Фуријеови редови

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cdot\cos\frac{2n\pi}{b-a}\cdot x+b_n\cdot\sin\frac{2n\pi}{b-a}x\right)$$

$$a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2\pi n}{b-a}x\cdot dx\ ,\ n\in N_0$$

$$b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi}{b-a}\cdot x\cdot dx\ ,\ n\in N$$

$[a,b]$

1. Разложити на фуријеов ред функцију $f(x)=x$ на $[-\pi,\pi]$

$a=-\pi \qquad b=\pi$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cdot\cos\frac{2n\pi}{\pi-(-\pi)}x+b_n\sin\frac{2n\pi}{2\pi}x\right)$$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cdot\cos nx+b_n\cdot\sin nx\right)$$

$$a_0=\frac{2}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=0$$

f - је непарна
$f(-x)=-x$

$$a_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}\underbrace{f(x)}_{\text{неп}}\cdot\underbrace{\cos x}_{\text{parna}}\cdot dx=0$$

Scanned by TapScanner

напомена

Када је функција непарна $a_n = 0 \quad \forall n \in \mathbb{N}_0$

Када је функција парна тада је $b_n = 0$, $\forall n \in \mathbb{N}$

$$b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{неп}} \cdot \underbrace{\sin nx}_{\text{неп.}}}_{\text{парна}} dx = \frac{1}{\pi} \cdot 2 \int_0^{\pi} f(x) \cdot \sin nx \, dx =$$

$$\frac{2}{\pi} \cdot \int_0^{\pi} x \cdot \sin nx \, dx = \left[\begin{array}{ll} u = x & du = dx \\ dV = \sin nx \, dx & \int dV = \int \sin x \, dx \\ & V = -\frac{1}{n} \cdot \cos x \end{array} \right]$$

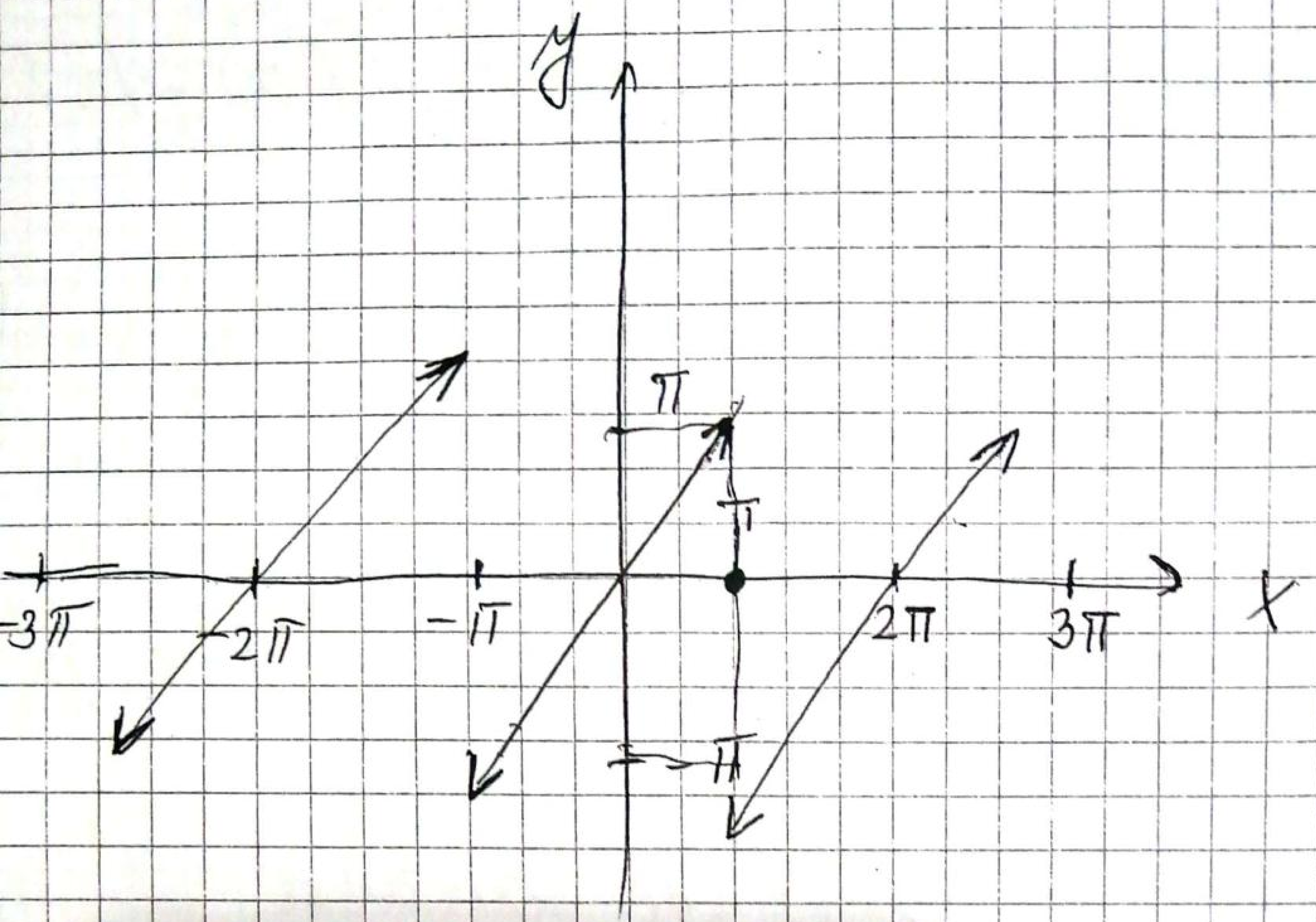
$$\frac{2}{\pi} \left(\underbrace{-\frac{x}{n} \cdot \cos nx \Big|_0^{\pi}}_{0} \right) + \frac{1}{n} \int_0^{\pi} \cos nx \, dx$$

