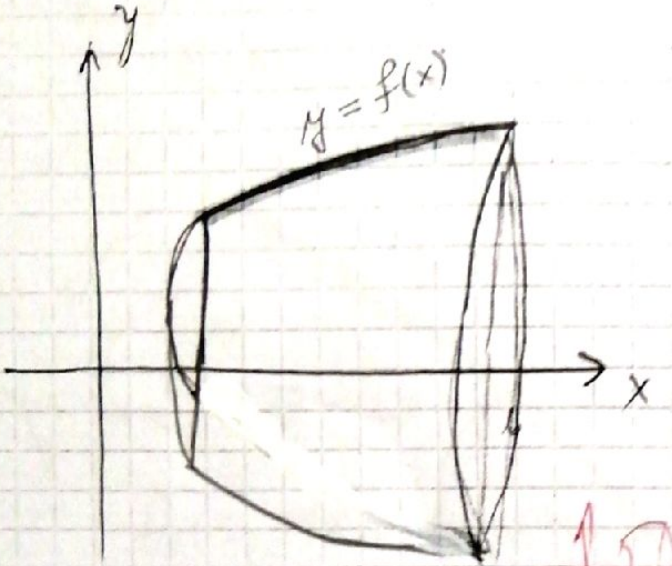
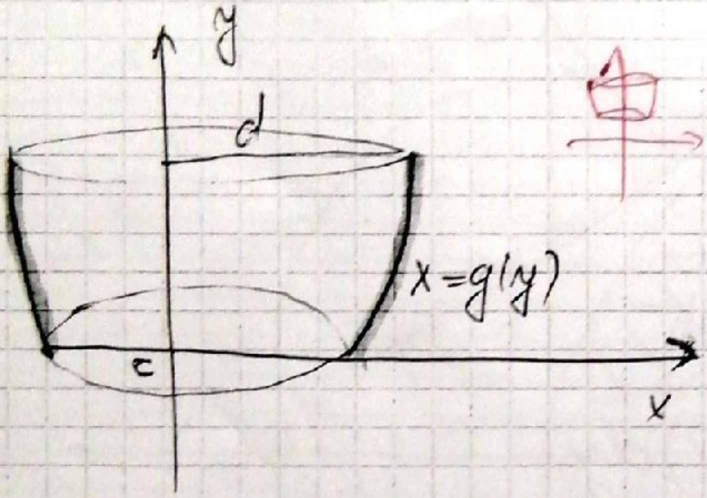
Површина ротационог тијела

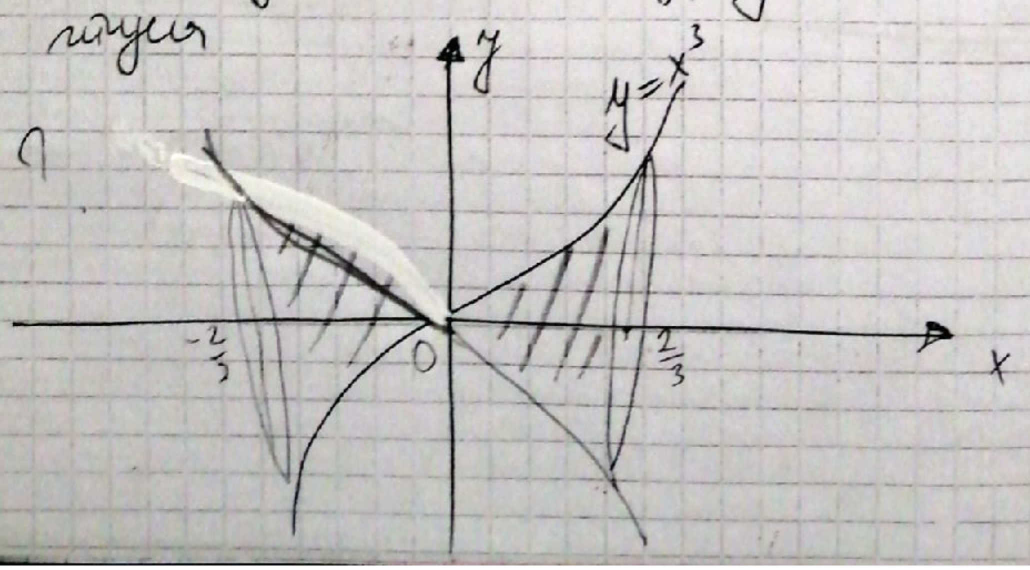


$$P = 2\pi \int_a^b f(x) \cdot \sqrt{1 + f'^2(x)}\, dx$$



$$P = 2\pi \int_c^d g(y) \cdot \sqrt{1 + g'^2(y)}\, dy$$

● Лук криве $y = x^3$ ротира око Ox-осе на сегменту $\left[-\frac{2}{3}, \frac{2}{3}\right]$. Одредити површину ротације



