8. Нати повтанни диференціал I реда за інтуїтивною задану функцію

$$\frac{x}{z} = \ln\frac{z}{y} + 1 \qquad z = z(x, y)$$

$$\boxed{dz = \frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy}$$

$$F(x, y, z) = \frac{x}{z} - \ln\frac{z}{y} - 1$$

$$F(x, y, z) = 0$$

$$\frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}\cdot 0 + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$z'_x = \frac{-F'_x}{F'_z}$$

$$\frac{\partial F}{\partial x}\cdot 0+\frac{\partial F}{\partial y}\cdot 1+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y}=0$$

$$\frac{\partial z}{\partial y}=-\frac{F'y}{F'z}$$

$$F'(x)=\frac{1}{z}$$

$$F'(y)=-\frac{y}{z}\left(-\frac{z}{y^2}\right)=\frac{1}{y}$$

$$F'(z)=-\frac{x}{z^2}-\frac{y}{z}:\frac{1}{y}=-\frac{x+z}{z^2}$$

$$\frac{\partial z}{\partial x}=-\frac{\frac{1}{z}}{-\frac{x+z}{z^2}}=\frac{z}{x+z}$$

$$\frac{\partial z}{\partial y}=\frac{-\frac{1}{y}}{\frac{x+z}{z^2}}=\frac{z^2}{y(x+z)}$$

$$\boxed{dz=\frac{z}{x+z}\,dx+\frac{z^2}{y(x+z)}\cdot dy}$$

① Трансформисати једначину $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ увођењем нових променљивих $u = x + y$

$v = x - y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \overset{=1}{\frac{\partial u}{\partial x}} + \frac{\partial z}{\partial v} \cdot \overset{=1}{\frac{\partial u}{\partial x}} \; ; \; \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \overset{=1}{\frac{\partial u}{\partial y}} + \frac{\partial z}{\partial v} \cdot \overset{=-1}{\frac{\partial v}{\partial y}}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$\cancel{\frac{\partial z}{\partial u}} + \frac{\partial z}{\partial v} = \cancel{\frac{\partial z}{\partial v}} - \frac{\partial z}{\partial v}$$

$$\frac{2\partial z}{\partial v} = 0$$

Трансформисати једначину увођењем нових променљивих

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{x}{z}$$

$$u = 2x - z^2 \qquad v = \frac{y}{z} \qquad z = z(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(2 - 2z \cdot \frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial v} \cdot y(-1) \cdot z^{-2} \cdot \frac{\partial z}{\partial x}$$

$$\left(1 + 2z \frac{\partial z}{\partial u} + \frac{y}{z^2} \cdot \frac{\partial z}{\partial v}\right) \cdot \frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial u}$$

$$\frac{\partial z}{\partial x} = \frac{2 \cdot \frac{\partial z}{\partial u}}{1 + 2z \cdot \frac{\partial z}{\partial u} + \frac{y}{z^2} \cdot \frac{\partial z}{\partial v}}$$

$$\frac{\partial z}{\partial x} = \frac{2 \cdot \frac{\partial z}{\partial u}}{1 + 2z \cdot \frac{\partial z}{\partial u} + \frac{y}{z^2} \cdot \frac{\partial z}{\partial v}}$$

$$\frac{\partial z}{\partial v} = \frac{2 \cdot \frac{\partial z}{\partial u}}{1 + 2z \frac{\partial z}{\partial u} + \frac{y}{z^2} \cdot \frac{\partial z}{\partial v}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{2 \cdot \frac{\partial z}{\partial u}}{1 + 2z \frac{\partial z}{\partial u} + \frac{v}{z} \cdot \frac{\partial z}{\partial v}}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(-2z \frac{\partial z}{\partial y}\right) + \frac{\partial z}{\partial x} \frac{z - y \cdot \frac{\partial z}{\partial y}}{z^2}$$

$$\left(1 + 2z \frac{\partial z}{\partial u} + \frac{}{z^2} \frac{\partial z}{\partial v}\right) \cdot \frac{\partial z}{\partial y} = \frac{1}{z} \cdot \frac{\partial z}{\partial v}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{\frac{1}{z} \cdot \frac{\partial z}{\partial v}}{1 + 2z \frac{\partial z}{\partial u} + \frac{v}{z} \cdot \frac{\partial z}{\partial v}}}$$

