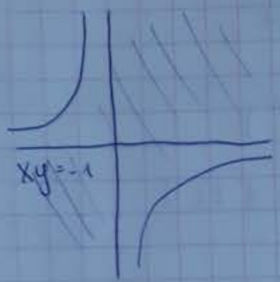
Vježbe 2

① Odrediti domen f-je $f(x,y) = \ln(1+xy)$.



② Dokazati po definiciji da $f(x,y) = x \cdot y$ je neprekidna f-ja u $\mathbb{R}^2$.

Fiksiramo tačku $(x_0, y_0) \in \mathbb{R}^2$ (proizvoljno)

$$|f(x,y) - f(x_0,y_0)| = |xy - x_0 y_0| = |((x-x_0)+x_0)((y-y_0)+y_0) - x_0 y_0|$$

$$|f(x,y) - f(x_0,y_0)| \leq |x-x_0||y-y_0| + |x_0-x||y_0| + |x_0||y-y_0| \leq$$

$$\leq |y_0| \cdot \sqrt{|x_0-x|^2 + |y_0-y|^2} + |x_0| \sqrt{|x-x_0|^2 + |y-y_0|^2} +$$

$$+ \|(x,y) - (x_0,y_0)\|^2 =$$

$$= \underbrace{|x_0| + |y_0|}_{b} \cdot \underbrace{\|(x,y) - (x_0,y_0)\|}_{z} + \underbrace{\|(x,y) - (x_0,y_0)\|^2}_{z^2}$$

ε, δ, $\|(x,y) - (x_0,y_0)\| < \delta \Rightarrow |f(x,y) - f(x_0,y_0)| < \varepsilon$

$$zb - z^2 - \varepsilon < 0$$

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 + 4\varepsilon}}{2}$$

Biramo pozitivne $\Rightarrow \delta \in \left(0, \frac{\sqrt{b^2+4\varepsilon}}{2} - \frac{b}{2}\right)$

$b = (|x_0| + |y_0|)$

$$\sqrt{\left(\frac{b}{2}\right)^2 + \varepsilon} - \frac{b}{2} > 0$$

③ Ispitati ravnomjernu neprekidnost za f-ju:

a) $f(x,y,z) = \sin(x^2+y^2+z^2)$, $(x,y,z) \in \mathbb{R}^3$.

$(x,y,z) \mapsto \|(x,y,z)\|^2 \to \sin(\|x,y,z\|^2)$

$| \|(x,y,z)\| - \|(u,v,w)\| | \leq \|(u,v,w) - (x,y,z)\|$

Def F-ja f - ravnomjerno neprekidna ako:

$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)(\forall x, y \in \mathbb{R}^3)\ (\|x-y\| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon)$

negacija:

$(\exists \varepsilon_0 \geq 0)(\forall \delta > 0)(\exists x, y \in \mathbb{R}^3)(\|x-y\| < \delta \wedge |f(x) - f(y)| \geq \varepsilon_0)$

$\{(x_n, y_n, z_n)\}_{n \geq 1}$

$\{(x_n', y_n', z_n')\}_{n \geq 1}$

$\|(x_n, y_n, z_n) - (x_n', y_n', z_n')\| \to 0$, $n \to \infty$

$|f(x_n, y_n, z_n) - f(x_n', y_n', z_n')| \geq \varepsilon_0$

$(x_n, y_n, z_n) = \left(\sqrt{\frac{\pi}{2} + 2n\pi}, 0, 0\right)$, $n \in \mathbb{N}$

$(x_n', y_n', z_n') = \left(\sqrt{2n\pi}, 0, 0\right)$, $n \in \mathbb{N}$

$$\|(x_n, y_n, z_n) - (x_n', y_n', z_n')\|^2 = \sqrt{\frac{\pi}{2} + 2n\pi} - \sqrt{2n\pi} =$$

$$= \frac{\frac{\pi}{2}}{\sqrt{\frac{\pi}{2} + 2n\pi} + \sqrt{2n\pi}}$$

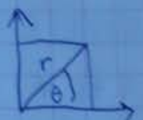
$\|(x_n, y_n, z_n) - (x_n', y_n', z_n')\| \to 0$, $n \to \infty$