$$P = 2\cdot 2\pi \int_0^{2/3} y(x)\cdot\sqrt{1+y'^2(x)}\cdot dx$$

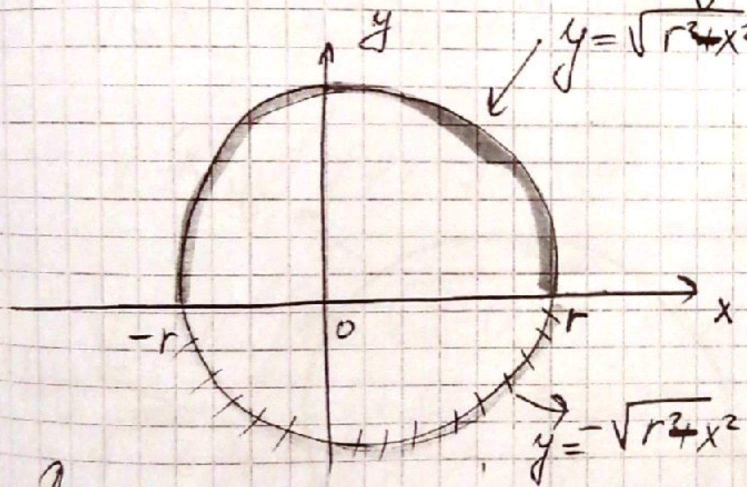
$$y(x) = x^3$$

$$y'(x) = 3x^2$$

$$P = 4\pi\int_0^{2/3} x^3\sqrt{1+9x^4}\,dx = \quad \begin{array}{l} 1+9x^4 = t \\ 36x^3dx = dt \\ x^3dx = \frac{1}{36}t \end{array} \quad \begin{array}{c|c|c} x & 0 & \frac{2}{3} \\ \hline t & 1 & \frac{25}{9} \end{array}$$

$$= 4\pi\cdot\frac{1}{36}\int_1^{\frac{25}{9}}\sqrt{t}\cdot dt = \frac{\pi}{9}\cdot\frac{2}{3}\sqrt{t^3}\,\Big|_1^{\frac{25}{9}} = \frac{196}{729}\pi$$

● Израчунати површину сфере



$x^2+y^2=r^2$

центар $(0,0)$, r

Лопта настаје ротацијом круга око Ox-са

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2-x^2}$$

$$P = 2\cdot 2\pi\cdot\int_0^r y(x)\cdot\sqrt{1+y'^2(x)}\cdot dx, \quad y(x) = \sqrt{r^2-x^2}$$

$$y'(x) = \frac{1}{2\sqrt{r^2-x^2}}\cdot -2x = \frac{-x}{\sqrt{r^2-x^2}}$$

$$P = 4\pi\cdot\int_0^r\sqrt{r^2-x^2}\cdot\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx =$$

$$4\pi\int_0^r\sqrt{r^2-x^2}\cdot\sqrt{\frac{r^2-\cancel{x^2}+\cancel{x^2}}{r^2-x^2}}\,dx =$$

?

$$= 4\pi r \int_0^H dx = 4\pi r \cdot x \Big|_0^H = 4\pi H^2$$

3. Израчунати запремину и површину тијела које настаје ротацијом криве $x^2+y^2-8x-8y+23$ око Ox-осе