$$\frac{2x \cdot \frac{\partial z}{\partial u} + \frac{y}{z} \frac{\partial z}{\partial x}}{1 + 2z \frac{\partial z}{\partial u} + \frac{v}{z} \frac{\partial z}{\partial v}} = \frac{x}{z}$$

$$\frac{y}{z} = v \qquad x = \frac{u + z^2}{2}$$

$$\frac{\partial z}{\partial v} = \frac{z}{v} \cdot \frac{u + z^2}{z^2 - u}$$

Naći $d^2 z$ ako je $z = f(t)$ gdje je
$t = x^2 + y^2$

$$d^2 z = \frac{\partial^2 t}{\partial x^2} \cdot dx^2 + \frac{\partial^2 z}{\partial y^2} \cdot dy^2 \cdot 2 \frac{\partial z}{\partial x \cdot \partial y} \cdot dx \cdot dy$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$\frac{\partial z}{\partial x} = z'_t \cdot 2x$$

$$\frac{\partial z}{\partial y} = z'_t \cdot 2y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2x \cdot z'_t) = 2z'_t + 2x \cdot z''_t \cdot \underbrace{\frac{\partial t}{\partial x}}_{=2x} = \boxed{= 2z'_t + 4x^2 z''_t}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(2y z'_t) = 2z'_t + 2y \cdot z''_t \cdot \frac{\partial t}{\partial y} = \boxed{= 2z'_t + 4y^2 z''_t}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(2y \cdot z'_t) = 2y \cdot z''_t \cdot \frac{\partial t}{\partial x} =$$

$$= 4xy \cdot z''_t$$

$$d^2 z = (2z'_t + 4x^2 z''_t) \cdot dx^2 + (\ xy z''_t) \cdot dx \cdot dy +$$

$$+ (2z'_t + 4y^2 z''_t) \cdot dy^2$$

13. Под претпоставком да су g и h диференцијабилне довољан број пута доказати да важи:

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = 0,$$

ако је

$$u = g\left(\frac{y}{x}\right) + x \cdot h\left(\frac{y}{x}\right).$$

Рјешење:

$$t = \frac{y}{x}$$

$$g = g(t) \qquad g'(t) = g'$$

$$h = h(t) \qquad h'(t) = h'$$

$$\frac{\partial u}{\partial x} = g' \frac{dt}{dx} + 1 \cdot h + x \cdot h' \frac{dt}{dx}$$

$$\frac{\partial u}{\partial x} = -\frac{y}{x^2} g' - \frac{y}{x} \cdot h'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2y}{x^3} \cdot g' - \frac{y}{x^2} \cdot g'' \cdot \frac{dt}{dx} + h' \frac{dt}{dx} + \frac{y}{x^2} \cdot h' - \frac{y}{x} \cdot h'' \cdot \frac{dt}{dx}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2y}{x^3} \cdot g' + \frac{y^2}{x^4} \cdot g'' \cancel{- \frac{y}{x^2} \cdot h'} \cancel{+ \frac{y}{x^2} \cdot h'} + \frac{y^2}{x^3} \cdot h''$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2y}{x^3}\cdot g' + \frac{y^2}{x^4}\cdot g'' + \frac{y^2}{x^3}\cdot h''$$

$$\frac{\partial u}{\partial y} = g \cdot \qquad + x\cdot h'\cdot \frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y} = g'\cdot \frac{1}{x} + x\cdot h'\cdot \frac{1}{x} = \frac{1}{x}\cdot g' + h'$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x}\cdot g''\cdot \frac{\partial t}{\partial y} + h''\cdot \frac{\partial t}{\partial y} = \frac{1}{x}\cdot g''\cdot \frac{1}{x} + h''\cdot \frac{1}{x}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x^2}\cdot g'' + \frac{1}{x}\cdot h''$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\cdot\left(\frac{1}{x}\cdot g' + h'\right) =$$