$$\frac{2}{\pi} \left(-\frac{\pi}{n} \cdot \cos n\pi + 0 + \frac{1}{n} \cdot \frac{1}{n} \underbrace{\sin nx \Big|_0^{\pi}}_{\substack{\parallel \\ 0}} \right) =$$

$$= -\frac{2}{n} \cdot \underline{(-1)^n} =$$

$$= \frac{2}{n} (-1)^{n-1}$$

$$S(x) = \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^{n-1}}{n} \cdot \sin nx \quad , \quad |x| < +\infty$$



$$f(x) = S(x), \quad x \in (-\pi, \pi)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \cdot \sin nx, \quad x \in (-\pi, \pi)$$

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \cdot \sin x, \quad \underline{\underline{x \in (-\pi, \pi)}}$$

1. Развити у Фуријеов ред функцију $f(x)=|x|$ на интервалу $(-\pi,\pi)$

Р/

$a=-\pi$

$a=\pi$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cdot\cos\left(\frac{2n\pi}{b-a}x\right)+b_n\cdot\sin\left(\frac{2n\pi}{b-a}\right)x\right)dx$$

$f(-x)=|-x|=|x|=f(x) \Rightarrow f$ је парна функција $b_n=0$, $\forall n\in N$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0=\frac{2}{2\pi}\cdot\int_{-\pi}^{\pi}\underbrace{f(x)}_{\text{парна}}dx=\frac{1}{\pi}\cdot2\cdot\int_0^{\pi}f(x)dx=$$

$$=\frac{2}{\pi}\cdot\int_0^{\pi}|x|\cdot dx=$$

$$=\frac{2}{\pi}\cdot\int_0^{\pi}x\,dx=\frac{\cancel{2}}{\pi}\cdot\frac{x^2}{\cancel{2}}\Big|_0^{\pi}=\pi$$

Scanned by TapScanner

$$a_n = \frac{2}{2\pi} \cdot \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{parna}} \cdot \underbrace{\cos nx}_{\text{parna}}}_{\text{parna}} \cdot dx = \frac{1}{\pi} \cdot 2 \int_0^{\pi} |x| \cdot \cos nx \, dx =$$

$$= \frac{2}{\pi} \cdot \int_0^{\pi} x \cdot \cos nx \, dx = \left[\begin{array}{l} u = x \Rightarrow du = dx \\ V = \int \cos nx = \frac{1}{n} \sin nx \end{array} \right]$$

$$\frac{2}{\pi} \cdot \left(\underbrace{\frac{x}{n} \cdot \sin nx \Big|_0^{\pi}}_{=0} - \frac{1}{n} \int_0^{\pi} \sin nx \, dx \right) =$$

$$= -\frac{2}{n\pi} \left(-\frac{1}{n}\right) \cdot \cos nx \Big|_0^{\pi} =$$

$$= \frac{2}{n^2\pi} \left(\cos n\pi - \cos 0 \right) = \frac{2}{n^2\pi} \left((-1)^n - 1 \right)$$