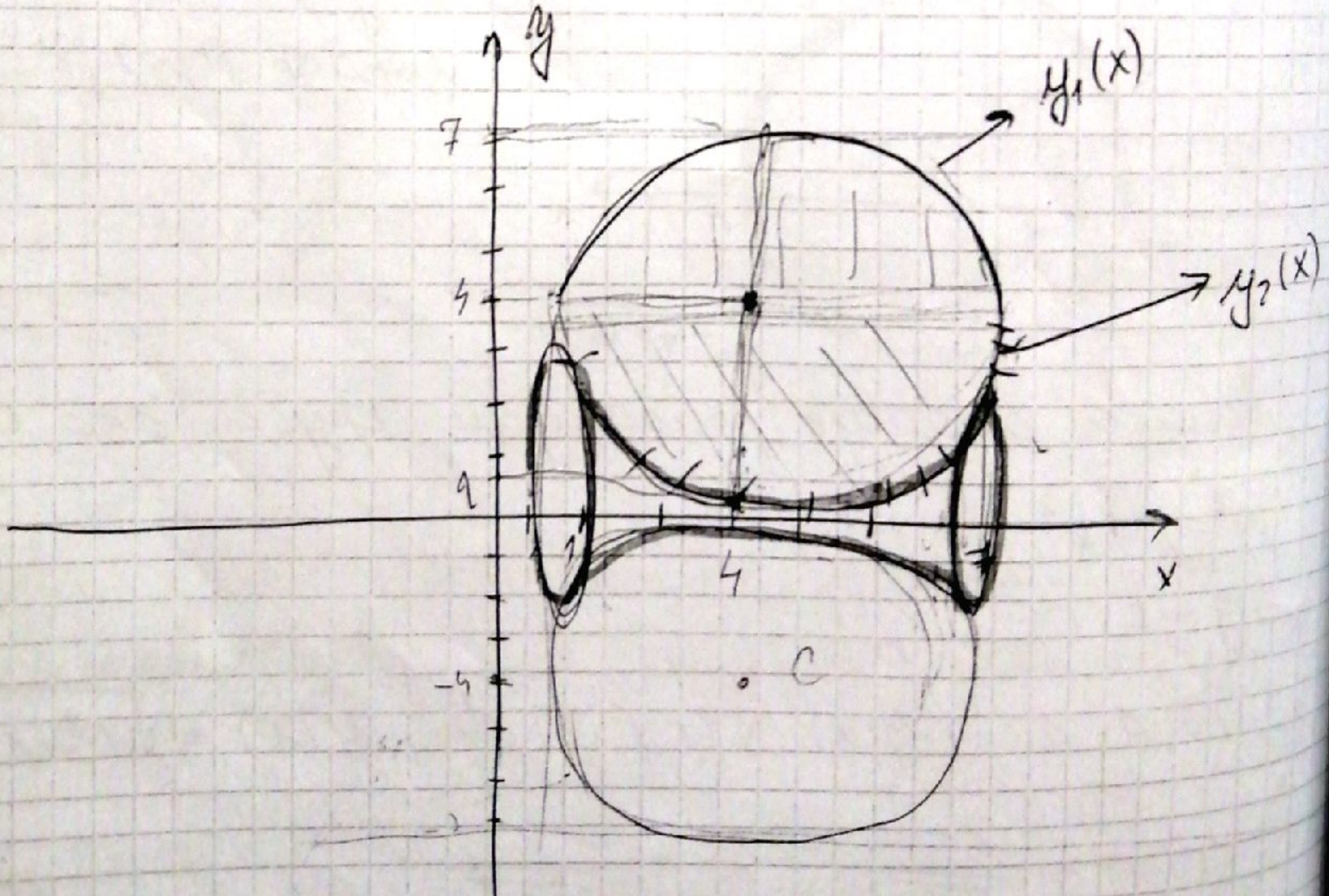
Р:

$$x^2+y^2-8x-8y+23=0$$

$$\dot{x}^2-8\dot{x}+16+\dot{y}^2-8\dot{y}+16+23=0$$

$$(x-4)^2+(y-4)^2=9$$

Круг са центром $C(4,4)$ и полупречника $r=3$



$(x-4)^2+(y-4)^2=9$

$(y-4)^2=9-(x-4)^2$

$y-4=\pm\sqrt{9-(x-4)^2}$

$y=4\pm\sqrt{9-(x-4)^2}$

круг са центром $C(4,4)$, $r=3$

$y_1(x)=4+\sqrt{9-(x-4)^2}$

$y_2(x)=4-\sqrt{9-(x-4)^2}$

$V=V_1-V_2$

$V=\pi\int_1^7(y_1^2(x)-y_2^2(x))\,dx$

$V=\pi\int_1^7\left(4+\sqrt{9-(x-4)^2}\right)^2-\left(4-\sqrt{9-(x-4)^2}\right)^2dx=$

