$$2\varphi' = -a_1(x)\varphi$$

$$\rightarrow \frac{d\varphi}{\varphi} = -\frac{1}{2}a_1(x)\,dx \rightarrow \varphi = e^{-\int \frac{a_1(x)}{2}dx}$$

SISTEMI DIF. J-NA

→ radimo sisteme 1. reda;

→ u opštem obliku sistem prvog reda je:

$$(1)\quad \begin{cases} y_1' = f_1(x, y_1, \dots, y_n) \\ \vdots \\ y_n' = f_n(x, y_1, \dots, y_n) \end{cases} \rightarrow \text{normalni oblik}$$

$y_1, \dots, y_n \rightarrow$ nepoznate

$f_1, \dots, f_n \rightarrow$ definisane u $G \subset \mathbb{R}^{n+1}$

→ prelazimo na matrični oblik:

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} ; \quad F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \rightarrow Y' = F(x, Y) \quad (2)$$

Definicija $y_1 = \varphi_1(x), y_2 = \varphi_2(x), \dots, y_n = \varphi_n(x)$ su rješenja j-ne (1) ako:

1) $\varphi_i \in C^1(I)$

2) $\forall x \in I \ (x, \varphi_1(x), \dots, \varphi_n(x)) \in G$

3) $\forall x \in I; \quad \varphi_i' = f_i(x, \varphi_1(x), \dots, \varphi_n(x))$,

$i = 1, 2, \dots, n$

→ Dova def. u matričnom obliku

Koš. zad. prije $y(x_0) = y_0$

→ sad: $Y(x_0) = Y_0 \Leftrightarrow y_1(x_0) = y_0^1$, $y_2(x_0) = y_0^2$ ... $y_n(x_0) = y_0^n$

$$Y_0 = \begin{pmatrix} y_0^1 \\ y_0^2 \\ \vdots \\ y_0^n \end{pmatrix}$$

$\underbrace{(4) \quad K.z \quad (3)}$

Teoreme o egzistenciji i jedinstvenosti:

T(PEANO): $F \in C(D)$, $(x_0, Y_0) \in D \Rightarrow$ K.z (2)-(4) ima rješenje

→ $f_1, \dots, f_n \in C(D)$, $(x_0, y_0^1, \dots, y_0^n) \in D \Rightarrow$
$\Rightarrow$ K. z (1)-(3) ima rj.

T2: $F, \dfrac{\partial F}{\partial y_i} \in C(D)$, $(x_0, Y_0) \in D$

$\Rightarrow$ K. z. (2)-(4) ima jedinstveno rješenje

T3: PIKAR: Neka su ispunjeni uslovi:

1) $F \in C(D)$

2) $\exists k > 0 \rightarrow \left| \dfrac{\partial F}{\partial y_i} \right| \leq k$

3) $(x_0, Y_0) \in D$

→ Tada K. z (2)-(4) ima jedinstveno rješenje

2) $\dfrac{\partial f_1}{\partial y_i}$; $\dfrac{\partial f_2}{\partial y_i}$, ..., $\dfrac{\partial f_n}{\partial y_i}$

POSEBAN SLUČAJ ZA LINEARNE

→ sistem glasi:

$$\begin{cases} y_1' = a_{11}(x)y_1 + \dots + a_{1n}(x)y_n + b_1(x) \\ \vdots \\ y_n' = a_{n1}(x)y_1 + \dots + a_{nn}(x)y_n + b_n(x) \end{cases}$$

→ u matričnom obliku

$$Y' = A(x) \cdot Y + B(x)$$

A → matrica sistema

$$A(x) = \begin{pmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \vdots & & \\ a_{n1}(x) & \dots & a_{nn}(x) \end{pmatrix} \qquad B(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{pmatrix}$$

$\underbrace{A, B \in C(I)}_{\text{neprek. na } I}$; Tada je oblast egzistenc. $\underbrace{D = I \times \mathbb{R}^n}$

