} +$$

$$\underbrace{+ \; a_1(x)\left(c_1(x)\varphi_1^{(n-1)}(x) + \dots + c_n(x)\cdot\varphi_n^{(n-1)}(x)\right)}_{a_1\cdot y^{(n-1)}} +$$

$$\dots + \underbrace{a_n(x)\left(c_1(x)\varphi_1^{(n-1)}(x) + \dots + c_n(x)\varphi_n^{(n-1)}(x)\right)}_{a_n\cdot y} = b(x)$$

$$C_1'(x)\cdot \varphi_1^{(n-1)}(x) + \ldots + C_n'(x)\,\varphi_n^{(n-1)}(x) +$$

$$+ C_1(x)\Big[\overbrace{\boxed{\varphi_1^{(n)}(x) + a_1(x)\varphi_1^{(n-1)}(x) + \ldots + a_n(x)\varphi_1(x)}}^{L(\varphi_1(x))=0}\Big] +$$

$$\ldots + C_n(x)\underbrace{\left(\varphi_n^{(n)}(x) + \ldots + a_n(x)\varphi_n(x)\right)}_{L(\varphi_n)=0}$$

$\rightarrow$ pretpostaviti da je rj. $\varphi_1, \ldots, \varphi_n$;

ako je to rj; postavimo da je:

$$C_1'(x)\varphi_1^{(n-1)}(x) + \ldots + C_n'(x)\cdot\varphi_n^{(n-1)}(x) = b(x)$$

$\rightarrow$ postaviti uslove koje moraju da zadovoljavaju $C_1, \ldots, C_n$

$$\begin{cases} C_1'(x)\varphi_1(x) + \ldots + C_n'(x)\varphi_n(x) = 0 \\ C_1'(x)\varphi_1'(x) + \ldots + C_n'(x)\varphi_n'(x) = 0 \\ \vdots \\ C_1'(x)\varphi_1^{(n-2)}(x) + \ldots + C_n'(x)\varphi_n^{(n-2)}(x) = 0 \\ C_1'(x)\varphi_1^{(n-1)}(x) + \ldots + C_n'(x)\varphi_n^{(n-1)}(x) = b(x) \end{cases}$$

da bi bilo rješenje $C_1, \ldots, C_n$ moraju ovo da zadovoljavaju

$W(x) \neq 0 \Rightarrow \exists!$ rješ. sist.

$$C_i' = \frac{W_i(x)}{W(x)}; \quad i = 1, \ldots, n$$

$\rightarrow$ Kramerovo pravilo $\rightarrow$ i-tu kolonu zamijenimo kolonom slob. članova

$\rightarrow$ dobili smo dif. j-nu sa razdv. promj.

$$C_i(x) = \int \frac{W_i(x)}{W(x)}\,dx + r_i, \quad i = 1, \dots, n$$

$$y = C_1(x)\varphi_1(x) + \dots + C_n(x)\varphi_n(x) =$$

$$= \left(\int \frac{W_1(x)}{W(x)}\,dx + r_1\right)\cdot\varphi_1(x) + \dots + \left(\int \frac{W_n(x)}{W(x)}\,dx + r_n\right)\cdot\varphi_n(x) =$$

$$= \underbrace{r_1\varphi_1(x) + \dots + r_n\varphi_n(x)}_{y_H \ (1)} + \underbrace{\varphi_1(x)\cdot\int \frac{W_1(x)}{W(x)}\,dx + \dots + \varphi_n(x)\int \frac{W_n(x)}{W(x)}\,dx}_{y_p}$$

$$L(y) = \underbrace{P_m(x)\cdot e^{r x}}_{b(x)}$$

$\to y^{(n)} + a_1 y^{(n-1)} + \dots + a_n \cdot y$ $\to$ najčešće se radi o konst. koef.