$$a_n = \frac{2}{n^2\pi} \left((-1)^n - 1 \right)$$

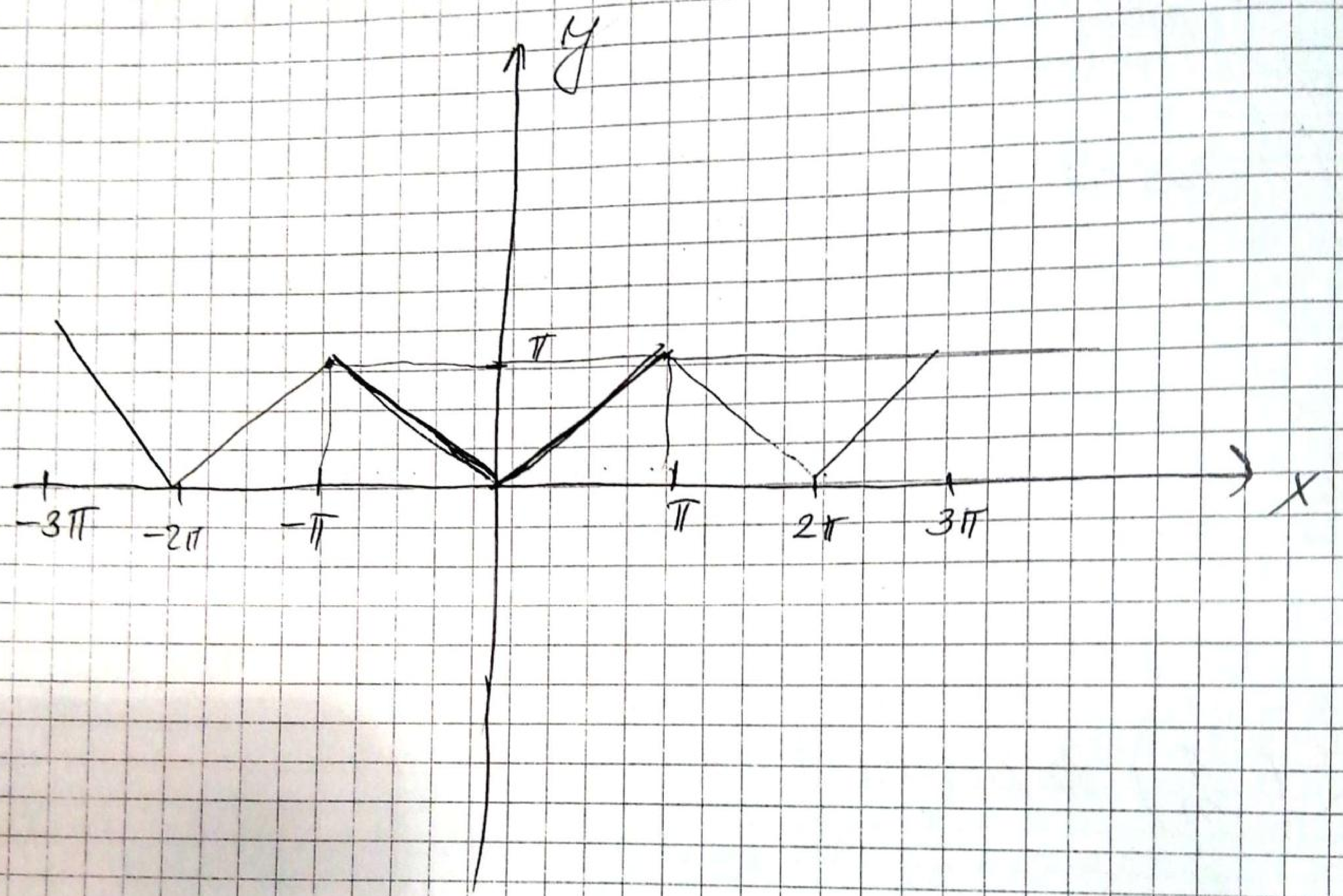
$$a_n = \begin{cases} 0 & n - \text{парно} \\ -\frac{4}{n^2\pi} & n - \text{непарно} \end{cases}$$

$$S(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \cdot \pi} \left((-1)^n - 1 \right) \cos nx$$

$$S(x) = \frac{\pi}{2} + \sum \frac{-4}{(2n-1)^2 \pi} \cdot \cos(2n-1)x$$

$f(x) = S(x) \quad , x \in [-\pi, \pi]$

$$|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2\pi} \cdot \cos(2n-1)x \quad , x \in [-\pi, \pi]$$



b) $x = 0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cdot \cos 0$$

$$\frac{\pi}{2} - \frac{4}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 0$$

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

② Функцију $f(x)=2x-x^2$ развиjи у Фуријеов ред на $[0,3]$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cdot\cos\left(\frac{2n\pi}{b-a}x\right)+b_n\cdot\sin\left(\frac{2n\pi}{b-a}x\right)\right)$$

$$a_n=\frac{2}{b-a}\int_a^b f(x)\cdot\cos\left(\frac{2n\pi}{b-a}\right)dx$$

$$b_n=\frac{2}{b-a}\int_a^b f(x)\cdot\sin\left(\frac{2n\pi}{b-a}x\right)dx$$

$a=0$

$b=3$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cdot\cos\frac{2n\pi}{3}x+b_n\sin\frac{2n\pi}{3}x\right)$$