$\Leftrightarrow a_{ij}, b_i \in C(I)$; $i, j = 1, 2, \dots, n$

---

dif-jna n-tog reda u normalnom obliku može da se prevede u sistem u normalnom obliku

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

smjena $z_1 = y$

$z_2 = z_1' = y'$

$z_3 = z_2' = y''$

$\vdots$

$z_n = y^{(n-1)}$

$z_n' = y^{(n)}$

$$\begin{cases} z_1' = z_2 \\ z_2' = z_3 \\ \vdots \\ z_{n-1}' = z_n \\ z_n' = y^{(n)} = f(x, z, z_2, \dots, z_{n-1}) \end{cases}$$

→ sistem u norm. obliku

6.11.18. **PREDAVANJA 07**

METOD ELIMINACIJE

- sistem u opštem obliku:

$$\begin{cases} y_1' = f_1(x, y_1, \dots, y_n) \\ \vdots \\ y_n' = f_n(x, y_1, \dots, y_n) \;/' \end{cases} \qquad y_1 = y_1(x),\ y_2 = y_2(x), \dots, y_n = y_n(x)$$

→ izaberemo jednu i diferenciramo do n-tog reda

$$y_n'' = \frac{\partial f_n}{\partial x} + \frac{\partial f_n}{\partial y_1} \cdot \underbrace{\frac{dy_1}{dx}}_{f_1} + \dots + \frac{\partial f_n}{\partial y_n} \cdot \frac{dy_n}{dx} =$$

$$= \frac{\partial f_n}{\partial x} + \frac{\partial f_n}{\partial y_1} \cdot f_1 + \dots + \frac{\partial f_n}{\partial y_n} \cdot f_n$$

$$\vdots$$

$$y_n^{(n)} = g_n(x, y_1, y_2, \dots, y_n)$$

$$\left.\begin{aligned} y_n' &= f(x, y_1, y_2, \dots, y_n) \\ y_n'' &= g_2(x, y_1, \dots, y_n) \\ &\vdots \end{aligned}\right\}$$ od prvih $n-1$ j-na izdvojimo $y_1, \dots, y_n$

$$\boxed{y_n^{(n)} = g_n(x, y_1, \dots, y_n)}$$ ← uvrstimo

$$\left.\begin{aligned} y_1 &= h_1(x, y_n, y_n', \dots, y_n^{(n-1)}) \\ &\vdots \\ y_{n-1} &= h_{n-1}(x, y_n, y_n', \dots, y_n^{(n-1)}) \end{aligned}\right\} (*)$$

$$\Rightarrow y_n^{(n)} = g_n(x, h_1(x, y_n, y_n', \dots, y_n^{(n-1)}), \dots, y_n)$$

$$y^{(n)} = G\left(x, y_n, y_n', \dots, y_n^{(n-1)}\right)$$

→ dif. jna n-tog reda po y_n

→ $y_n = \varphi_n(x, c_1, \dots, c_n)$

→ kad dobijemo y_n vratimo u *

(*) →
$$y_1 = \varphi_1(x, c_1, \dots, c_n)$$
$$\vdots$$
$$y_n = \varphi_n(x, c_1, \dots, c_n)$$

Primjer – Riješiti sistem:

$$\begin{cases} x y_1' = -y_1 + x y_2 \\ x^2 y_2' = -2y_1 + x y_2 \end{cases}$$

prvo napisati sistem u normalnom obliku

$x \neq 0$

$$\begin{cases} y_1' = -\frac{1}{x} y_1 + y_2 \quad /' \\ y_2' = -\frac{2}{x^2} y_1 + \frac{1}{x} y_2 \end{cases}$$

diferenciramo jednu od ove dvije j-ne

$$\Rightarrow y_1'' = \frac{1}{x^2} y_1 - \frac{1}{x} y_1' + y_2'$$

$$y_1'' = \frac{1}{x^2} y_1 - \frac{1}{x} y_1' \underbrace{- \frac{2}{x^2} y_1 + \frac{1}{x} y_2}$$