$\to$ kada je $b(x)$ ovog oblika $\to$ $b(x)$ je kvazipolinom

$L(y) = 0$; imamo polinom

$P(\lambda) = 0 \to \lambda_1, \lambda_2, \dots, \lambda_n$

$y_h = \dots$

y_p tražimo na sl. način:

ako je $r \in \{\lambda_1, \dots, \lambda_n\} \to s$ - višestrukost r kao korijena karak. polinoma.

tada je $y_p = x^s \cdot Q_m(x) \cdot e^{r x}$ → samo ovo nepoznato

$\to$ ovo sad zamijenimo u $\int$ nu:

$$L(y) = P_m(x)\cdot e^{r x}$$

$\to r$ u opštem slučaju $\in \mathbb{C}$

ali mi razdvajamo

$r = \alpha + i\beta$

$b(x) = e^{\alpha x}\left(P_{1m}(x)\cos\beta x + P_{2\ell}(x)\sin\beta x\right)$

$s \to$ višestrukost korjena $\alpha \pm i\beta$ karakt. polinoma

$y_p = x^s \cdot e^{\alpha x}\left(Q_{1k}(x)\cos\beta x\right) + Q_{2k}(x)\sin\beta x)$

$k = \max\{m, \ell\}$

$L(y) = P_{1m}(x)\cdot e^{r_1 x} + \dots + P_{s m_s}(x) e^{r_s x}$

$$\boxed{y_p = y_{p_1} + \dots + y_{p_s}}$$

y_{p_1}: $L(y) = P_{1m_1}(x) e^{r_1 x}$

$\vdots$

$L(y) = P_{s m_s}(x)\cdot e^{r_s x}$

$\boxed{\text{# Primjer}}$ nehomogena dif. j-na sa konst. koef:

$$\boxed{y'' - 2y' + y = \frac{e^x}{x}, \quad x > 0}$$

$y'' - 2y' + y = 0$

$P(\lambda) = \lambda^2 - 2\lambda + 1 = 0 \to (\lambda - 1)^2 = 0$; $\lambda_1 = \lambda_2 = 1$;

$s = 1$

$\lambda = 1 \to r = 1 \to \varphi_1(x) = e^x$

$\varphi_2(x) = xe^x$

$y_H = C_1 e^x + C_2 x e^x$

part. rj. tražimo tako što pretpostavljamo:

$$\begin{cases} C_1'(x)\cdot e^x + C_2'(x)\cdot xe^x = 0 \\ C_1'(x)\underbrace{e^x}_{\text{izvod } e^x} + C_2'(x)\underbrace{e^x(x+1)}_{\text{izvod } xe^x} = \dfrac{e^x}{x} \end{cases}$$

$$(1)\quad \left.\begin{aligned} &C_1'(x) + C_2'(x)\cdot x = 0 \\ &C_1'(x) + C_2'(x)(x+1) = \frac{1}{x} \end{aligned}\right\}$$

$$C_2'(x) = \frac{1}{x}$$

$$C_2(x) = \int \frac{1}{x}\,dx = \ln x + \mathcal{C}_2$$

zamijenimo u (1)

$$\longrightarrow C_1'(x) = -x C_2'(x)$$

$$C_1'(x) = -1 \longrightarrow C_1(x) = -x + \mathcal{C}_1$$

$$y = (-x + \mathcal{C}_1)e^x + (\ln x + \mathcal{C}_2)xe^x$$

$$y = \underbrace{\mathcal{C}_1 e^x + \mathcal{C}_2 xe^x}_{y_H} \; \underbrace{- xe^x + xe^x \ln x}_{y_p}$$

$\boxed{\# \text{ Primjer}}$ $\quad y'' - 4y' + 8y = \underbrace{e^{2x} + \sin 2x}_{\text{zbir kvazi polinoma}}$

homogeno rješenje:

$$y'' - 4y' + 8y = 0\,; \qquad P(\lambda) = \lambda^2 - 4\lambda + 8 = 0$$

$$\lambda_{1,2} = 2 \pm 2i$$

$$y_H = C_1 e^{2x}\cos 2x + C_2 e^{2x}\sin^{2x} = e^{2x}(C_1\cos 2x + C_2 \sin 2x)$$

$$y_p = y_{p_1} + y_{p_2}$$

$\longrightarrow$ za part. rj. 1 $\longrightarrow$ rješavamo j-nu:

$$y_{p_1}: \quad y'' - 4y' + 8y = e^{2x}$$

$$2 \notin \{2 \pm 2i\} \Rightarrow S = 0$$

$$y_{p_1} = x^0 e^{2x}\cdot a = ae^{2x}$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\hookrightarrow$ polinom nultog stepena konstantu dobijamo kad zamijenimo y_{p_1} - - -

$y_{p_1}' = 2ae^{2x}$; $y_{p_1}'' = 4ae^{2x}$

$4ae^{2x} - 4\cdot 2\cancel{ae^{2x}} + 8\cancel{ae^{2x}} = e^{2x} \quad / : e^{2x}$

