Функције више променљивих

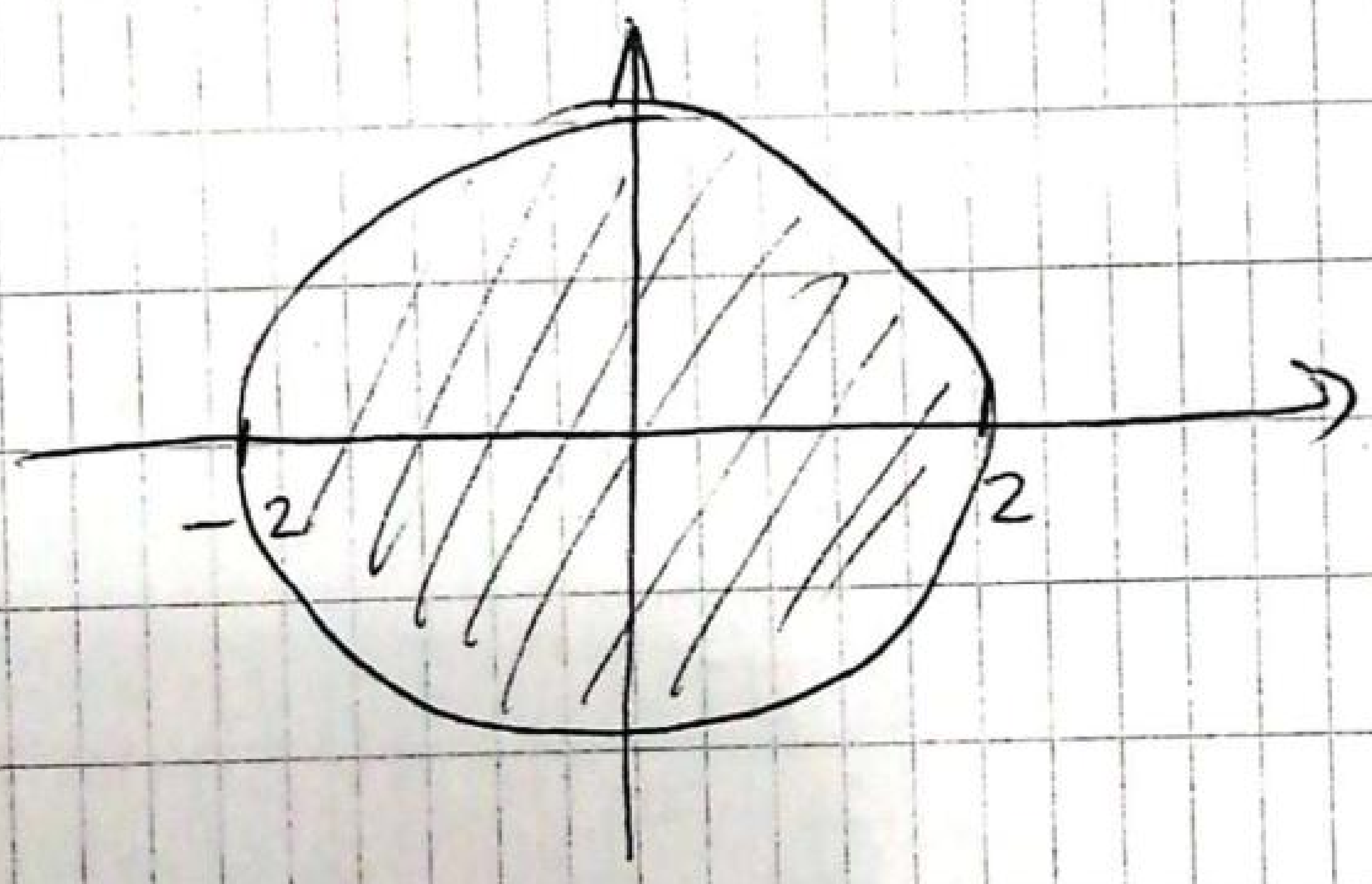
2. $f(x,y) = \sqrt{4 - x^2 - y^2}$. Одредити област дефинисаности ф-је.

$$4 - x^2 - y^2 \geq 0$$
$$x^2 + y^2 \leq 4$$

$$D = \{(x,y) \mid x^2 + y^2 \leq 4\}$$

$$x^2 + y^2 = 4$$

једначина круга са центром у $O(0,0)$ и $r=2$



D – све тачке круга и кружне обручи

3. $f(x,y) = \sqrt{\dfrac{x+y+1}{x-1}}$

$$* \quad x+y+1 > 0 \quad \land \quad x-1 \neq 0$$
$$y \geq -x-1 \quad \land \quad x \neq 1$$