$$a_0=\frac{2}{3}\int_0^3 f(x)dx=\frac{2}{3}\int_0^3(2x-x^2)dx=$$

$$=\frac{2}{3}\cdot x^2\Big|_0^3-\frac{2}{9}x^3\Big|_0^3=6-6=0$$

$$a_n=\frac{2}{3}\int_0^3(2x-x^2)\cdot\cos\frac{2n\pi}{3}x\cdot dx=$$

$u=2x-x^2\Rightarrow du\,(2-2x)dx$

$du=2(1-x)dx$

$v=\int\cos\frac{2n\pi}{3}x\,dx$

$=\frac{3}{2n\pi}\cdot\sin\frac{2n\pi}{3}x$

$$=\frac{2}{3}\left(\underbrace{\frac{3}{2n\pi}\cdot(2x-x^2)\cdot\sin\frac{2n\pi}{3}x\Big|_0^3}_{0}-\frac{3}{n\pi}\int_0^3(1-x)\cdot\sin\frac{2n\pi}{3}x\right)dx$$

$$-\frac{2}{n\pi}\int_0^3 (1-x)\sin\frac{2n\pi}{3}x\,dx = \quad \begin{array}{l} u = 1-x \Rightarrow du = -dx \\ V = \int \sin\frac{2n\pi}{3}x\,dx \\ \Rightarrow V = -\frac{3}{2n\pi}\cos\frac{2n\pi}{3}x \end{array}$$

$$= -\frac{2}{n\pi}\cdot\left(-\frac{3}{2n\pi}\cdot(1-x)\cdot\cos\frac{2n\pi}{3}\Big|_0^3 + \frac{3}{2n\pi}\int_0^3 \cos\frac{2n\pi}{3}x\,dx\right) =$$

$$= \frac{3}{n^2\pi^2}\cdot\left(-2\cos 2n\pi - 1\cdot\cos 0\right) \underbrace{-\frac{3}{n^2\pi^2}\cdot\frac{3}{2n\pi}\sin\frac{2n\pi}{3}x\Big|_0^3}_{0}$$

$$= \frac{3}{n^2\pi^2}\left(-2\cos 2n\pi - 1\cos 0\right)\Big|_0^3$$

$$= \frac{3}{n^2\pi^2}\cdot\left(-2\cdot 1 - 1\right) = -\frac{9}{n^2\pi}, \quad n\in N$$

$$b_n = \frac{2}{3}\int_0^3 (2x-x^2)\cdot\sin\frac{2n\pi}{3}x\,dx = \ldots$$

$$\frac{R}{=}\ \frac{3}{n\pi}, \quad n\in N$$

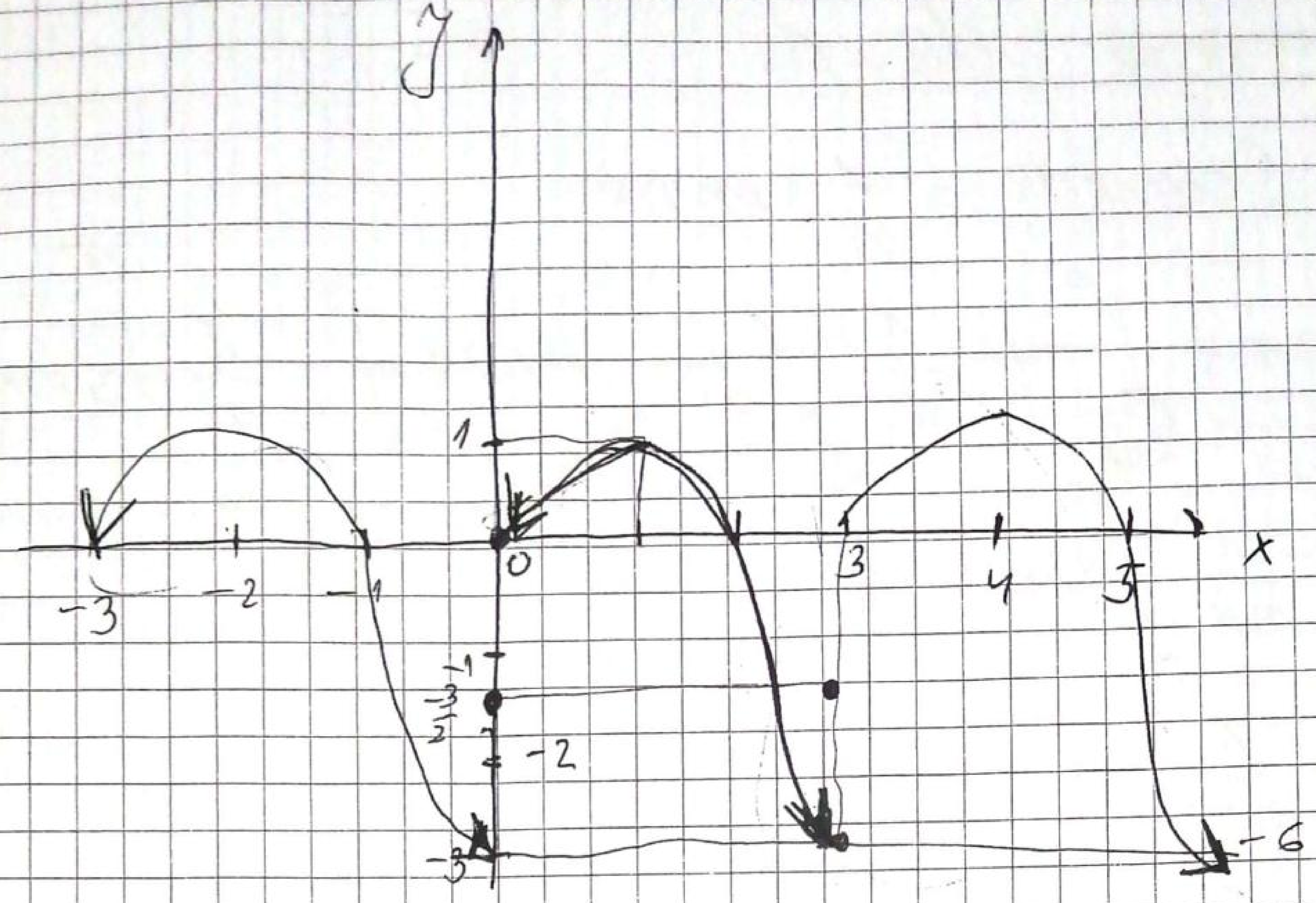
$$S(x) = \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cdot\cos\frac{2n\pi}{3} + \frac{3}{n\pi}\cdot\sin\frac{2n\pi}{3}x\right)$$