→ dobijamo:

$$y_1'' = -\frac{1}{x^2} y_1 - \frac{1}{x} y_1' + \underbrace{\frac{1}{x} y_2} \to y_1' + \frac{1}{x} y_1$$

$$\begin{cases} y_1' = -\frac{1}{x} y_1 + y_2 \Rightarrow y_2 = y_1' + \frac{1}{x} y_1 \\ y_1'' = -\frac{1}{x^2} y_1 - \frac{1}{x} y_1' + \frac{1}{x_2} \end{cases}$$

→

$$y_1'' = -\frac{1}{x^2}y_1 - \frac{1}{x}y_1' + \frac{1}{x}\cdot\left(y_1' + \frac{1}{x}y_1\right)$$

$$y_1'' = -\frac{1}{x^2}y_1 - \frac{1}{x}y_1' + \frac{1}{x}y_1' + \frac{1}{x^2}y_1$$

$$y_1'' = 0 \Rightarrow \boxed{y_1 = C_1 x + C_2}$$

$$y_2 = y_1' + \frac{1}{x}y_1 = C_1 + C_1 + \frac{1}{x}C_2 = 2C_1 + \frac{1}{x}C_2$$

→ opšte rješenje poč. dif. jne

$$\begin{cases} y_1 = C_1 x + C_2 \\ y_2 = 2C_1 + \frac{1}{x}C_2 \end{cases}$$

HOMOGENI SISTEMI DIF. J-NA

→ to je sistem oblika:

$$\begin{cases} y_1' = a_{11}(x)\,y_1 + \dots + a_{1n}(x)\,y_n \\ \vdots \\ y_n' = a_{1n}(x)\,y_1 + \dots\ a_{nn}(x)\,y_n \end{cases} \quad (1)$$

→ Košijev zadatak isti kao kod svih sistema

→ Oblast egzistencije; potrebno:

$a_{ij} \in C(I)$, $i,j = 1, \dots, n$

$$Y' = A(x)\cdot Y; \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad A(x) = \begin{pmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \vdots & & \\ a_{n1}(x) & \dots & a_{nn}(x) \end{pmatrix}$$

T: Skup rješenja sistema (1) obrazuje vektorski prostor dimenzije n

ovo je bilo prije

$\alpha_1 \varphi_1(x) + \dots + \alpha_n \varphi_n(x) = 0 \Rightarrow \alpha_1 = \dots = \alpha_n = 0$

$\Rightarrow \varphi_1(x), \dots, \varphi_n(x) \rightarrow$ lin. nez.

$\varphi_1, \dots, \varphi_n \rightarrow$ sad vektorske funkcije

$$\varphi_1(x) = \begin{pmatrix} \varphi_{11}(x) \\ \vdots \\ \varphi_{n1}(x) \end{pmatrix} \quad \dots \quad \varphi_n(x) = \begin{pmatrix} \varphi_{1n}(x) \\ \vdots \\ \varphi_{n}(x) \end{pmatrix}$$

$\rightarrow$ i ovdje uvodimo det. Vronskog

$$W(x) = W(\varphi_1, \dots, \varphi_n) = \begin{vmatrix} \varphi_{11}(x) & \dots & \varphi_{1n}(x) \\ \vdots & & \vdots \\ \varphi_{n1}(x) & \dots & \varphi_{nn}(x) \end{vmatrix} =$$

$$= \left| \begin{array}{ccc} \text{1. kol.} & & \text{n. kol.} \\ \left[\varphi_1(x), \right. & \dots, & \left. \varphi_n(x)\right] \end{array} \right|$$

T: Sledeći uslovi su ekvivalentni:

1) $\forall x \in I, \quad W(x) = 0$
2) $\exists x_0 \in I, \quad W(x_0) = 0$
3) rješ. $\varphi_1(x), \dots, \varphi_n(x)$ su linearno ~~ne~~zavisna