$$D(f) = \{(x,y) \mid y \geq -x-1 \ \land \ x \neq 1\}$$

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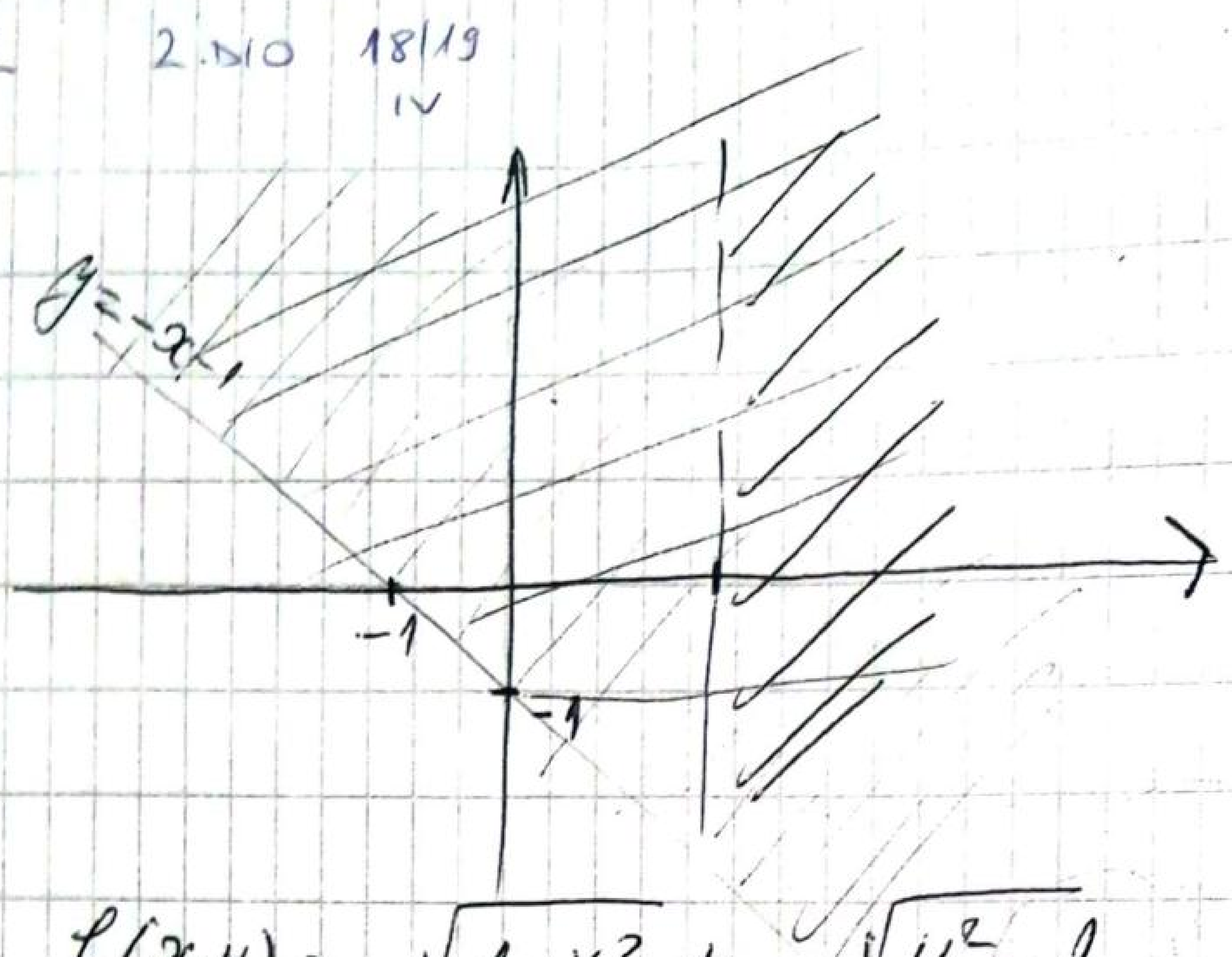
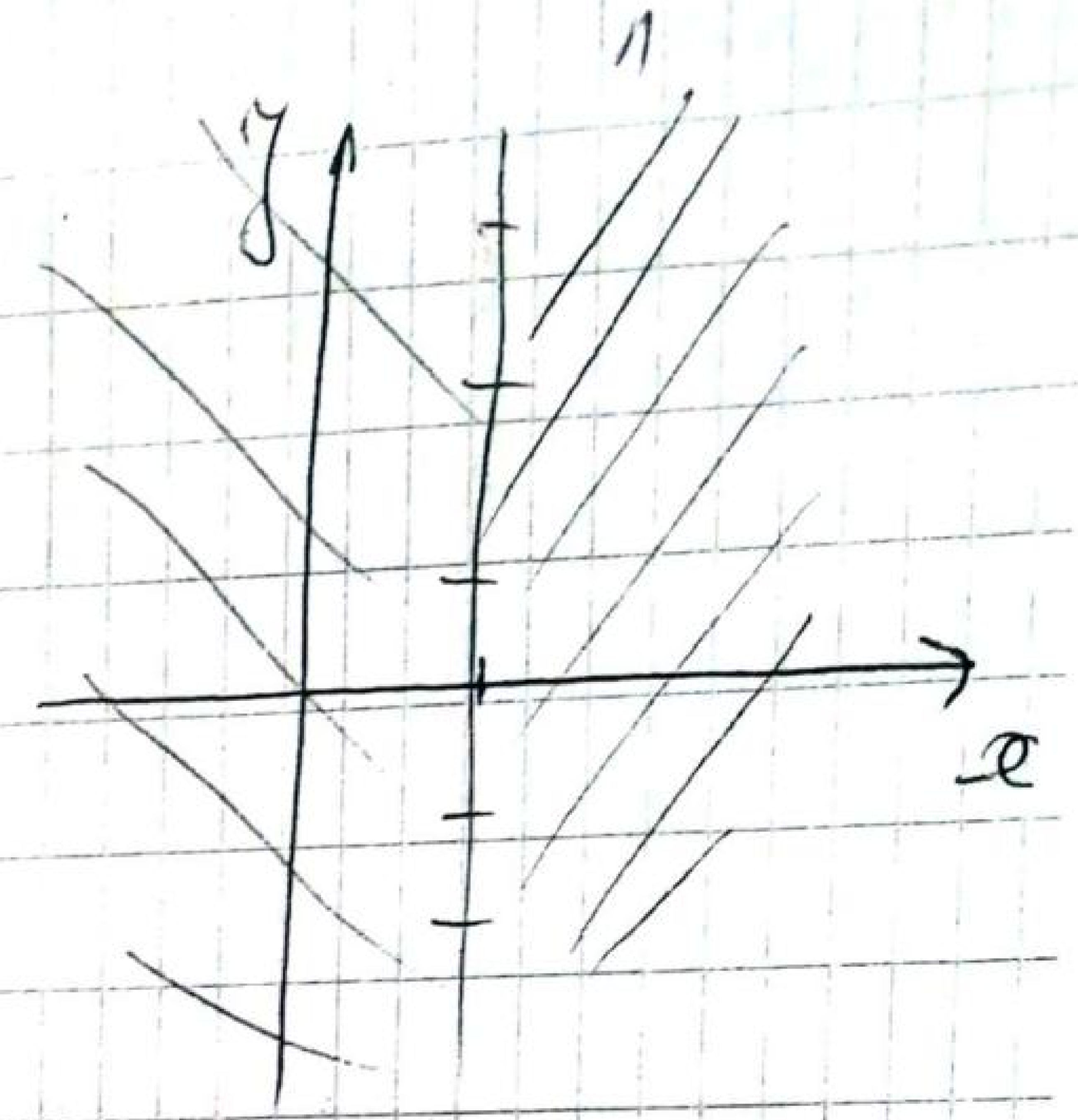
IV