$$f(x) = S(x) \qquad x\in[0,3]$$

$$2x - x^2 = \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cdot\cos\frac{2n\pi}{3}x + \frac{3}{n\pi}\cdot\sin\frac{2n\pi}{3}x\right)$$

$$x\in[0,3)$$

Scanned by TapScanner



③ Функцију $f(x) = \begin{cases} \pi, & -\pi \le x < 0 \\ x, & 0 \le x \le \pi \end{cases}$

развити у Фуријеов ред на $[-\pi, \pi]$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi}{b-a} x + b_n \sin \frac{2n\pi}{b-a} x \right)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos nx + b_n \cdot \sin nx \right)$$

$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx \right)$$

$$= \frac{1}{\pi} \cdot \left(\int_{-\pi}^{0} \pi \cdot dx + \int_{0}^{\pi} x \cdot dx \right) = \frac{1}{\pi} \left(\pi \cdot x \Big|_{-\pi}^{0} + \frac{x^2}{2} \Big|_{0}^{\pi} \right) =$$

$$= \frac{1}{\pi} \left(\pi^2 + \frac{\pi^2}{2} \right) = \frac{1}{\pi} \cdot \frac{3\pi^2}{2} = \frac{3}{2}\pi$$

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx\,dx = \frac{1}{\pi} \cdot \int_{-\pi}^{0} \pi \cdot \cos nx\,dx + \frac{1}{\pi} \int_{0}^{\pi} x \cdot \cos nx\,dx =$$

$= \left[u = x \Rightarrow du = dx \right.$

$V = \int \cos nx\, dx = \frac{1}{n} \sin nx$

$\underbrace{\frac{1}{n} \sin nx \Big|_{-\pi}^{0}}_{0} + \frac{1}{\pi} \cdot \left(\underbrace{\frac{x}{n} \cdot \sin nx \Big|_{0}^{\pi}}_{0} - \frac{1}{n} \int_0^{\pi} \sin nx\, dx \right) =$

$= -\frac{1}{n\pi} \int_0^{\pi} \sin nx\, dx =$

$= -\frac{1}{n\pi} \cdot \left(\frac{-1}{n} \right) \cdot \cos nx \Big|_0^{\pi} =$

$= \frac{1}{n^2 \cdot \pi} \left(\cos n\pi - \cos 0 \right)$

$\frac{1}{n^2 \pi} \cdot \left((-1)^n - 1 \right)$

$a_n = \frac{1}{n^2 \cdot \pi} \left((-1)^n - 1 \right)$

$a_n = \begin{cases} 0, & n - \text{парно} \\ -\frac{2}{n^2 \pi}, & n - \text{непарно} \end{cases}$