$4a = 1 \longrightarrow a = \frac{1}{4} \longrightarrow \boxed{y_{p_1} = \frac{e^{2x}}{4}}$

y_{p_2}: $\quad y'' - 4y' + 8y = \underbrace{\sin 2x}$ → polinom nultog stepena; $\cos 2x$ → konst je 0 ali ne smijemo sin u rješ. zaboraviti

$\alpha + i\beta = 2i \notin \{2 \pm 2i\} \Rightarrow s = 0;$

$y_{p_2} = (a\cos 2x + b\sin 2x) \cdot \underset{1}{x^s} \cdot \underset{1}{e^{\alpha x}}$

$y_{p_2} = a\cos 2x + b\sin 2x$

$y_{p_2}' = -2a\sin 2x + 2b\cos 2x$

$y_p'' = -4a\cos 2x - 4b\sin 2x$; zamijenimo sve u poč. j-nu

$\rightarrow -4a\cos 2x - 4b\sin 2x - 4(-2a\sin 2x + 2b\cos 2x) +$

$+ 8a\cos 2x + 8b\sin 2x = \sin 2x$

$\cos 2x(-4a - 8b + 8a) + \sin 2x(-4b + 8a + 8b) = \sin 2x$

$\rightarrow \begin{matrix} 4a - 8b = 0 \\ 8a + 4b = 1 \end{matrix} \rightarrow a = 2b \rightarrow \boxed{\begin{matrix} b = \frac{1}{20} \\ a = \frac{1}{10} \end{matrix}}$

$y_{p_2} = \frac{1}{10}\cos 2x + \frac{1}{20}\sin 2x$

$y = y_h + y_{p_1} + y_{p_2} =$

$= e^{2x}(C_1\cos 2x + C_2\sin 2x) + \frac{e^{2x}}{4} +$

$+ \frac{1}{10}\cos 2x + \frac{1}{20}\sin 2x$

Snižavanje reda lin. dif. j-ne

→ kada su nam poznata part. rj možemo sniziti

$L(y)=0 \quad (1)$

→ neka su $\varphi_1(x), \dots, \varphi_m(x)$ linearno nezavisna rješenja ove j-ne (1) $(m<n)$

n lin. nez. rješ.

→ tražimo još $n-m$:

$\varphi_{m+1}(x) \dots \varphi_n(x) \rightarrow$? rješenja

$$\underbrace{\varphi_1(x), \dots, \varphi_m(x), \varphi_{m+1}(x), \dots, \varphi_n(x)}_{\text{baza vektorskog prostora}}$$

$P_n(\lambda)=0 \rightarrow$ ako znamo m korijena

$$\frac{P_n(\lambda)}{(\lambda-\lambda_1)\dots(\lambda-\lambda_m)} = Q_{n-m}(\lambda) = 0$$

→ uvodimo smjenu:

$y = \varphi_1 \cdot z$; $\boxed{z = z(x)} \rightarrow$ nova fja

$y' = \varphi_1' z + \varphi_1 z'$

$y'' = \varphi_1'' z + 2\varphi_1' z' + \varphi_1 z''$

$y''' = \varphi_1''' z + 3\varphi_1'' z' + 3\varphi_1' z'' + \varphi_1 z'''$

PASKALOV TROUGAO

$\vdots$

$$y^{(n)} = \sum \binom{n}{k} \varphi_1^{(k)} \cdot z^{(n-k)}$$

$L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y = 0$

$$\sum_{k=0}^{n} \binom{n}{k} \varphi_1^{(k)} \cdot z^{(n-k)} + a_1(x) \sum_{k=0}^{n-1} \binom{n-k}{k} \varphi_1^{(k)} \cdot z^{(n-1-k)} + \dots$$

$$+ \dots + a_n(x) \cdot \varphi_n \cdot z = 0$$

$$\underbrace{}_{L(\varphi_1)=0}$$

$$\to z\,\boxed{\left(\varphi_1^{(n)} + a_1(x)\cdot\varphi_1^{(n-1)} + \ldots + a_n(x)\cdot\varphi_1\right)} +$$

$$+\varphi_1\, z^{(n)} + b_1(x)\cdot z^{(n-1)} + \ldots + b_{n-1}(x)\cdot z' = 0 \;\Big/ : \varphi_1 \neq 0$$