$\| f(x_n, y_n, z_n) - f(x_n', y_n', z_n') \| = 1 \geq \frac{1}{2}$

Dva niza se približavaju, ali razlika im je konstanta, tako da f nije ravnomj. neprekidna

b) $f(x,y) = \begin{cases} \sqrt{x^2+y^2} \cdot \sin \frac{1}{\sqrt{x^2+y^2}} &, (x,y) \neq (0,0) \\ 0 &, (x,y) = (0,0) \end{cases}$

Za $(x,y) \neq (0,0) \rightarrow f \in C(\mathbb{R}^2 \setminus \{(0,0)\})$



$r = \sqrt{x^2+y^2}$

$x = r \cdot \cos\theta \qquad 0 \leq r < +\infty$

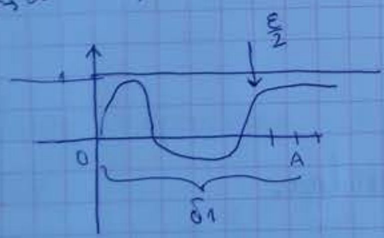
$y = r \cdot \sin\theta \qquad 0 \leq \theta < 2\pi \qquad ([-\pi, \pi])$

$f(x,y) = f(r\cdot\cos\theta,\ r\cdot\sin\theta) = r\cdot\sin\frac{1}{r} = \phi(r)$

$\lim\limits_{r\to 0} \phi(r) = \lim\limits_{r\to 0} r\cdot\sin\frac{1}{r} = 0 = f(0,0)$

$\phi : [0, +\infty] \rightarrow \mathbb{R}$

$\lim\limits_{r\to\infty} \frac{\sin\frac{1}{r}}{\frac{1}{r}} = 1 , \quad \lim\limits_{n\to\infty} \phi(r) = 1$



$\varepsilon > 0,\ \exists A > 0,\quad r > A \Rightarrow |1 - \phi(r)| < \varepsilon$

$A_0 ,\ r > A_0 \Rightarrow |1 - \phi(r)| < \frac{\varepsilon}{2}$

$\phi \in C([0, A_0])$ – ϕ je ravnomjerno neprekidna na $[0, A_0]$

Za $r, r' \in (A_0, +\infty)$

$|\phi(r) - \phi(r')| \leq |\phi(r) - 1| + |\phi'(r) - 1| < \varepsilon$

4) Ispitati neprekidnost funkcije:

a) $f(x,y) = \begin{cases} \dfrac{x^3 y}{x^6+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

b) $f(x,y,z) = \begin{cases} x+z \cdot \dfrac{\sin(x^2+y^2+z^2)}{x^2+y^2+z^2}, & (x,y,z) \neq (0,0,0) \\ 0, & (x,y,z) = (0,0,0) \end{cases}$

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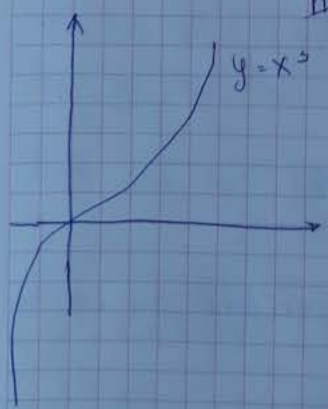
b) sami (smjena $t \to 0$)

a) Za $(x,y) \neq (0,0)$ - f- neprekidna kao količnik neprekidnih f-ja ($f \in C(\mathbb{R}^2 \setminus \{(0,0)\})$)