$V=\pi\int_1^7 16\cdot\sqrt{9-(x-4)^2}=16\pi\int_1^7\sqrt{9-(x-4)^2}\,dx$

$\int_{-3}^{3}\sqrt{a^2-t^2}$

$x-4=t$

$dx=dt$

x	1	7
t	-3	3

$=16\pi\int_{-3}^{3}\sqrt{9-t^2}\cdot dt=$

$t=3\cdot\sin z$

$dt=3\cdot\cos z\,dz$

$\sin z=\frac{t}{3}$

$z=\arcsin\frac{t}{3}$

t	-3	3
z	$-\frac{\pi}{2}$	$\frac{\pi}{2}$

$=48\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{9-9\sin^2 z}\cdot\cos z\cdot dz=$

$$144\pi \int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{1-\sin^2 z} \cdot \cos z \cdot dz = 144\pi \cdot \int_{-\frac{\pi}{2}}^{\pi/2} |\cos z| \cdot \cos z \, dz =$$

$$\Big[\cos z \geq 0, \; \forall z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Big]$$

$$144\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2z}{2} dz = 72\pi z \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 72\pi \frac{1}{2} \sin 2z \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= 72\pi \cdot \pi + 0 = 72\pi^2$$

$$P = P_1 + P_2$$

$$P_1 = 2\pi \int_1^7 y_1(x) \cdot \sqrt{1 + y_1'^2(x)} \cdot dx$$

$$P_2 = 2\pi \int_1^7 y_2(x) \cdot \sqrt{1 + y_2'^2(x)} \cdot dx$$

$$y_1(x) = 4 + \sqrt{9-(x-4)^2}$$

$$y_1'(x) = \frac{1}{2\sqrt{9-(x-4)}} \cdot (-2(x-4)) = -\frac{x-4}{\sqrt{9-(x-4)^2}}$$

$$P_1 = 2\pi \int_1^7 (4 + \sqrt{9-(x-4)^2} \cdot \sqrt{1 + \frac{(x-4)^2}{9-(x-4)^2}} \, dx =$$

$$2\pi \int_1^7 (4 + \sqrt{9-(x-4)^2}) \frac{3}{\sqrt{9-(x-4)^2}} \, dx =$$

$$= 24\pi \int_1^7 \frac{dx}{\sqrt{9-(x-4)^2}} + 6\pi \int_1^7 dx =$$

$$= \Bigg[\begin{array}{l} x-4 = 3z \\ z = \frac{x-4}{3} \end{array} \qquad dx = 3\,dz \quad \begin{array}{c|c|c} x & 1 & 7 \\ \hline z & -1 & 1 \end{array} \Bigg]$$

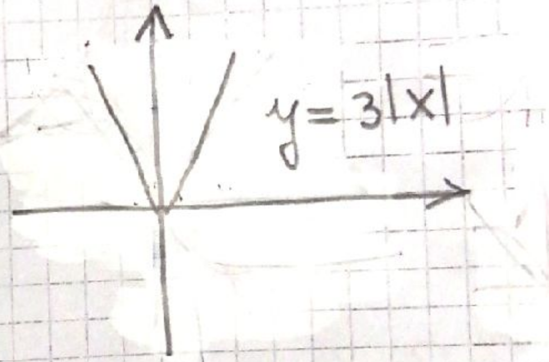
$$= 72\pi \int_{-1}^{1} \frac{dz}{\sqrt{9-9z^2}} + 6\pi x \Big|_1^7 = 72\pi \cdot \frac{1}{3}\int_{-1}^{1}\frac{dz}{\sqrt{1-z^2}} + 6\pi \cdot 6 =$$

$$= 24\cdot\pi\cdot \arcsin z \Big|_{-1}^{1} + 36\pi = 24\pi(\arcsin 1 - \arcsin(-1))$$

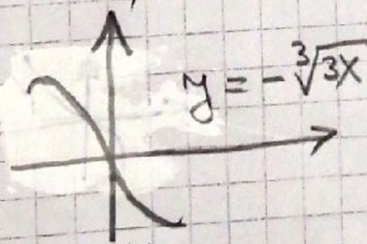
$$+ 36\pi = 24\pi\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) + 36\pi =$$

$$\underline{= 24\pi^2 + 36\pi}$$

Фигура F ограничена кривама $y = -\sqrt[3]{3x}$ и $y = 3|x|$ ротира око Oy-осе. Скицирати фигуру F и израчунати површину ротационог тијела