1. $f(x,y) = \sqrt{1-x^2} + \sqrt{y^2-1}$

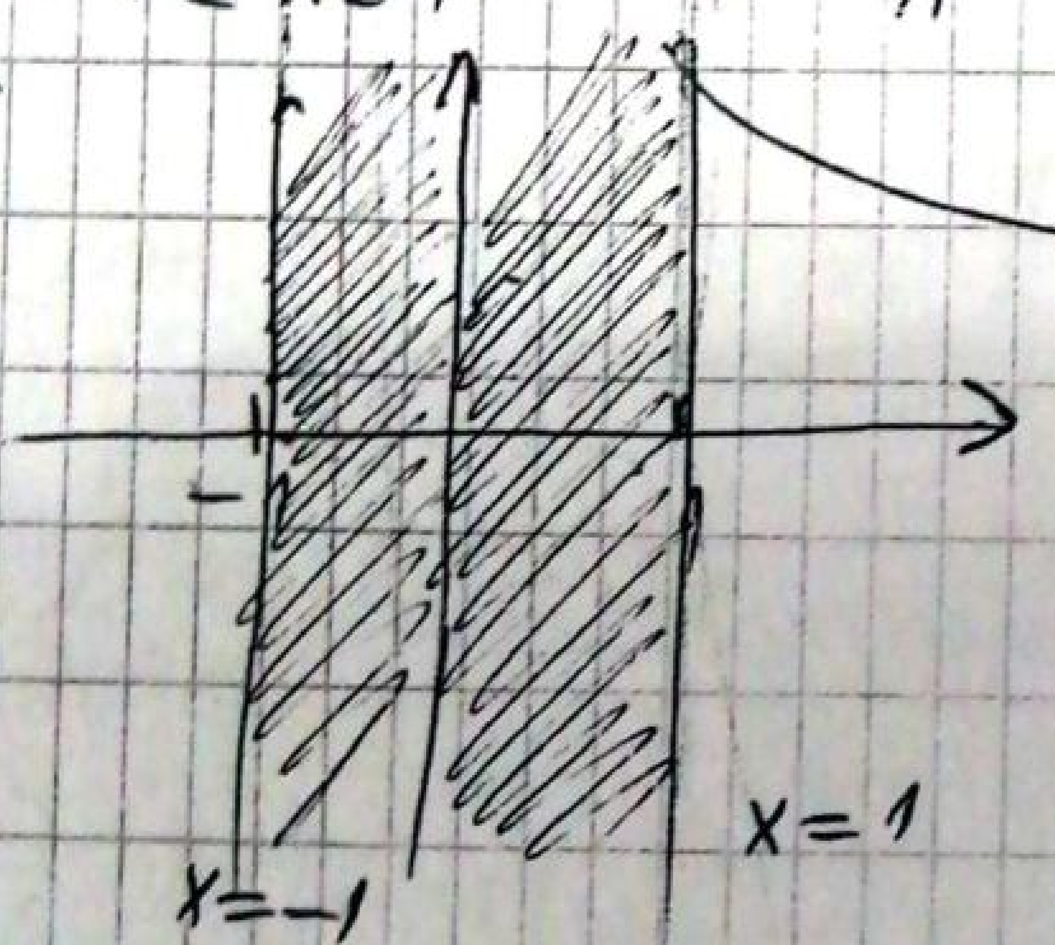
D: $1-x^2 \geq 0 \quad \wedge \quad y^2-1 \geq 0$