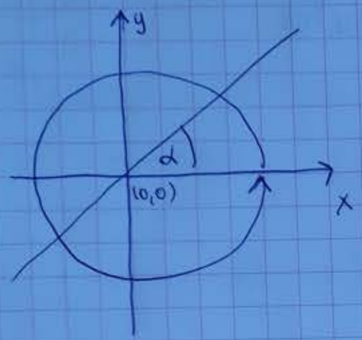
$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^3}\right) \to (0,0), \quad n \to \infty$

$f(x_n, y_n) = \dfrac{\frac{1}{n^6}}{\frac{2}{n^6}} = \frac{1}{2} \neq f(0,0)$



Nije neprekidna

5. Dokazati da je f-ja $f(x,y)=\begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$ neprekidna u $(0,0)$ duž svakog pravca iako nije neprekidna u $(0,0)$.



$$(\underbrace{t\cdot\cos\alpha}_{x},\ \underbrace{t\cdot\sin\alpha}_{y}),\quad \alpha\in[0,2\pi]$$

$$0\le t\le+\infty$$

$$\lim_{t\to0} f(t\cdot\cos\alpha,\ t\cdot\sin\alpha)=\lim_{t\to0}\frac{t^3\cdot\cos^2\alpha\cdot\sin\alpha}{t^4\cdot\cos^4\alpha+t^2\cdot\sin^2\alpha}=$$

$$=\lim_{t\to0}\frac{t\cdot\cos^2\alpha\cdot\sin\alpha}{t^2\cdot\cos^4\alpha+\sin^2\alpha}=\begin{cases}0, & \alpha=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}\\ 0, & \alpha\neq0,\frac{\pi}{2},\pi,\frac{3\pi}{2}\end{cases}$$

Jeste neprekidna na svakom pravcu, ali po paraboli nije.

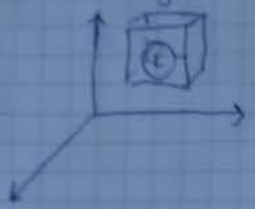
$$(x_n,y_n)=\left(\frac{1}{n},\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} f(x_n,y_n)=\lim_{n\to\infty}\frac{\frac{1}{n^4}}{\frac{2}{n^4}}=\frac{1}{2}\neq f(0,0)$$

⑥ Ako je $K \subset \mathbb{R}^n$ kompaktan skup, dokazati da svaki niz $\{x_k\}_{k \geq 1}$, $x_k \in K$ $(k \in \mathbb{N})$ ima konvergentan podniz koji konvergira u K.

$K \subset \mathbb{R}^n \to K$ je zatvoren i ograničen

$\exists I^{\circ}_{a,b}$



$I^n = \{ x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \}$

Za niz $x_k = (x_1^k, \dots, x_n^k)$, $k \in \mathbb{N}$, $x_k \in K$

$a_1 \leq x_1^k \leq b_1$, $k \in \mathbb{N}$

$a_2 \leq x_2^k \leq b_2$, $k \in \mathbb{N}$

$\vdots$

$a_n \leq x_n^k \leq b_n$, $k \in \mathbb{N}$

$\Rightarrow \exists x_1^{k_m}$, $\exists \lim_{m \to \infty} x_1^{k_m} = x_1^{\circ} \in [a_1, b_1] \Rightarrow a_2 \leq x_2^{k_m} \leq b_2$

$\Rightarrow \{\exists x_2^{k_{m_j}}\}$, $\lim_{j \to \infty} x_2^{k_{m_j}} = x_2^{\circ} \in [a_2, b_2]$

Ponavljajući postupak n puta $\Rightarrow$

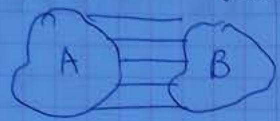
$\{x_{k_j}\}_{j \geq 1} \subset \{x_k\}_{k \geq 1}$

$\exists \lim_{k_j \to \infty} x_i^{k_j} = x_i^{\circ}$, $1 \leq i \leq n \Rightarrow \lim_{k_j \to \infty} x_{k_j} = x_0$, $x_0 = (x_{01}^{\circ}, \dots, x_n^{\circ})$