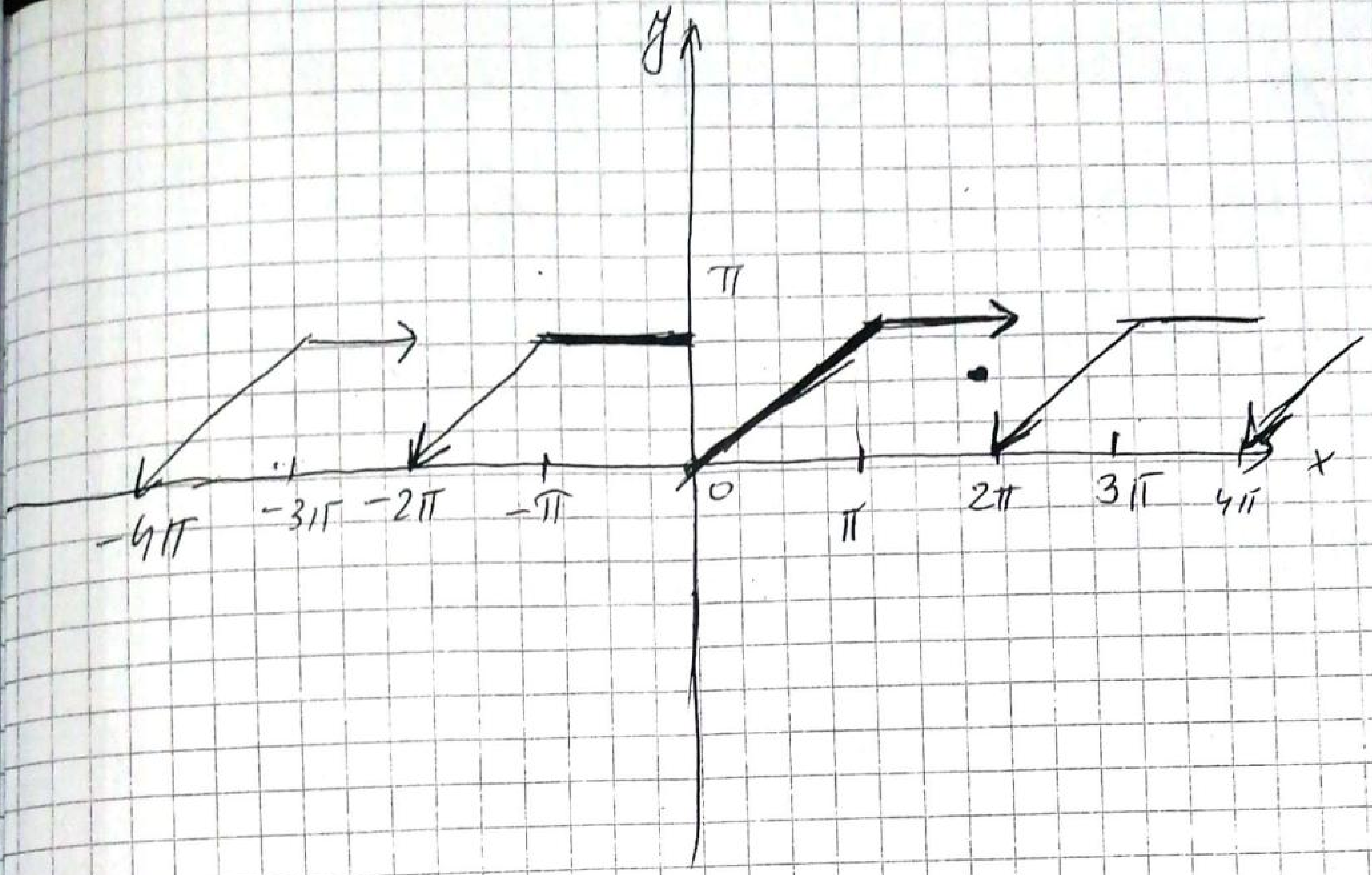
$b_n = \frac{2}{2\pi} \int_0^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \cdot \int_{-\pi}^{0} \pi \cdot \sin nx\, dx$

$+ \frac{1}{\pi} \int_0^{\pi} x \cdot \sin nx\, dx = \ldots = -\frac{1}{n}$

$f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \frac{-2}{(2n-1)^2 \pi} \cdot \cos(2n-1)x + \sum_{n=1}^{\infty} \left(-\frac{1}{n} \right) \cdot \sin nx$

$x \in [-\pi, \pi]$

Scanned by TapScanner



* Ако функцију треба развити у синусни Фуријеов ред то значи да функцију треба непарно проширити $F(x)$ и онда $F(x)$ развијати у Фуријеов ред!

* Ако функцију треба развити у косинусни Фуријеов ред, прво је парно проширимо и функцију $G(x)$ и онда развијамо $G(x)$ и Фуријеов ред.

$$F(x) = \begin{cases} f(x), & 0 \leq x \leq \pi \\ -f(-x), & -\pi \leq x < 0 \end{cases}$$

непарно проширење

парно проширење

$$G(x) = \begin{cases} f(x), & 0 \leq x \leq \pi \\ f(-x), & -\pi \leq x < 0 \end{cases}$$