$x^2 \leq 1 \quad \wedge \quad y^2 \geq 1$

$|x| < 1 \quad \wedge \quad |y| \geq 1$

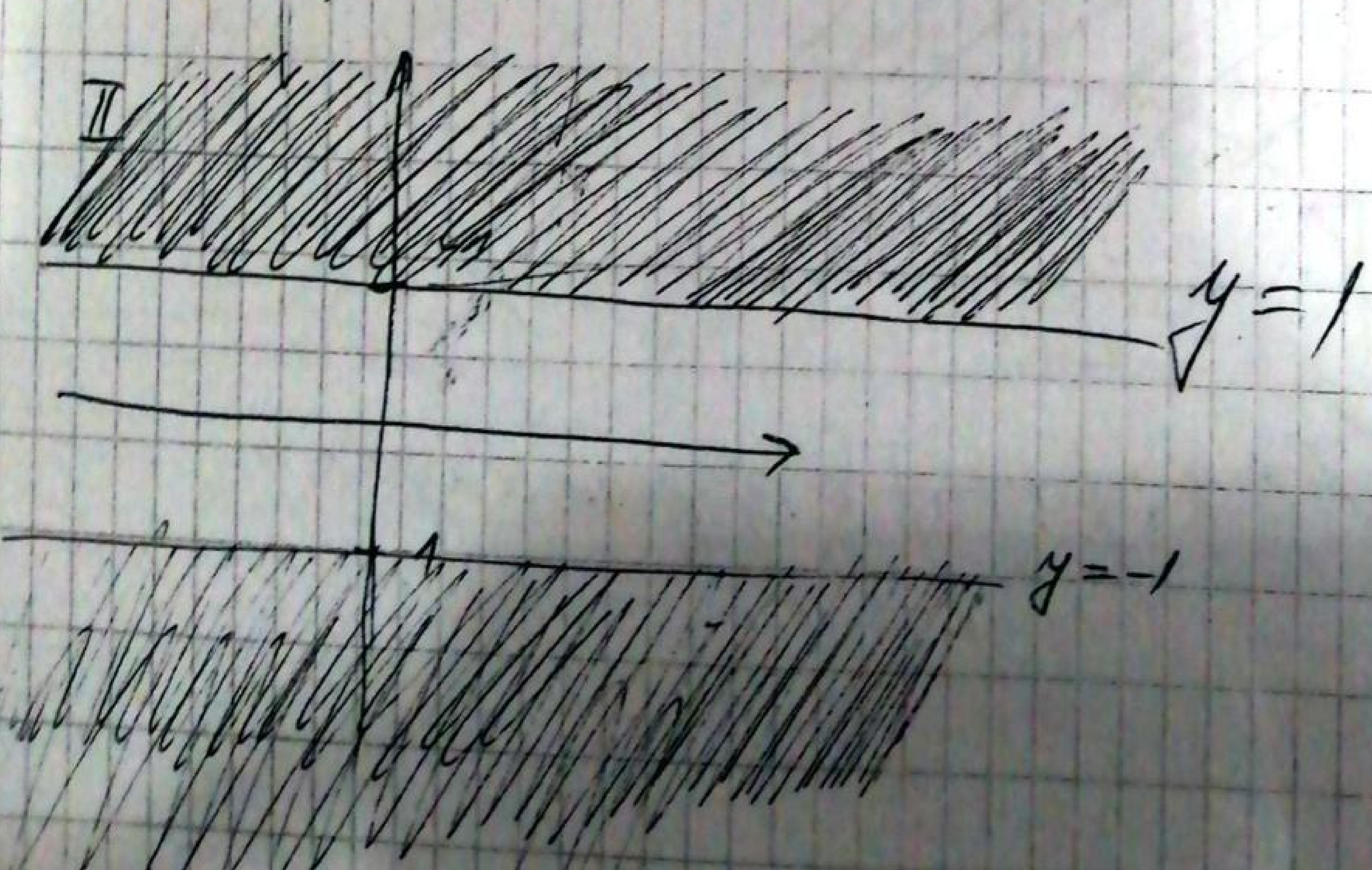
$-1 \leq x \leq 1 \quad \wedge \quad y \in (-\infty,-1) \cup (1,+\infty)$

I

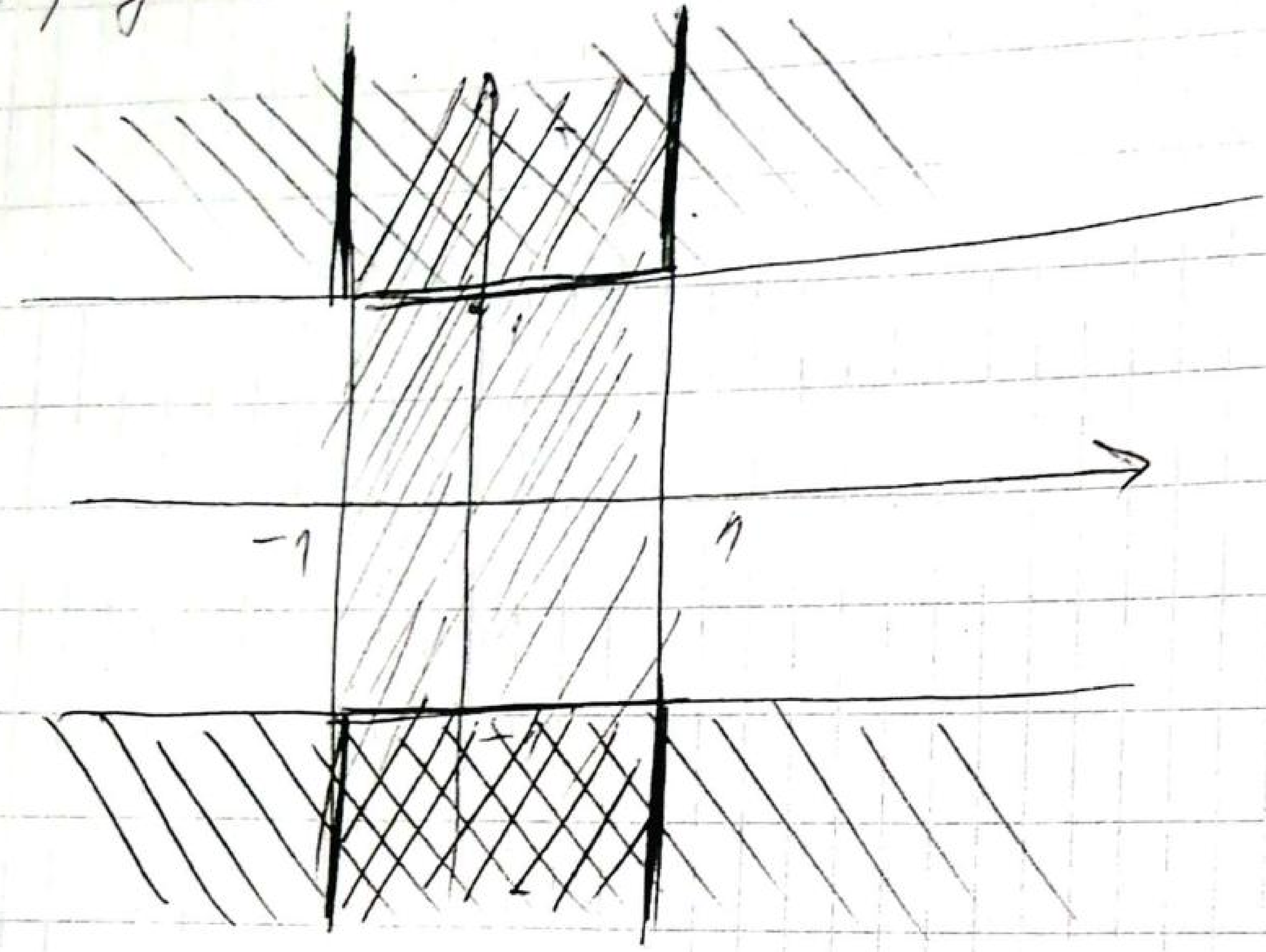


$D(f) = \{(x,y) \mid x \in [-1,1]$
$\wedge$
$y \in (-\infty,-1] \cup [1,+\infty)\}$