jer je dio lin. nez. skupa

$\to$ imamo $z', z'', \ldots, z^{(n)}$

$\to$ uvodimo smjenu: $z' = u \to$

$$\to u^{(n-1)} + p_1(x)\cdot u^{(n-2)} + \ldots + p_{n-1}(x)u = 0$$

---

znamo $n-m$ rješenja

$$\underbrace{\left(\frac{\varphi_2}{\varphi_1}\right)', \left(\frac{\varphi_3}{\varphi_1}\right)', \ldots, \left(\frac{\varphi_m}{\varphi_1}\right)'}_{\text{ima izvedeno u knjizi da su nezavisna; odraditi kući;}} \to \text{lin nez. rješ.}$$

$$y = \varphi_1 \cdot z \to z' = \left(\frac{y}{\varphi_1}\right)' = u$$

ponavlja se ... $\to$ sve dok ne dobijemo j-nu reda $\boxed{n-m}$

$n=2$: $\quad y'' + a_1(x)\cdot y' + a_2(x)\cdot y = 0$

$a_1, a_2 \in C(I)$; $\varphi_1(x)$ rješ;

$$\left.\begin{array}{l}\varphi_1(x) \to \text{rješ}\\ \varphi_2(x) \to ?\end{array}\right\}\ \text{linearno nezavis.}$$

$$W(x) = \begin{vmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{vmatrix} = \varphi_1\varphi_2' - \varphi_1'\varphi_2 = 0$$

φ_1 rješ: $\quad \varphi_1'' + a_1(x)\varphi_1' + a_2(x)\varphi_1 = 0 \;/\varphi_2$

φ_2 rješ: $\quad \varphi_2'' + a_1(x)\varphi_2' + a_2(x)\varphi_2 = 0 \;/\varphi_1$

$$\rightarrow \underbrace{\varphi_1'' - \varphi_2 - \varphi_2''\varphi_1}_{-W'(x)} + a_1(x)\cdot\underbrace{(\varphi_1'\varphi_2 - \varphi_1\varphi_2')}_{-W(x)} = 0$$

$$W'(x) = \varphi_1\varphi_2' + \varphi_1\varphi_2'' - \varphi_1'\varphi_2' - \varphi_1''\varphi_2 = \varphi_1\varphi_2'' - \varphi_1''\varphi_2$$

$$\boxed{W'(x) + a_1(x)\,W(x) = 0}$$

$$\frac{dW}{W} = -a_1(x)\,dx \rightarrow W(x) = Ce^{-\int a_1(x)\,dx}$$

$$\varphi_1\varphi_2' - \varphi_1'\varphi_2 = Ce^{-\int a_1(x)\,dx}$$

$\rightarrow = 1$ jer nama treba samo jedno

dif. jna po φ_2 ... $\rightarrow \varphi_2$

$$\rightarrow \boxed{y = c_1\varphi_1 + c_2\varphi_2} \rightarrow \text{opšte rj.}$$

2. način

$$y'' + a_1(x)\cdot y' + a_2(x)\cdot y = 0$$

$\rightarrow$ koef. uz prvi izvod $=0$; namještamo

$y = \varphi\cdot z \rightarrow$ nove fje

$$y' = \varphi' z + \varphi z'$$

$$y'' = \varphi'' z + 2\varphi' z' + \varphi z''$$

$\rightarrow$ zamijenimo sve u početnu jnu i dobijamo

$$\varphi'' z + 2\varphi' z' + \varphi z'' + a_1(x)\varphi' z + a_1(x)\varphi z' + a_2(x)\varphi z = 0$$

$$\varphi z'' + (2\varphi' + a_1(x)\cdot\varphi)z' + (\varphi'' + a_1(x)\varphi' + a_2(x)\varphi)z = 0$$

$\rightarrow$ postavljamo da je koef. uz 1. izvod $=0$ $\rightarrow$

$$\rightarrow \boxed{2\varphi' + a_1(x)\varphi = 0}$$

$$2\varphi' = -a_1(x)\varphi$$

$$\to \frac{d\varphi}{\varphi} = -\frac{1}{2}a_1(x)\,dx \to \varphi = e^{-\int \frac{a_1(x)}{2}dx}$$

SISTEMI DIF. J-NA

→ radimo sisteme 1. reda;