$$= -\frac{1}{x^2}\cdot g' + \frac{1}{x}\cdot g''\cdot \frac{\partial t}{\partial x} + h''\cdot \frac{\partial t}{\partial x} =$$

$$= -\frac{1}{x^2}\cdot g' + \frac{1}{x}\cdot g''\cdot\left(-\frac{y}{x^2}\right) + h''\cdot\left(-\frac{y}{x^2}\right) =$$

$$= -\frac{1}{x^2}\cdot g' - \frac{y}{x^3}\cdot g'' - \frac{y}{x^2}\cdot h''$$

Уврстимо у једначину добијене изводе

$$x^2\cdot\left(\frac{2y}{x^3}\cdot g' + \frac{y^2}{x^4}\cdot g'' + \frac{y^2}{x^3}\cdot h''\right) +$$

$$+ 2xy\cdot\left(-\frac{1}{x^2}\cdot g' - \frac{y}{x^3}\cdot g'' - \frac{y}{x^2}\cdot h''\right) +$$

$$+ y^2\cdot\left(\frac{1}{x^2}\cdot g'' + \frac{1}{x}\cdot h''\right) =$$

$$= \cancel{\frac{2y}{x}\cdot g'} + \cancel{\frac{y^2}{x^2}\cdot g''} + \frac{y^2}{x}\cdot h'' - \cancel{\frac{2y}{x}g'} - \cancel{\frac{2y^2}{x^2}\cdot g''} -$$

$$- \cancel{\frac{2y^2}{x}\cdot h''} + \cancel{\frac{y^2}{x^2}\cdot g''} + \frac{y^2}{x}\cdot h'' = 0$$

5. Neka je $z = z(x,y)$ i $F\left(x + \frac{z}{y}, y + \frac{z}{x}\right) = 0$ i F diferencijabilna funkcija. Naći $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$

→ z je fja od x i y → zadata je implicitno

Neka je $U = x + \frac{z}{y}$ a $V = y + \frac{z}{x}$

$F(U, V) = 0$

$$\boxed{\begin{aligned} z'_x &= -\frac{F'_x}{F'_z} \\ z'_y &= -\frac{F'_y}{F'_z} \end{aligned}}$$

→ ovo važi kako god da je zadata F

impl fja

$$F'_x = \frac{\partial F}{\partial U} \cdot \frac{\partial U}{\partial x} + \frac{\partial F}{\partial V} \cdot \frac{\partial V}{\partial x}$$

$$F'_x = \frac{\partial F}{\partial U} \cdot 1 + \frac{\partial F}{\partial V} \cdot \left(-\frac{z}{x^2}\right)$$

$$F'_y = \frac{\partial F}{\partial U} \cdot \frac{\partial U}{\partial y} + \frac{\partial F}{\partial V} \cdot \frac{\partial V}{\partial y}$$

$$F'_y = -\frac{z}{y^2} \cdot \frac{\partial F}{\partial U} + 1 \cdot \frac{\partial F}{\partial V} \quad !!!$$

$$F'_z = \frac{\partial F}{\partial U} \cdot \frac{\partial U}{\partial z} + \frac{\partial F}{\partial V} \cdot \frac{\partial V}{\partial z}$$

$$F'_z = \frac{1}{y} \cdot \frac{\partial F}{\partial U} + \frac{1}{x} \cdot \frac{\partial F}{\partial V}$$

→ imamo sve, samo uvrstimo

$$\frac{\partial Z}{\partial x} = -\frac{\frac{\partial F}{\partial u} - \frac{z}{x^2} \cdot \frac{\partial F}{\partial V}}{\frac{1}{y} \cdot \frac{\partial F}{\partial u} + \frac{1}{x} \cdot \frac{\partial F}{\partial V}}$$

$$\frac{\partial Z}{\partial y} = -\frac{-\frac{z}{y^2} \cdot \frac{\partial F}{\partial u} + \frac{\partial F}{\partial V}}{\frac{1}{y} \cdot \frac{\partial F}{\partial u} + \frac{1}{x} \cdot \frac{\partial F}{\partial V}}$$