II



пресјек



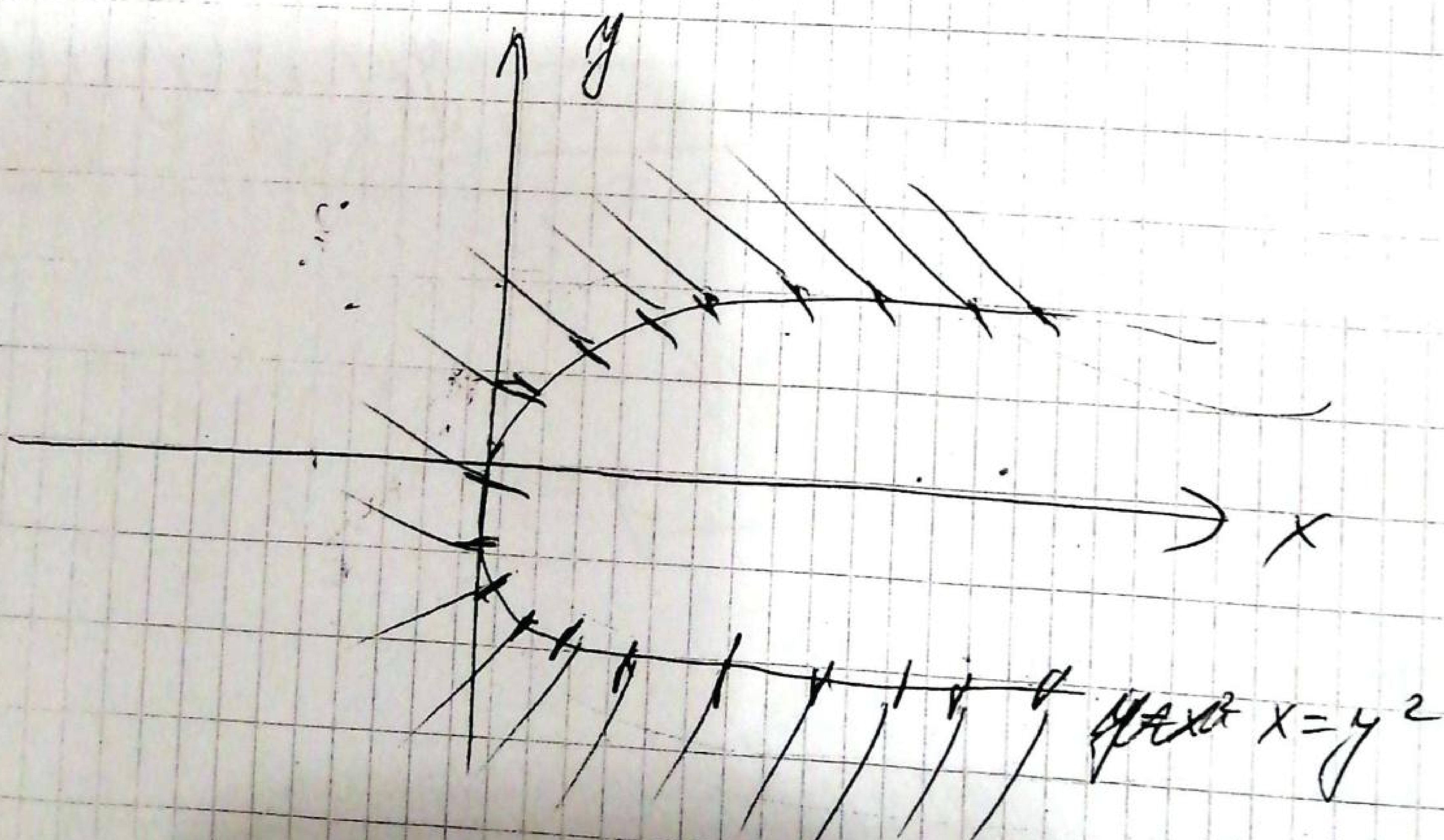
5. $f(x,y) = x \cdot \ln(y^2 - x)$

D: $y^2 - x > 0$

$y^2 > x$

$x < y^2$

$D(f) = \{(x,y) \mid x < y^2\}$



6. Израчунати граничну вриједност низа

а) $\lim\limits_{n\to\infty} \left(\frac{1}{n}, -\frac{1}{n}\right)$

б) $\lim\limits_{n\to\infty} \left(\left(\frac{n}{n+1}\right)^n, \frac{1}{n}-1\right)$