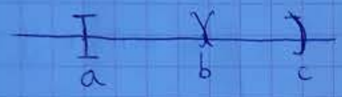
$K = K' \Rightarrow x_0$ - tačka nagomilavanja $\Rightarrow x_0 \in K$

Def. Za $A, B \subset X$, (X,d) - metrički prostor, rastojanje

$$d(A,B) = \inf_{\substack{x \in A \\ y \in B}} d(x,y)$$



7. Dokazati da ako su $K, F \subset \mathbb{R}^n$, K- kompaktan, F- zatvoren i $K \cap F = \emptyset \Rightarrow d(K,F) > 0$.

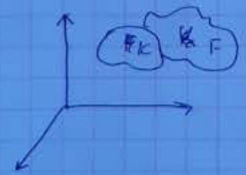


$$d([a,b), (b,c)) = 0$$

$$[a,b) \cap (b,c) = \emptyset$$

Dk.

Pretpostavimo suprotno, tj. K- kompaktan, F- zatvoren, $d(K,F) = 0$.



Za $k \in \mathbb{N}$ postoje $x_k \in K$, $y_k \in F$, $d(x_k, y_k) < \frac{1}{k} \Rightarrow$

$$\Rightarrow \lim_{k \to \infty} d(x_k, y_k) = 0.$$

$$\{x_k\}_{k \geq 1} \subset K \Rightarrow \exists \{x_{k_m}\} \subset \{x_k\}_{k \geq 1}$$

$$\exists \lim_{m \to \infty} x_{k_m} = x_0 \in K$$

$$x_{k_m} \longleftrightarrow y_{k_m} \in F$$

$$d(x_{k_m}, y_{k_m}) < \frac{1}{k_m} \quad \Big| \; m \to \infty$$

fiksirano

$$\Downarrow$$

$\Downarrow$

$d(x_0, y_{km}) \leq \frac{1}{km}$

$\lim_{m\to\infty} d(x_0, y_{km}) = 0,$

$y_{km} \to x_0, \quad m \to \infty$

$\Rightarrow$

$x_0 \in F \Rightarrow \{x_0\} \subset F \cap K = \emptyset \quad \to$ kontradikcija $\lightning$

$\Rightarrow \quad \underline{d(K, F) > 0}$

$$\boxed{\begin{aligned} &d(x,y) = \|x-y\| \\ &\big| \|x-y\| - \|z-y\| \big| \leq \|x-z\| \\ &|d(x,y) - d(z,y)| \leq d(x,z) \end{aligned}}$$

8) Dokazati da za f-ju $f(x,y) = \begin{cases} x \sin\frac{1}{y} + y, & y \neq 0, x \in R \\ 0, & y = 0, x \in R \end{cases}$ ne postoje ponovljeni limesi, ali postoji $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$

Rj. $|f(x,y)| \leq |x| + |y|$

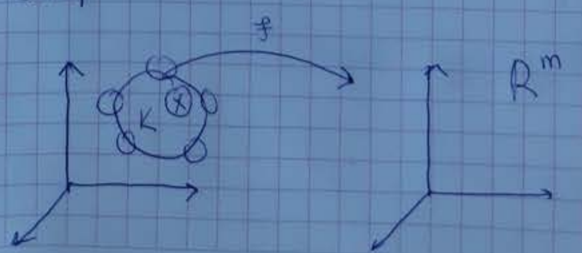
$\lim_{\substack{x\to 0\\ y\to 0}} |f(x,y)| = 0$

$\lim_{x\to 0} \left(\lim_{y\to 0} f(x,y) \right) \to$ ne postoji

$\lim_{y\to 0} \left(\lim_{x\to 0} x \cdot \sin\frac{1}{y} + y \right) = 0$

9) Dokazati da je svaka neprekidna funkcija: $K \subset R^n \to R^m$ ravnomjerna na K, K - kompaktan skup.

Rj.