T: Sledeći uslovi su ekvivalenti

1) $\forall x \in I, \quad W(x) \neq 0$
2) $\forall x_0 \in I, \quad W(x_0) \neq 0$
3) rješenja $\varphi_1(x), \dots, \varphi_n(x)$ su linearno nezavisna

$\rightarrow$ opšte rješenje sistema linearna kombinacija baze

$$y = C_1 \varphi_1(x) + \dots + C_n \varphi_n(x)$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = C_1 \begin{pmatrix} \varphi_{11}(x) \\ \vdots \\ \varphi_{n1}(x) \end{pmatrix} + \dots + C_n \begin{pmatrix} \varphi_{1n}(x) \\ \vdots \\ \varphi_{nn}(x) \end{pmatrix}$$

$$\begin{cases} y_1 = c_1 \varphi_{11}(x) + \dots + c_n \varphi_{1n}(x) \\ \vdots \\ y_n = c_1 \varphi_{n1}(x) + \dots + c_n \varphi_{nn}(x) \end{cases}$$

$$Y = \begin{pmatrix} \varphi_{1n}(x) & \dots & \varphi_{1n}(x) \\ \vdots & & \\ \varphi_{n1}(x) & \dots & \varphi_{nn}(x) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$\phi(x) = \begin{pmatrix} \varphi_{11}(x) & \dots & \varphi_{1n}(x) \\ \vdots & & \\ \varphi_{n1}(x) & \dots & \varphi_{nn}(x) \end{pmatrix} ; \quad C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$Y = \phi(x) \cdot C$

→ fundamentalna matrica sistema

opšte rješenje

det. fundamentalne matrice je det. Vronskog

$Y' = A(x) \cdot Y$, a znamo $\phi(x)$

$$Y = \phi(x) \cdot C \longrightarrow Y' = \phi'(x) \cdot C = A(x) \cdot \phi(x) \cdot C$$

$$\underbrace{\left(\phi'(x) - A(x) \cdot \phi(x)\right)}_{} \cdot \underbrace{C}_{\neq 0} = 0$$

$\phi'(x) = A(x)\,\phi(x) \;/ \cdot \phi^{-1}(x)$ → postoji inverzna ako su $\neq 0$

$$\Rightarrow \boxed{A(x) = \phi'(x) \cdot \phi^{-1}(x)}$$

HOMOGENI SISTEMI LIN. DIF. J-NA SA KONST. KOEF.

→ imamo sistem oblika

$$\begin{cases} y_1' = a_{11}y_1 + \dots + a_{1n}y_n \\ \vdots \\ y_n' = a_{n1}y_1 + \dots + a_{nn}y_n \end{cases} \qquad a_{ij} \in \mathbb{R},\ j,i = 1,\dots,n$$

$$Y' = AY \ ; \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

→ ovaj sistem možemo metodom eliminacije ali i Ojlerovim metodom

OJLEROV METOD

→ rješenje tražimo u obliku $e^{\lambda x}$, λ moraju biti korijeni karakterističnih polinoma ovako prije

→ slično ovdje; samo sad rješenje tražimo u obliku: $Y = h \cdot e^{\lambda x}$

→ radićemo sve u matričnom obliku

$$Y' = \lambda \cdot h \cdot e^{\lambda x}$$

$$Y' = A \cdot Y$$

$$\lambda h e^{\lambda x} = A h e^{\lambda x} \ / : e^{\lambda x}$$

$$\underline{A \cdot h = \lambda \cdot h}$$

matrica

h → sopstveni vektor

λ → sopstvena vrijednost

$$A h - \lambda h = 0$$

$$(A - \lambda E) \cdot h = 0$$

→ homogeni sist. algeb. j-na uvijek ima trivijalno rješenje; kad ima netriv?