Рј. а) општи члан низа

$$x_n = \left(\frac{1}{n}, -\frac{1}{n}\right)$$

$$x_1 = (1, -1)$$
$$x_2 = \left(\frac{1}{2}, -\frac{1}{2}\right)$$
$$x_3 = \left(\frac{1}{3}, -\frac{1}{3}\right)$$

$$\lim_{n\to\infty} x_n = \left(\lim_{n\to\infty}\frac{1}{n}, \lim_{n\to\infty} -\frac{1}{n}\right) = (0,0)$$

$$\lim_{n\to\infty} x_n = x_0 \quad , x_0 = (0,0)$$

б) $x_n = \left(\left(\frac{n}{n+1}\right)^n, \frac{1}{n}-1\right)$

$$\lim_{n\to\infty} x_n = \left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n, \lim_{n\to\infty}\left(\frac{1}{n}-1\right)\right) =$$

$$= \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^{-1}\right)^n, \lim_{n\to\infty}\left(\frac{1}{n}-1\right) =$$

$$= \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^{-1}, \lim_{n\to\infty}\left(\frac{1}{n}-1\right) = \left(\frac{1}{e}, -1\right)$$

$$\lim_{n\to\infty} x_n = x_0 \quad , x_0 = \left(\frac{1}{e}, -1\right)$$

(4) Доказати да за функцију $f(x,y)=\frac{x^2y^2}{x^2y^2+(x-y)^2}$

важи:

а) $\lim\limits_{x\to0}\left(\lim\limits_{y\to0} f(x,y)\right)=0$ (узастопни лимеси)

б) $\lim\limits_{y\to0}\left(\lim\limits_{x\to0} (f(x,y))\right)=0$

в) не постоји $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

а) $\lim\limits_{x\to0}\left(\lim\limits_{y\to0} f(x,y)\right)=\lim\limits_{x\to0}\left(\lim\limits_{y\to0}\frac{x^2y^2}{x^2y^2+(x-y)^2}\right)=$

$=\lim\limits_{x\to0}(0)=0$

б) $\lim\limits_{y\to0}\left(\lim\limits_{x\to0} f(x,y)\right)=\lim\limits_{y\to0}\left(\lim\limits_{x\to0}\frac{x^2y^2}{x^2y^2+(x-y)^2}\right)=$

$=\lim\limits_{y\to0}(0)=0$

Хајнеови низови

$\lim\limits_{x\to x_0} f(x)=A$ када важи!

$x_n \xrightarrow[n\to\infty]{} x_0 \Longrightarrow f(x_n)\xrightarrow[n\to\infty]{} A$

$\forall x_n \to$ ће да важи.

$\nexists \lim\limits_{x\to x_0} f(x)$

постоје бар два низа (x_n) и (y_n) такви да $\lim\limits_{n\to\infty} x_n = x_0$, $\lim\limits_{n\to\infty} y_n = x_0$, али је

[redacted]

$$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$$

Неka je $x_n = \left(\frac{1}{n}, \frac{1}{n}\right)$

$y_n = \left(\frac{1}{n}, -\frac{1}{n}\right)$

1° $\lim_{n\to\infty} x_n = (0,0)$

$\lim_{n\to\infty} y_n = (0,0)$

2° $x_1 = (1,1)$ $\quad f(x_1) = f(1,1) = 1$

$x_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ $\quad f(x_2) = f\left(\frac{1}{2}, \frac{1}{2}\right) = \dfrac{\frac{1}{4}\cdot\frac{1}{4}}{\frac{1}{4}\cdot\frac{1}{4}+0} = 1$

$x_n = \left(\frac{1}{n}, \frac{1}{n}\right)$ $\quad f(x_n) = f\left(\frac{1}{n}, \frac{1}{n}\right) = \dfrac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^2}\cdot\frac{1}{n^2}+0} = 1$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^2}\cdot\frac{1}{n^2}+0} = 1$$

$y_1 = (1,-1)$ $\quad f(y_1) = f(1,-1) = \dfrac{1}{1+4} = \dfrac{1}{5}$

$y_2 = \left(\frac{1}{2}, -\frac{1}{2}\right)$ $\quad f(y_2) = f\left(+\frac{1}{2}, -\frac{1}{2}\right) = \dfrac{\frac{1}{4}\cdot\frac{1}{4}}{\frac{1}{4}\cdot\frac{1}{4}+1} = \dfrac{\frac{1}{\cancel{16}}}{\frac{2}{\cancel{16}}} = \dfrac{1}{2}$

$y_n = \left(\frac{1}{n}, -\frac{1}{n}\right)$ $\quad f(y_n) = \dfrac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^2}\cdot\frac{1}{n^2}+\frac{4}{n^2}} = \dfrac{\frac{1}{\cancel{n^4}}}{\frac{1+4n^2}{\cancel{n^4}}} =$

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, -\frac{1}{n}\right) = \frac{1}{4n^2+1} = \frac{1}{1+4n^2} = 0$$

$$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$$

Из (1) (и) (2) слиједи да не постоји

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

8. Израчунати граничну вриједност

а) $$\lim_{(x,y)\to(0,a)} \frac{\sin(xy)}{x} = \lim_{(x,y)\to(0,a)} \frac{\sin xy}{xy}\cdot y =$$

$$= \lim_{(x,y)\to(0,a)} \frac{\sin xy}{xy} \cdot \lim_{(x,y)\to(0,a)} y \overset{(*)}{=} 1\cdot a = a$$

(*): $$\lim_{\substack{x\to 0\\ y\to a}} \frac{\sin xy}{xy} = \left[\begin{array}{l} xy = t \\ (x,y)\to(0,a) \Rightarrow t\to 0 \end{array}\right]$$

$$= \lim_{t\to 0} \frac{\sin t}{t} = \underline{1}$$

б) $$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x^2+y^2}{x^4+y^4}$$

$$f(x,y) = \frac{x^2+y^2}{x^4+y^4}$$

$$0 \le f(x,y) = \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4} \le \frac{x^2}{x^4} + \frac{y^2}{y^4} = \frac{1}{x^2}+\frac{1}{y^2} \quad (1)$$

$$0 \xrightarrow[y\to+\infty]{x\to+\infty} 0 \qquad \frac{1}{x^2}+\frac{1}{y^2} \xrightarrow[y\to+\infty]{x\to+\infty} 0$$

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \left(\frac{1}{x^2}+\frac{1}{y^2}\right) = 0 \quad (2)$$