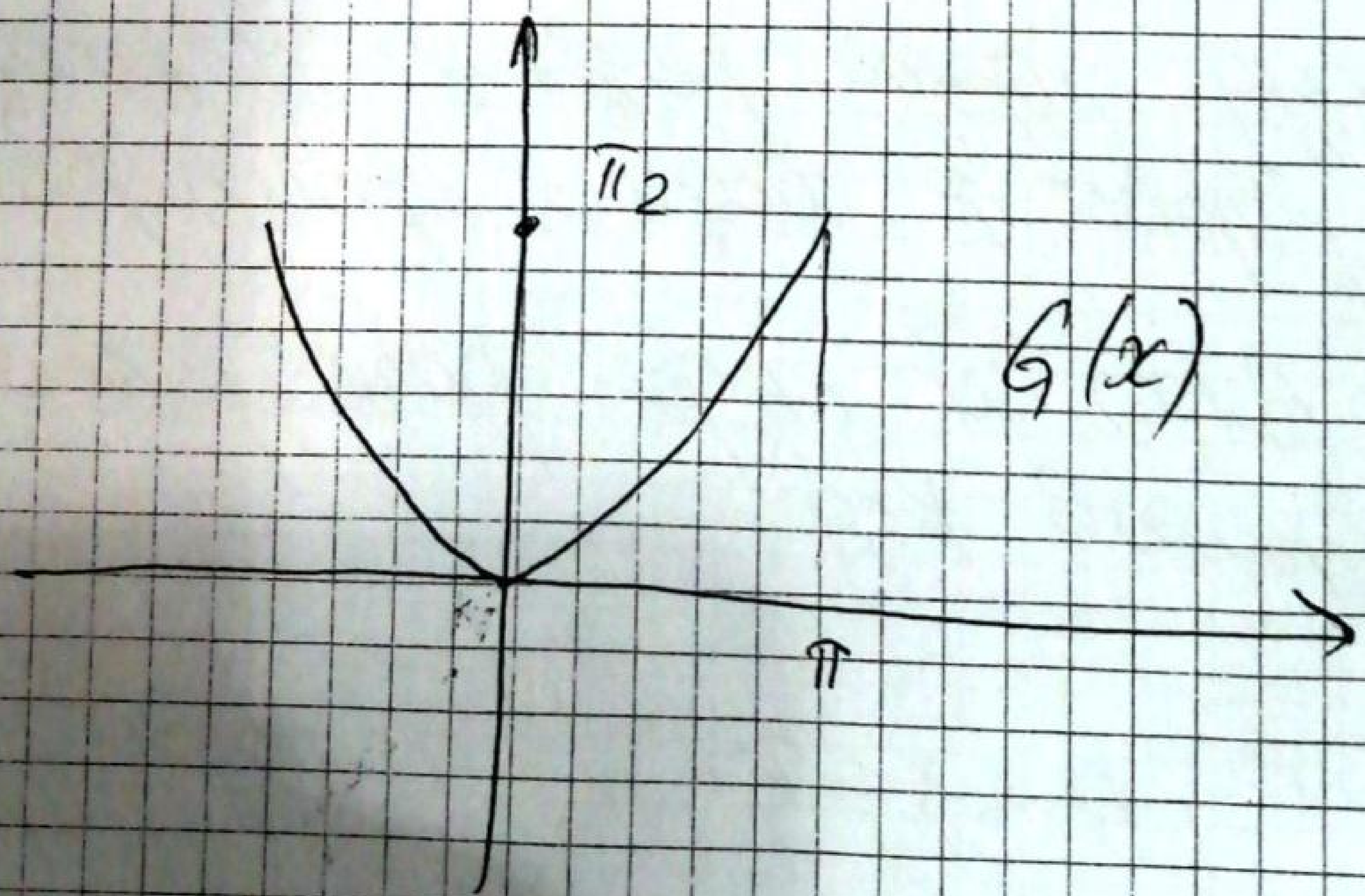
1. Функцију $f(x)=x^2$ развити у косинусни Фуријеов ред на интервалу од $[0,\pi]$. Развој искористити и израчунати

$$\sum_{n=1}^{\infty}\frac{1}{n^2}, \quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}, \quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Функцију $f(x)$ проширујемо парно на $[-\pi,\pi]$

$$G(x)=\begin{cases} f(x), & 0\le x\le \pi \\ f(-x), & -\pi\le x<0 \end{cases}$$

$$G(x)=\begin{cases} x^2, & 0\le x\le \pi \\ (-x)^2, & -\pi\le x\le 0 \end{cases} = x^2 \quad x\in[-\pi,\pi]$$



Функцију $G(x)$ развијамо у Фуријеов ред на $[-\pi,\pi]$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cdot \cos x + b_n \cdot \sin nx)$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} G(x)\cdot\cos nx\, dx$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \underbrace{G(x)}_{par}\cdot\underbrace{\sin nx}_{nep}\, dx = 0$$

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} G(x)\cdot dx = \frac{1}{\pi}\cdot 2\int_{0}^{\pi} G(x)\cdot dx =$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\cdot\frac{x^3}{3}\Big|_{0}^{\pi} = \frac{2\pi^2}{3}$$