→ nas interesuje n lin. nez. rj

→ 0 uvijek zavisan

→ det. sistema $= 0 \Rightarrow$ ima netrivijalna rješenja
treba da važi da je:

$$\underbrace{P(\lambda) = \det(A - \lambda E) = 0}$$ → ovo je sad karakterist. polinom

→ polinom stepena n

$\boxed{\text{1. SLUČAJ}}$

$\lambda_1, \ldots, \lambda_n$ → prosti korijeni karak. polinoma

→ $\lambda_1 \to h_1 : \underbrace{(A - \lambda_1 E) \cdot h = 0}_{\text{ima besk. rješ}}$ → tražimo jedno $\quad \varphi_1(x) = h_1 e^{\lambda_1 x}$

$\vdots$

$\lambda_n \to h_n : (A - \lambda_n E) \cdot h = 0 \quad \varphi_n(x) = h_n e^{\lambda_n x}$

↑ rješ. sistema

→ pitanje da li su rješenja lin. nez ?

$$W(x) = \begin{vmatrix} h_{11} e^{\lambda_1 x} & \ldots & h_{1n} e^{\lambda_n (x)} \\ \vdots & & \\ h_{n1} e^{\lambda_1 x} & \ldots & h_{nn} e^{\lambda_n (x)} \end{vmatrix} =$$

$h_1 \quad \ldots \quad h_n$

$$= e^{(\lambda_1 + \lambda_2 + \ldots + \lambda_n)} \cdot \underbrace{\begin{vmatrix} h_{11} & \ldots & h_{1n} \\ \vdots & & \\ h_{n1} & \ldots & h_{nn} \end{vmatrix}}_{\neq 0}$$

$\boxed{\text{2. SLUČAJ}}$ λ → korijen višestrukosti $r > 1$

$n - \text{rang}(A - \lambda E) = k$ → broj lin. nez. sopstv. vektora koji odgovaraju sopstvenoj vrijednosti λ

$\boxed{1)\ k=r}$

$\lambda: \quad h_1, \ldots, h_r \rightarrow$ lin. nez. sopstv. vekt.

$$\left.\begin{aligned} \varphi_1(x) &= h_1 e^{\lambda x} \\ &\vdots \\ \varphi_r(x) &= h_r \cdot e^{\lambda x} \end{aligned}\right\} \rightarrow \text{lin. nez.} \rightarrow \text{čine dio baze prostora}$$

$\boxed{2)\ k<r}$ (k ne može biti $>r$)

→ naći ćemo k lin. nez., a ostalih $r-k$?

→ nađemo k lin. nez. sopstv. vektora $h_1, \ldots h_k$

$$\varphi_1(x) = h_1 e^{\lambda x}, \ldots, \varphi_k(x) = h_k e^{\lambda x}$$

→ ostalih $r-k$ → ?

→ pridruženi vektori

$$\varphi_{k+1} = (p_0 + p_1 x + \ldots + p_m \cdot x^m) e^{\lambda x}$$

$$\vdots$$

$$\varphi_r = (g_0 + g_1 x + \ldots + g_m x^m) e^{\lambda x}$$

 # Primjer

$$\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= 3y_1 + 4y_2 \end{aligned} \qquad A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda E) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = 0 \rightarrow \text{karak. pol}$$

$$(2-\lambda)(4-\lambda) - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0 \ ; \quad \lambda_{1,2} = 3 \pm 2$$

$$\boxed{\lambda_1 = 1; \quad \lambda_2 = 5}$$

→

$\lambda_1 = 1; \quad r_1 = 1 \longrightarrow h_1: \quad (A - \lambda E) \cdot h_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} = 0 \qquad \begin{aligned} & h_{11} + h_{21} = 0 \\ & h_{11} = -h_{21} \end{aligned}$$