$\varepsilon > 0, \ x \in K$

$\frac{\varepsilon}{2}, \ (\exists\, \delta(x, \varepsilon)) = \delta_x > 0$

$\|x - y\| < \delta_x \Rightarrow \|f(x) - f(y)\| < \frac{\varepsilon}{2}$

$\{B(x, \delta_x) \mid x \in K\}$

$\bigcup_{x \in K} B(x, \frac{\delta_x}{2}) \mid \supset K \Rightarrow \exists \{B(x_i, \frac{\delta_{x_i}}{2}) \mid i = \overline{1,m}\},$

$K \subset \bigcup_{i=1}^{n} B(x_i, \frac{\delta_i}{2})$

Neka je $\delta = \min_{1 \le i \le m} \frac{\delta_{x_i}}{2}$

Neka su:

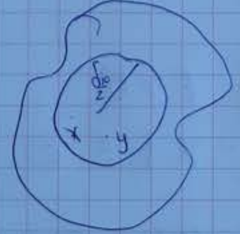
$x, y \in K, \ \|x - y\| < \delta$

Za $x \in K$ postoji $i_0 \in \{1, \dots, m\}$

$x \in B(x_{i_0}, \frac{\delta_{i_0}}{2})$

$\|x_{i_0} - y\| \le \|x_{i_0} - x\| + \|x - y\| < \delta_{i_0} + \delta \le \frac{\delta_{i_0}}{2} + \frac{\delta_{i_0}}{2} = \delta_{i_0}$

$\Rightarrow \ x, y \in B(x_{i_0}, \frac{\delta_{i_0}}{2})$



$\|f(x) - f(y)\| \le \overbrace{\|f(x) - f(x_{i_0})\|}^{< \frac{\varepsilon}{2}} +$

$+ \underbrace{\|f(x_{i_0}) - f(y)\|}_{< \frac{\varepsilon}{2}} < \varepsilon$

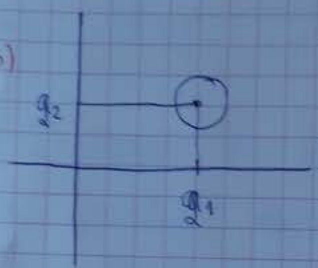
10. Data je funkcija $f(x,y) = \begin{cases} 0, & (x,y) \in R^2 \setminus Q^2 \\ \frac{1}{n^3}, & (x,y) \in Q^2, \\ & x = \frac{m}{n}, \\ & (m,n) = 1 \end{cases}$

Dokazati:

a) f - neprekidna u svim tačkama skupa $R \setminus Q^2$

b) f ima prekid u svim tačkama skupa Q^2

b)



$(g_1, g_2) \in Q^2$

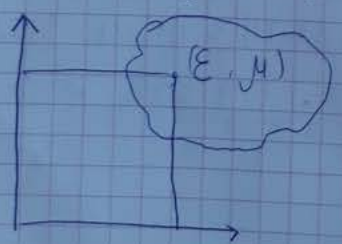
$\forall\, O((g_1, g_2))$, postoji $(i_1, i_2) \in R^2 \setminus Q^2$

$(i_1, i_2) \in O$

$f(i_1, i_2) = 0$

$f(g_1, g_2) = \frac{1}{n^3}$, $g_1 = \frac{m}{n}$

a) Neka je $(\xi, \mu) \in R^2 \setminus Q^2$



Za $\varepsilon > 0$

$|\xi - x| < \varepsilon$

$f(\xi, \mu) = 0$

$|\underbrace{f(\xi, \mu)}_{=0} - \underbrace{f(x,y)}_{=\frac{1}{n^3}}| < \varepsilon_0 \quad (*)$

$\frac{1}{n^3} < \varepsilon_0$

$n > \frac{1}{\sqrt[3]{\varepsilon_0}}$

$n \leq \frac{1}{\sqrt[3]{\varepsilon_0}}$ - tačke kod kojih $(*)$ nije zadovoljeno ima konačno mnogo.