Из (1) и (2) слиједи да је $\lim_{\substack{x\to+\infty\\ y\to+\infty}} f(x,y) = 0$

$$\text{⑫ Naći } \lim_{(x,y)\to(0,2)} (1+xy)^{\frac{2}{x^2+xy}}$$

$$\lim_{(x,y)\to(0,2)} (1+xy)^{\frac{2}{x^2+xy}} = \lim_{\substack{x\to 0\\ y\to 2}} \left((1+xy)^{\frac{1}{xy}} \right)^{\frac{2xy}{x^2+xy}} =$$

$$= \lim_{\substack{x\to 0\\ y\to 2}} \left((1+xy)^{\frac{1}{xy}} \right)^{\frac{2y}{x+y}} \overset{(*)}{=} e^{\lim\limits_{\substack{x\to 0\\ y\to 2}} \frac{2y}{x+y}} = e^2$$

$$(*) \quad \lim_{\substack{x\to 0\\ y\to 2}} (1+xy)^{\frac{1}{xy}} = \begin{bmatrix} xy = t \\ (x,y)\to(0,2) \\ \Rightarrow t\to 0 \end{bmatrix} = \lim_{t\to 0} (1+t)^{\frac{1}{t}} =$$

$$\text{⑬ Naći } \lim_{\substack{x\to +\infty\\ y\to +\infty}} \underbrace{(x^2+y^2)}_{\geq 0} \cdot \underbrace{e^{-(x+y)}}_{\geq 0} \quad \text{Ograničavamo}$$

$$0 \leq (x^2+y^2)\cdot e^{-(x+y)} \leq (x^2+2xy+y^2)\cdot e^{-(x+y)} =$$

$$= (x+y)^2 \cdot e^{-(x+y)} \quad (1) \qquad \frac{\text{polinom}}{\text{exp}} \to 0$$

$$\lim_{\substack{x\to +\infty\\ y\to +\infty}} (x+y)^2 \cdot e^{-(x+y)} = \begin{bmatrix} x+y=t \\ (x,y)\to(+\infty,+\infty) \\ \Rightarrow t\to +\infty \end{bmatrix} =$$

$$= \lim_{t\to +\infty} t^2 \cdot e^{-t} = \lim_{t\to +\infty} \frac{t^2}{e^t} = 0 \quad (2)$$

Iz (1) i (2) $\Rightarrow \lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} f(x,y) = 0$

na osnovu teoreme o uklještenju

14) Naći $\lim\limits_{\substack{x \to +\infty \\ y \to +\infty}} \left(\dfrac{xy}{x^2+y^2} \right)^{x^2}$

Ograničavamo:

$$(x-y)^2 \geq 0 \qquad x^2+y^2 \geq 2xy \quad / \cdot \frac{1}{x^2+y^2}$$

$$x^2 - 2xy + y^2 \geq 0$$

$$1 \geq \frac{2xy}{x^2+y^2} \quad / \cdot \frac{1}{2}$$

$$\frac{1}{2} \geq \frac{xy}{x^2+y^2} \rightarrow \frac{xy}{x^2+y^2} \leq \frac{1}{2}$$

$$0 \leq \left(\frac{xy}{x^2+y^2} \right)^{x^2} \leq \left(\frac{1}{2} \right)^{x^2} \quad (1)$$

$$\lim_{(x,y) \to (+\infty,+\infty)} \left(\frac{1}{2} \right)^{x^2} = \left[\lim_{n \to \infty} q^n = 0 \text{ ako } |q| < 1 \right] = 0 \quad (2)$$

Iz (1) i (2) na osnovu teoreme o ukliještenju slijedi $\Rightarrow \lim\limits_{\substack{x \to +\infty \\ y \to +\infty}} \left(\dfrac{xy}{x^2+y^2} \right)^{x^2} = 0$

---

15) Ispitati da li postoje:

a) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^2-y^2}{x^2+y^2}$ → cilj pokazati da fja nema gr. vr. u $(0,0)$

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

→ Neka je $x_n = \left(\frac{1}{n}, \frac{1}{n} \right)$ i $y_n = \left(\frac{1}{n}, \frac{2}{n} \right)$

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \left(\frac{1}{n}, \frac{1}{n} \right) = (0,0)$$

$$\lim_{n\to\infty}(y_n) = \lim_{n\to\infty}\left(\frac{1}{n}, \frac{2}{n}\right) = (0,0)$$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2} - \frac{1}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}} = 0$$

Konst. niz

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{2}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2} - \frac{4}{n^2}}{\frac{5}{n^2}} = -\frac{3}{5}$$

Konstantan niz

1° $\lim_{n\to\infty} x_n = (0,0) = \lim_{n\to\infty}(y_n)$

2° $\underbrace{\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)}$

ne postoji gr. vr. $\lim_{(x,y)\to(0,0)} f(x,y)$ → f-ja nije neprekidna

b) $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = 1$

$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = -1$

Izvesti zaključak ⇒ nema gr. vr. u $(0,0)$

---

16. Naći $\lim_{(x,y)\to(+\infty,+\infty)} \frac{x+y}{x^2+xy+y^2}$

$$0 \leq \frac{x+y}{x^2+xy+y^2} \leq \frac{x+y}{x\cdot y} = \frac{x}{xy} + \frac{y}{xy} = \frac{1}{y} + \frac{1}{x} \; (1)$$

$$\lim_{\substack{x\to\infty \\ y\to\infty}}\left(\frac{1}{y} + \frac{1}{x}\right) = 0 \quad (2)$$

iz (1) i (2) slijedi da je $\lim_{\substack{x\to+\infty \\ y\to+\infty}} \frac{x+y}{x^2+xy+y^2} = 0$