$$y = 3|x| = \begin{cases} 3x, & x \geq 0 \\ -3x, & x < 0 \end{cases}$$



$y = -\sqrt[3]{3x}$

1° За $x > 0$ криве немају пресјек

за $x < 0$ пресјек кривих

$y = -3x$

$y = -\sqrt[3]{3x}$

$3x = \sqrt[3]{3x} \quad /^3$

$27x^3 = 3x$

$9x^3 = x$

$9x^3 - x = 0$

$x(9x^2 - 1) = 0$

$y = -\sqrt[3]{3x}$

$y' = -\frac{1}{3}(3x)^{-\frac{2}{3}} \cdot 3 = -\frac{1}{\sqrt[3]{9x^2}}$

$\forall x \neq 0$

$\underline{y' < 0}$

$y'' = -\left(-\frac{2}{3}\right)\cdot(3x)^{-\frac{5}{3}} \cdot 3 =$

$= \frac{2}{\sqrt[3]{(3x)^5}}$

$y'' > 0 \Leftrightarrow x > 0$

$y'' < 0 \Leftrightarrow x < 0$

$x=0$

$y=0$

$O(0,0)$

$x^2 = \frac{1}{9}$

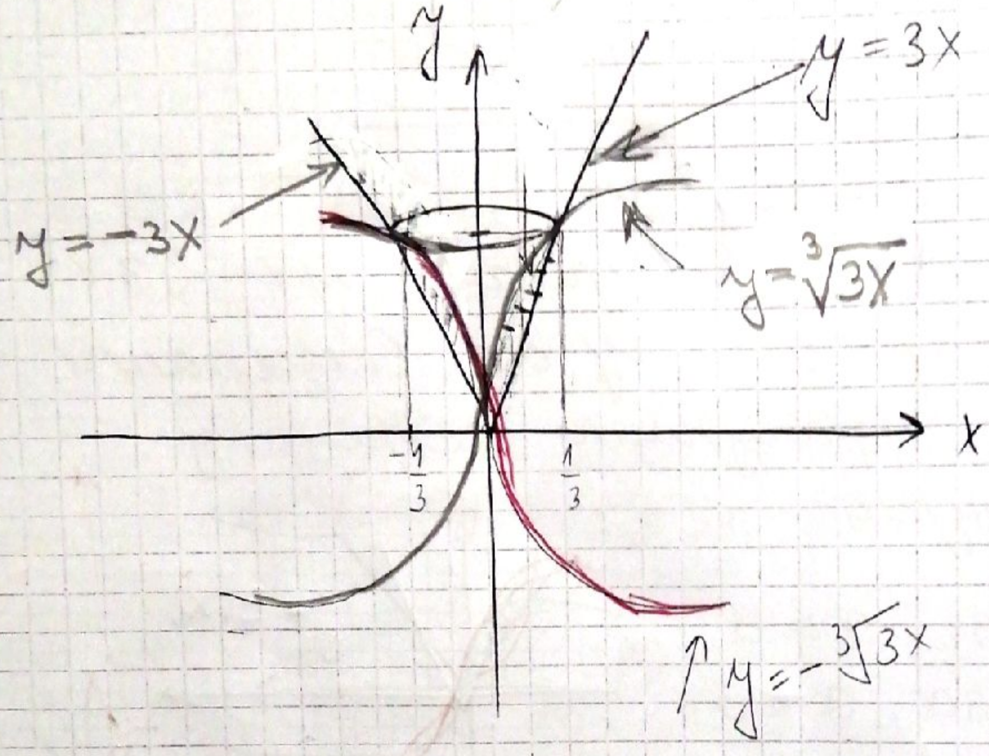
$x = -\frac{1}{3}$ $\quad \cancel{x = \frac{1}{3}}$

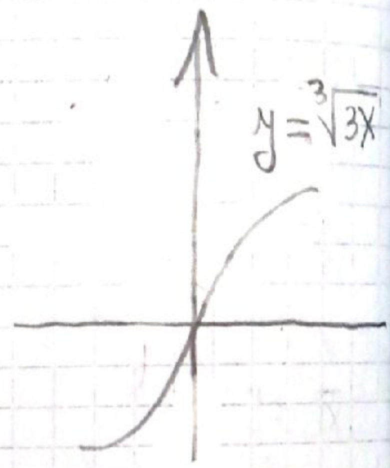
$y = 1$

$A\left(-\frac{1}{3}, 1\right)$

$y = \sqrt[3]{3x} \Rightarrow y^3 = 3x$

$\Rightarrow \boxed{x_2 = \frac{1}{3} y^3}$





$P = P_1 + P_2$

$P_1 = 2\pi \int_0^1 x_1(y) \cdot \sqrt{1 + x_1'^2(y)}\, dy \qquad x_1(y) = \frac{1}{3} y$