$h_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ možemo bilo koji ve ktor $\neq 0$ $\longrightarrow$ $\boxed{h_{11} = 1}$

$$\varphi_1(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^x$$

$\lambda_2 = 5, \quad r_2 = 1 \qquad h_2: \quad (A - 5E) \cdot h_2 = 0$

$$\begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix} = 0 \qquad 3h_{12} = h_{22}$$

za $h_{12} = 1$ dobijamo:

$$h_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \longrightarrow \varphi_2(x) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5x}$$

$$y = C_1 \varphi_1(x) + C_2 \varphi_2(x)$$

$$y = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^x + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5x}$$

$$\phi(x) = \begin{pmatrix} e^x & e^{5x} \\ -e^x & 3e^{5x} \end{pmatrix}$$

$\longrightarrow$ fundamentalna matrica rješenja napisana po kolonama

Primjer

$$\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= -y_1 + 4y_2 \end{aligned}$$

$\longrightarrow$ tražimo matricu sistema i nule karak. polinoma

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda E) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0 \longrightarrow (2-\lambda)(4-\lambda)+1=0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda-3)^2 = 0 \longrightarrow \lambda = 3; \quad r=2$$

$\rightarrow$ provjeravamo koliko ima lin. nez. rješ.

$$\text{rang}\,(A-3E) = \text{rang}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = 1$$

$$n-(A-3E) = 2-1 = 1 < r = 2$$

$\rightarrow$ možemo naći samo 1 sopstveni vektor; tražimo ga iz uslova:

$$h_1: \quad (A-\lambda E)h_1 = 0; \quad \lambda = 3 \rightarrow (A-3E)h_1 = 0$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} = 0; \quad \underline{h_{11} = h_{21}}$$

$$\Rightarrow h_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \varphi_1(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3x}$$

$$y = (a+bx)e^{3x}$$

$$y' = 3(a+bx)e^{3x} + be^{3x} = (3a+3bx+b)e^{3x}$$

$$y' = Ay$$

$$(3a+3bx+b)\cdot e^{3x} = A(a+bx)e^{3x} \quad /: e^{3x}$$

$$3a+b+3b\,x = Aa + Abx$$

$$\boxed{\begin{aligned} A\cdot b &= 3b \\ A\cdot a &= 3a+b \end{aligned}}$$

b je h_1 jer je $(A-3E)h_1=0$
$\Leftrightarrow A h_3 = 3h_3$

$\boxed{b = h_1}$ $\rightarrow$ sopstveni vektor

$\rightarrow$ a je pridruženi vektor sopstvenom vektoru h

$\rightarrow$ pridruženi vektor tražimo uvijek na ovaj način

$$\boxed{(A-3E)\cdot a = h_1} \longrightarrow$$

$(A-3E)\,a = h_1$

$a_1 = 0 \rightarrow$ ovdje možemo da postavimo na 0; kod sopstvenog moramo $\neq 0$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad -a_1 + a_2 = 1 \qquad a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\varphi_2(x) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x\right) e^{3x}$$

$$y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3x} + c_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x\right) e^{3x}$$

$$\phi(x) = \begin{pmatrix} e^{3x} & xe^{3x} \\ e^{3x} & (x+1)e^{3x} \end{pmatrix}$$

$$y = (h_2 + h_1 x)\, e^{3x}; \qquad (A - \lambda E)\, h_2 = h_1$$

Primjer

$$\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= y_1 + 3y_2 - y_3 \\ y_3' &= -y_1 + 2y_2 + 3y_3 \end{aligned} \qquad A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda_1 = 2; \qquad \lambda_{2/3} = 3 \pm i$$

$$\lambda_1 = 2; \quad r_1 = 1 \longrightarrow h_1 : (A - 2E)\, h_1 = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} h_{11} \\ h_{21} \\ h_{31} \end{pmatrix} = 0 \longrightarrow \boxed{h_{21} = 0}$$

$$\begin{aligned} h_{11} + h_{21} - h_{31} &= 0 \\ -h_{11} + 2h_{21} + h_{31} &= 0 \\ h_{11} &= h_{31} \end{aligned}$$

za h_{31} biramo bilo šta $\neq 0$

$$\Rightarrow h_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\underbrace{\varphi_1(x) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2x}}$$