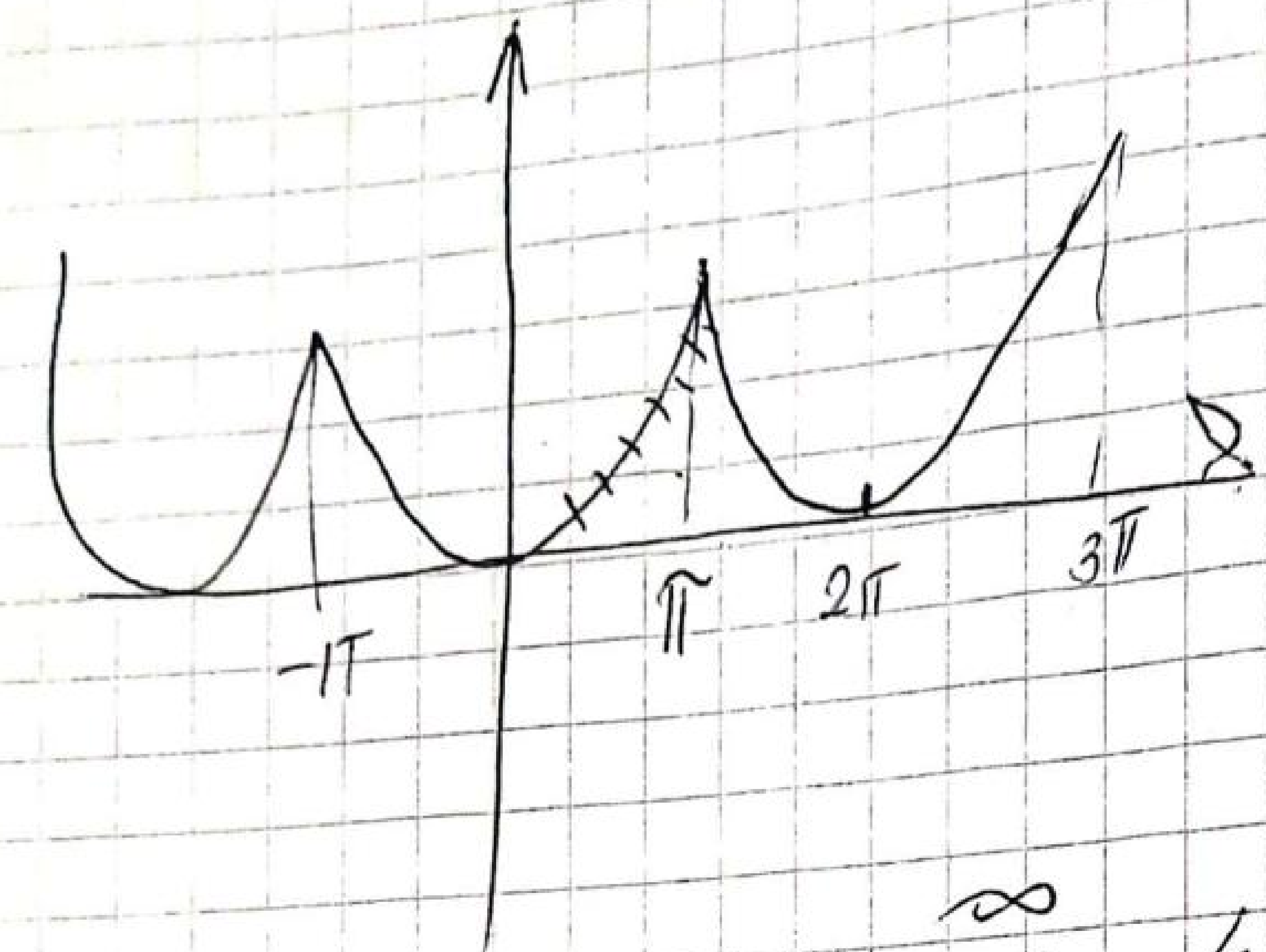
$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{G(x)}_{p}\cdot\underbrace{\cos x}_{p}}_{parno}\, dx = \frac{1}{\pi}\cdot 2\int_{0}^{\pi} G(x)\cdot\cos nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cdot\cos nx\cdot dx \quad \ldots\ldots$$ Парцијалн. интеграција

$$\overset{2}{=}$$

$$a_n = \frac{4}{n^2}\cdot\cos n\pi = \boxed{\frac{4}{n^2}(-1)^n}$$

$$S(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cdot\cos nx, \quad x\in(-\infty,+\infty)$$



$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cdot(-1)^n}{n^2}\cdot \cos nx \quad , x \in [0,\pi]$$

$$x = \pi : \quad \pi^2 = \frac{\pi^2}{3} + 4 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \cos n\pi$$

$$4 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n = \frac{2}{3}\pi^2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$x=0$

$$\frac{\pi^2}{3} + 4 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0$$

$$4 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

добили смо да је

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$