$x_1'(y) = \frac{1}{3}$

$P_1 = 2\pi \int_0^1 \frac{1}{3} y \cdot \sqrt{1 + \frac{1}{9}}\, dy =$

$P_1 = 2\pi \cdot \frac{\sqrt{10}}{9} \int_0^1 y \cdot dy = 2\pi \frac{\sqrt{10}}{9} \cdot \frac{y^2}{2} \Big|_0^1 =$

$= \frac{\pi\sqrt{10}}{9}$

$$P_2 = 2\pi \int_0^1 x_2(y) \cdot \sqrt{1 + x_2'^2(y)} \cdot dy \qquad x_2(y) = \sqrt[3]{3x}$$

$$= 2\pi \int_0^1 \frac{1}{3} y^3 \sqrt{1+y^4}\, dy \doteq \left[\begin{array}{l} 1+y^4 = t \\ y^3 dy = \frac{1}{4} dt \\ \begin{array}{c|c|c} y & 0 & 1 \\ \hline t & 1 & 2 \end{array} \end{array} \right]$$

$$= \frac{2\pi}{3} \cdot \frac{1}{4} \int_1^2 \sqrt{t} \cdot dt = \frac{\pi}{6} \cdot \frac{2}{3} \sqrt{t^3} \Big|_1^2 =$$

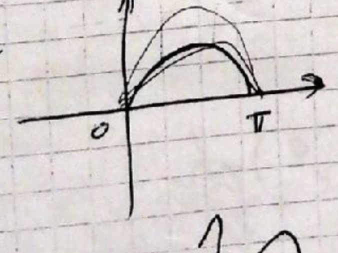
$$= \frac{\pi}{9} \cdot \left(\sqrt{2^3} - \sqrt{1}\right) = \frac{\pi}{9}\left(2\sqrt{2} - 1\right)$$

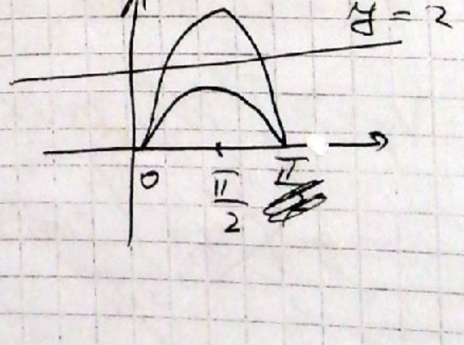
$$P = P_1 + P_2 = \frac{\pi\sqrt{10}}{9} + \frac{\pi}{9}\left(2\sqrt{2}-1\right)$$

5. Фигура ограничена луковима кривих $y = \sin x$ и $y = 2\sqrt{2} \sin x$ и правом $y = 2$ на интервалу $[0, \pi]$ ротира око O_x-осе. Израчунати запремину ротационог тела.

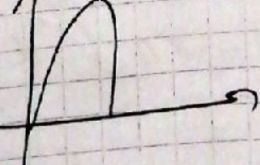
Р:

$y = \sin x$





$y = 2\sqrt{2} \cdot \sin x$



- пресек кривих

$$\left.\begin{array}{l} y = \sin x \\ y = 2\sqrt{2} \cdot \sin x \end{array}\right\} \Rightarrow x = 0 \;\vee\; x = \pi$$