$$\lambda = 3+i \longrightarrow h_2 : \quad (A - \underbrace{(3+i)}_{\lambda} E) \cdot h_2 = 0$$

$$\begin{pmatrix} -1-i & 1 & 0 \\ 1 & -i & -1 \\ -1 & 2 & -i \end{pmatrix} \begin{pmatrix} h_{12} \\ h_{22} \\ h_{32} \end{pmatrix} = 0$$

$$\begin{cases} (-1-i) h_{12} + h_{22} = 0 \\ h_{12} - i h_{22} - h_{32} = 0 \\ - h_{11} + 2 h_{22} - i h_{32} = 0 \end{cases}$$

$\longrightarrow$ treba riješiti sistem; besk. rj.

$$\underline{h_{22} = (1+i) h_{12}}$$

$$\longrightarrow h_{12} - i(1+i) h_{12} = h_{32} \Rightarrow \quad h_{32} = h_{12}(2-i)$$

$$- h_{12} + 2(1+i) h_{22} - i h_{32} \quad \longrightarrow i h_{32} = h_{12}(1+2i)$$

$$\boxed{h_{32} = h_{12}(2-i)}$$

$h_{12} \longrightarrow$ možemo postaviti bilo koji kompleksan broj

$$h_2 = \begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\longrightarrow$ pokazati kući da se za λ_3 dobija

$$h_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\lambda_2 = 3+i \quad ; \quad h_2 = \begin{pmatrix}1\\1\\2\end{pmatrix} + i\begin{pmatrix}0\\1\\-1\end{pmatrix} \Rightarrow \varphi_2 = h_2 \cdot e^{\lambda_2 x}$

$\lambda_3 = 3-i \quad ; \quad h_3 = \begin{pmatrix}1\\1\\2\end{pmatrix} - i\begin{pmatrix}0\\1\\-1\end{pmatrix} \Rightarrow \varphi_3 = h_3 \cdot e^{\lambda_3 x}$

$$\varphi_2 = \left(\begin{pmatrix}1\\1\\2\end{pmatrix} + i\begin{pmatrix}0\\1\\-1\end{pmatrix}\right) \cdot e^{(3+i)x} = e^{3x}\left(\begin{pmatrix}1\\1\\2\end{pmatrix} + i\begin{pmatrix}0\\1\\-1\end{pmatrix}\right)(\cos x + i\sin x) =$$

$$= \underbrace{e^{3x}\left(\begin{pmatrix}1\\1\\2\end{pmatrix}\cos x - \begin{pmatrix}0\\1\\-1\end{pmatrix}\sin x\right)}_{U(x)} + i\underbrace{e^{3x}\left(\begin{pmatrix}0\\1\\-1\end{pmatrix}\cos x + \begin{pmatrix}1\\1\\2\end{pmatrix}\sin x\right)}_{V(x)}$$

$$\varphi_3 = h_3 \cdot e^{\lambda_3 x} = e^{3x}\left(\begin{pmatrix}1\\1\\2\end{pmatrix} - i\begin{pmatrix}0\\1\\-1\end{pmatrix}\right)(\cos x - i\sin x) =$$

$$= \underbrace{e^{3x}\left[\begin{pmatrix}1\\1\\2\end{pmatrix}\cos x - \begin{pmatrix}0\\1\\-1\end{pmatrix}\sin x\right]}_{U(x)} - i\underbrace{e^{3x}\left[\begin{pmatrix}0\\1\\-1\end{pmatrix}\cos x + \begin{pmatrix}1\\1\\2\end{pmatrix}\sin x\right]}_{V(x)}$$

$\varphi_2(x) = U(x) + V(x)$

$\varphi_3(x) = U(x) - V(x)$

$U(x), V(x) \rightarrow$ rješ. sist. lin. nez;

$\Rightarrow$ opšte rj. sistema oblika:

$$Y = C_1\varphi_1(x) + C_2 U(x) + C_3 V(x)$$

NEHOMOGENI SISTEMI LIN. DIF. J-NA . SA ~~NE~~ KONST. KOEF.