8. $\lim_{(x,y)\to(\infty,a)} \left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}} \overset{1^\infty}{=} \lim_{\substack{x\to\infty \\ y\to a}} e^{\ln\left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}} = e^{l} = e^{1} = e$

$$l = \lim_{\substack{x\to\infty \\ y\to a}} \ln\left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}} = \lim_{\substack{x\to\infty \\ y\to a}} \left(\underbrace{\left(\frac{x}{x+y}\right)}_{l_1} \cdot \underbrace{\ln\left(1+\frac{1}{x}\right)^{x}}_{l_2}\right)$$

$$l_1 = \lim_{\substack{x\to\infty \\ y\to a}} \ln\left(1+\frac{1}{x}\right)^{x} = \ln e = 1$$

$$l_2 = \lim_{\substack{x\to\infty \\ y\to a}} \left(\frac{x}{x+y}\right) = \lim_{\substack{x\to\infty \\ y\to a}} \frac{1}{1+\underbrace{\frac{y}{x}}_{\to 0}} = 1$$

$\Rightarrow \varepsilon \to 0$

21) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{x}{y}$ $\quad a_n = \left(0, \frac{1}{n}\right)$ → za b_n ne možemo $\left(\frac{1}{n}, 0\right)$ jer $\frac{x}{y}$, $y \neq 0$

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} \frac{0}{\frac{1}{n}} = 0$$

$$b_n = \left(\frac{1}{n}, \frac{1}{n}\right)$$

$$\lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{n}} = 1$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = (0,0)$$

$$\lim_{n \to \infty} f(a_n) \neq \lim_{n \to \infty} f(b)$$ $\Rightarrow$ granična vrijednost ne postoji

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} x^2 \cdot e^{-(x-y^2)}$$

Vačimo 2 niza: $x_n=(n^2,n)$ i $y_n=(5n^2,n)$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = (\infty,\infty)$$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} n^4 \cdot e^{-(n^2-n^2)} = \lim_{n\to\infty} n^4 = \infty$$

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 25\cdot n^4 \cdot e^{-(5n^2-n^2)} = \lim_{n\to\infty} \frac{25n^4}{e^{4n^2}} = 0$$

Ne postoji $\lim_{\substack{x\to\infty\\ y\to\infty}} f(x,y)$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) =$$

---

25. Ispitati neprekidnost f-je $f(x,y) = \begin{cases} \frac{x^2 \cdot y}{x^4+y^2} & (x,y) \\ 0 & (x,y) \end{cases}$

Za $(x,y) \neq (0,0)$ f-ja je neprekidna kao količnik dvije neprekidne funkcije oblasti definisanosti.

Pokažimo da f nije neprekidna u $(0,0)$

Neka je $x_n = \left(\frac{1}{n}, \frac{1}{n}\right)$, a $y_n = \left(\frac{1}{n}, \frac{1}{n^2}\right)$.

Tada je:

1° $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = (0,0)$

2° $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot\frac{1}{n}}{\frac{1}{n^4}+\frac{1}{n^2}} =$

$= \lim_{n\to\infty} \frac{n}{1+n^2} = 0$

$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2\cdot n^2}}{\frac{1}{n^4}+\frac{1}{n^4}} = \frac{1}{2}$

$\to \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$

$\to$ iz 1° i 2° zaključujemo da granična vrijednost funkcije ne postoji u $(0,0)$, da postoji, morali bi biti jednaki $\lim_{n\to\infty} f(x_n)$ i $\lim_{n\to\infty} f(y_n)$

$\Rightarrow$ pa f-ja $f(x,y)$ nije neprekidna u $(0,0)$

(29.) Ispitati neprekidnost funkcije:

$$f(x,y) = \begin{cases} \dfrac{\sin(x+y)}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

→ u tačkama Euklidove ravni za koje je $(x,y) \neq (0,0)$ funkcija f je neprekidna kao kompozicija neprekidnih funkcija. Pokažimo da fja f ima prekid u $(0,0)$

$x_n = \left(0, \frac{1}{n}\right)$, $y_n = \left(0, -\frac{1}{n}\right)$

može i $\left(\frac{1}{n}, \frac{1}{n}\right)$ $\left(-\frac{1}{n}, \frac{1}{n}\right)$

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = (0,0)$

$\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} \dfrac{\sin\frac{1}{n}}{\frac{1}{n}} = \left[\begin{matrix}\frac{1}{n} = m \\ n\to\infty \\ m\to 0\end{matrix}\right] = \lim\limits_{m\to 0} \dfrac{\sin m}{m} =$

$\lim\limits_{n\to\infty} f(y_n) = \lim\limits_{n\to\infty} \dfrac{\sin\left(-\frac{1}{n}\right)}{\frac{1}{n}} = -\lim\limits_{n\to\infty} \dfrac{\sin\frac{1}{n}}{\frac{1}{n}} = -1$

$\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(y_n)$ → ne postoji $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

pa fja f u $O(0,0)$ ima prekid

(30.) Ispitati neprekidnost funkcije:

$$f(x,y) = \begin{cases} \dfrac{\sin^2(x+y)}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

→ Funkcija je neprekidna u tačkama za koje je $(x,y) \neq (0,0)$ kao kompozicija neprekidnih funkcija.

$$0 \le \sin^2(x+y) \le |x+y|^2 = (x+y)^2 = x^2+2xy+y^3 \le$$

$$\le x^2 + \underline{x^2+y^2} + y^2$$

$$0 \le \sin^2(x+y) \le 2(x^2+y^2)$$

$$0 \le \frac{\sin^2(x+y)}{\sqrt{x^2+y^2}} \le 2\sqrt{x^2+y^2} \quad (2)$$

$$\lim_{(x,y)\to(0,0)} 2\sqrt{x^2+y^2} = 0 \quad (3)$$

→ iz (2) i (3) slijedi da je $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

→ pa je fja f neprekidna u tački $(0,0)$

$(x-y)^2 \ge 0 \longrightarrow x^2-2xy+y^2 \ge 0$ $2xy \le x^2+y^2$	Zapamtiti! često se koristi