$y = 2$

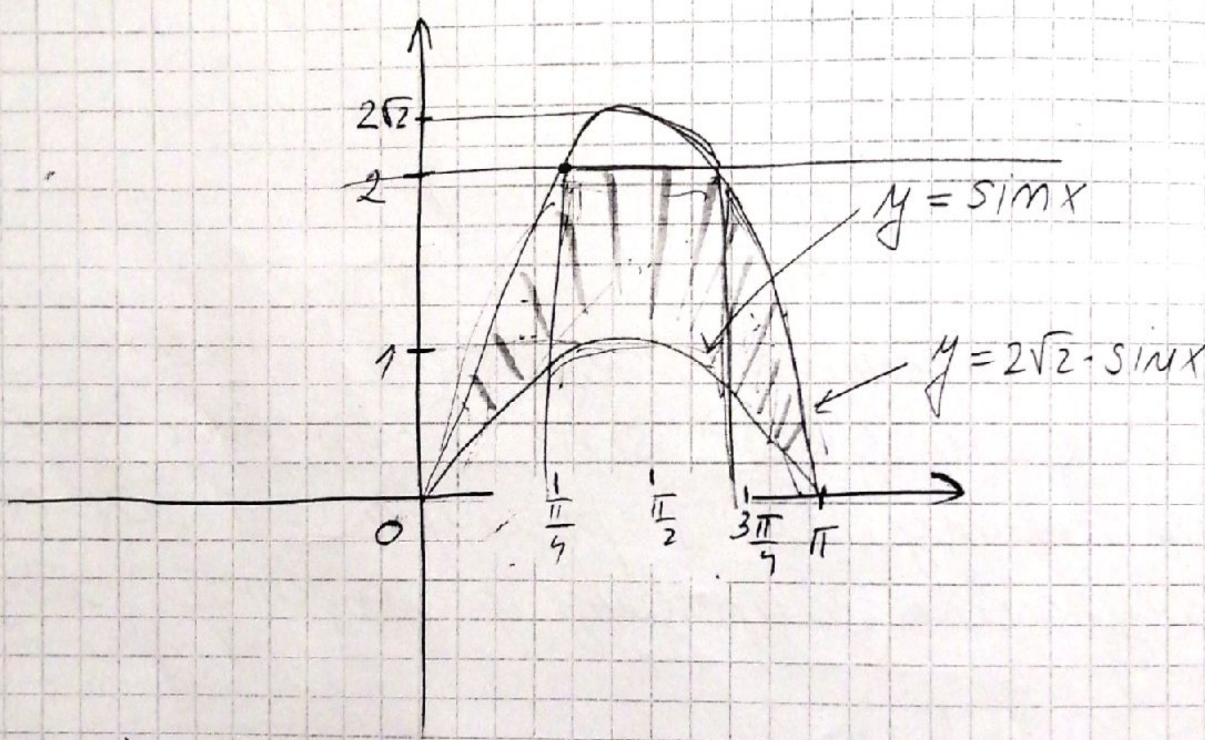
$y = 2\sqrt{2} \cdot \sin x$

$2\sqrt{2} \sin x = 2$

$\sqrt{2} \sin x = 1$

$\sin x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4} \lor x = \frac{3\pi}{4}$



$V = V_1 + V_2 + V_3$

$V_1 = \pi \int_0^{\frac{\pi}{4}} \left((2\sqrt{2} \sin x)^2 - \sin^2 x \right) dx =$

$= \cancel{\pi \int_0^{\frac{\pi}{4}}} \; 7\pi \int_0^{\pi/4} \sin^2 x \, dx = 7\pi \int_0^{\pi/4} \frac{1 - \cos 2x}{2} dx =$

$= \frac{7\pi}{2} \int_0^{\frac{\pi}{4}} dx - \frac{7\pi}{2} \int_0^{\frac{\pi}{4}} \cos 2x \cdot dx =$

$= \frac{7\pi}{2} \cdot x \Big|_0^{\frac{\pi}{4}} - \frac{7\pi}{2} \cdot \frac{1}{2} \sin 2x \Big|_0^{\frac{\pi}{4}} =$

$= \frac{7\pi^2}{8} - \frac{7\pi}{4}$

$V_1 = V_3$

$$V_2 = \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(2^2 - (\sin x)^2\right) dx$$

$$V_2 = 4\pi x \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} - \pi \cdot \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1-\cos 2x}{2}\, dx =$$

$$= 4\pi \cdot \frac{\pi}{2} - \frac{\pi}{2} x \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \frac{\pi}{2} \cdot \frac{1}{2} \cdot \sin 2x \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$2\pi^2 - \frac{\pi}{2} \cdot \frac{\pi}{2} + \frac{\pi}{4}\left((-1) - 1\right) =$$

$$= 2\pi^2 - \frac{\pi^2}{4} - \frac{\pi}{2}$$

$$V = 2 \cdot V_1 + V_2$$

$$V = \frac{7\pi^2}{4} - \frac{7\pi}{2} + \pi^2 - \frac{\pi^2}{4} - \frac{\pi}{2} =$$

$$= \frac{7\pi^2}{2} - 4\pi$$