$Y' = AY + B(x) \qquad (1)$

$\longrightarrow$ METOD VARIJACIJE KONSTANTI

(2) $\underline{Y' = AY}$ $\longrightarrow$ prvo rješimo homogeni dio

$Y_H = \phi(x) \cdot C$

$$\boxed{Y = \phi(x) \cdot C(x)}$$

tražimo tako što Y zamijenimo u sistem

$$y' = \phi'(x) \cdot C(x) + \phi(x) \cdot C'(x)$$

$$y' = AY + B(x)$$

$\phi' = A \cdot \phi$

$$\underbrace{\phi'(x) \cdot \cancel{C(x)}} + \phi(x) \cdot C'(x) = \underbrace{A \cdot \phi(x) \cdot \cancel{C(x)}} + B(x)$$

→ ostaje $\phi^{-1}(x) /\ \phi(x) \cdot C'(x) = B(x)$

$$C'(x) = \phi^{-1}(x) \cdot B(x)$$

$$\underbrace{C(x) = \int \phi^{-1}(x) B(x)\, dx + \gamma}_{\text{sve ovo u vektorskom obliku}}$$

$$Y = \phi(x) \cdot \left[\int \phi^{-1}(x) \cdot B(x)\, dx + \gamma\right] \Rightarrow$$

$$\Rightarrow Y = \underbrace{\phi(x) \cdot \gamma}_{Y_H} + \underbrace{\phi(x) \cdot \int \phi^{-1}(x) B(x)\, dx}_{Y_p}$$

→ opšte rješenje sistema (1) jednako je zbiru homogenog rješenja sistema (2) i bilo kojeg partikularnog rješenja.

Kada je $B(x)$ kvazipolinom tj. $B(x) = P_m(x) \cdot e^{\gamma x}$, provjeravamo da li je γ korijen karakterističnog polinoma

S → višestrukost korijena γ kao korijena karak. pol.

→ tada je part. rješ. oblika:

$$Y_p = P_{m+s}(x) \cdot e^{\gamma x}$$

→ razlika u odnosu na prije; sad se polinom uvećava za stepen s

Primjer

$$y_1' = y_2 + 2e^x \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B(x) = \begin{pmatrix} 2e^x \\ x^2 \end{pmatrix} =$$

$$y_2' = y_1 + x^2$$

$$= \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix} e^x}_{} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} x^2}_{}$$

zbir kvazi polinoma → part. rj.
zbir 2 part. rj.

$\rightarrow \det(A - \lambda E) = 0$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 = 1 \rightarrow \lambda = \pm 1$$

→ korijeni realni i različiti

$\lambda_1 = 1 \rightarrow (A - E) h_1 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} = 0 \Rightarrow \begin{array}{l} h_{11} = h_{21} \\ h_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}$$

$$\varphi_1(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^x$$

$\lambda_1 = -1 \rightarrow (A + E) h_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix} = 0 \rightarrow \begin{array}{l} h_{12} = -h_{22} \\ h_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{array}$$

$$\varphi_2(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x}$$

$y_{p1} : Y' = AY + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^x \quad (*)$

da li je $\gamma = 1$ (e^{1x}) korijen karak. pol

$1 \in \{1, -1\} \Rightarrow s = 1$

$$\boxed{Y_{p1} = \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right) e^x}$$

ovo part. rj. uvrstimo u $(*)$

$$y_{p_2} \ldots \cancel{Y} = AY + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x^2 e^{r x} \quad r = 0$$

$(*)$ $(*)$

$$r = 0 \notin \{\pm 1\} \Rightarrow s = 0$$

$$Y_{p_2} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} x^2 + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} x + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$\longrightarrow$ ovo rješ. uvrstimo u $(*)$ $(*)$

$$Y = Y_H + Y_{p_1} + Y_{p_2}$$