→ u opštem obliku sistem prvog reda je:

$$(1)\quad \begin{cases} y_1' = f_1(x, y_1, \dots, y_n) \\ \vdots \\ y_n' = f_n(x, y_1, \dots, y_n) \end{cases} \to \text{normalni oblik}$$

$y_1, \dots, y_n \to$ nepoznate

$f_1, \dots, f_n \to$ definisane u $G \subset \mathbb{R}^{n+1}$

→ prelazimo na matrični oblik:

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} ; \quad F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \to Y' = F(x, Y) \quad (2)$$

Definicija $y_1 = \varphi_1(x), y_2 = \varphi_2(x), \dots, y_n = \varphi_n(x)$ su rješenja j-ne (1) ako:

1) $\varphi_i \in C'(I)$

2) $\forall x \in I \; (x, \varphi_1(x), \dots, \varphi_n(x)) \in G$

3) $\forall x \in I; \quad \varphi_i' = f_i(x, \varphi_1(x), \dots, \varphi_n(x)),$

$i = 1, 2, \dots, n$

→ Dova def. u matričnom obliku

Koš. zad. prije $y(x_0) = y_0$

→ sad: $\underbrace{Y(x_0) = Y_0}_{(4)} \Leftrightarrow \underbrace{y_1(x_0) = y_0^1 \quad y_2(x_0) = y_0^2 \quad \dots \quad y_n(x_0) = y_0^n}_{(3)}$

$$Y_0 = \begin{pmatrix} y_0^1 \\ y_0^2 \\ \vdots \\ y_0^n \end{pmatrix}$$

(4) K. Z (3)

Teoreme o egzistenciji i jedinstvenosti:

T(PEANO): $F \in C(D)$, $(x_0, Y_0) \in D \Rightarrow$ K. Z (2)-(4) ima rješenje

→ $f_1, \dots, f_n \in C(D)$, $(x_0, y_0^1, \dots, y_0^n) \in D \Rightarrow$
$\Rightarrow$ K. Z (1)-(3) ima rj.

T2: $F, \dfrac{\partial F}{\partial y_i} \in C(D)$, $(x_0, Y_0) \in D$

$\Rightarrow$ K. Z. (2)-(4) ima jedinstveno rješenje

T3: PIKAR: Neka su ispunjeni uslovi:

1) $F \in C(D)$

2) $\exists k > 0 \rightarrow \left| \dfrac{\partial F}{\partial y_i} \right| \le k$

3) $(x_0, Y_0) \in D$

→ Tada K. Z (2)-(4) ima jedinstveno rješenje

2) $\dfrac{\partial f_1}{\partial y_i}$; $\dfrac{\partial f_2}{\partial y_i}$, $\dots$, $\dfrac{\partial f_n}{\partial y_i}$

POSEBAN SLUČAJ ZA LINEARNE

→ sistem glasi:

$$\begin{cases} y_1' = a_{11}(x)y_1 + \dots + a_{1n}(x)y_n + b_1(x) \\ \vdots \\ y_n' = a_{n1}(x)y_1 + \dots + a_{nn}(x)y_n + b_n(x) \end{cases}$$

→ u matričnom obliku

$$Y' = A(x) \cdot Y + B(x)$$

$A \to$ matrica sistema

$$A(x) = \begin{pmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \vdots & & \\ a_{n1}(x) & \dots & a_{nn}(x) \end{pmatrix} \qquad B(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{pmatrix}$$

$\underbrace{A, B \in C(I)}_{\text{neprek. na } I}$; Tada je oblast egzistenc.

$$\underbrace{D = I \times \mathbb{R}^n}$$

$\Leftrightarrow a_{ij}, b_i \in C(I)$; $i, j = 1, 2, \dots, n$

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dif-jna n-tog reda u normalnom obliku može da se prevede u sistem u normalnom obliku

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

smjena $z_1 = y$

$z_2 = z_1' = y'$

$z_3 = z_2' = y''$

$\vdots$

$z_n = y^{(n-1)}$

$z_n' = y^{(n)}$

$$\begin{cases} z_1' = z_2 \\ z_2' = z_3 \\ \vdots \\ z_{n-1}' = z_n \\ z_n' = y^{(n)} = f(x, z, z_2, \dots, z_{n-1}) \end{cases}$$

→ sistem u norm. obliku