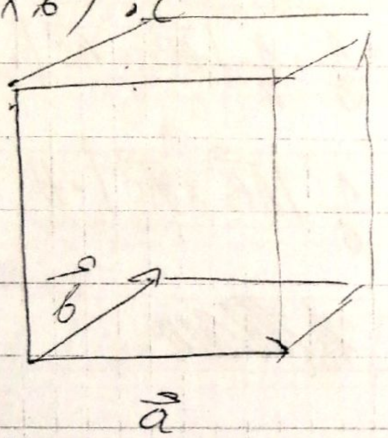
Мјешовити производ

Деф.

Мјешовити производ вектора $\vec{a}, \vec{b}, \vec{c}$ је $(\vec{a} \times \vec{b}) \cdot \vec{c}$

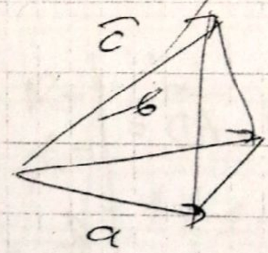


V запремина паралелопипеда

$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$

Vt - запремина тетраедра над векторима $\vec{a}, \vec{b}, \vec{c}$ одређ. је

$V_t = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|$



Својства

1) $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \iff \vec{a}, \vec{b}, \vec{c}$ компланарни

2) $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

3) $((\vec{a_1} + \vec{a_2}) \times \vec{b}) \cdot \vec{c} = (\vec{a_1} \times \vec{b}) \cdot \vec{c} + (\vec{a_2} \times \vec{b}) \cdot \vec{c}$

4) $(\vec{a} \times \vec{b}) \cdot (\lambda \cdot \vec{a}) = (\vec{a} \times \vec{b}) \cdot (\lambda \vec{b}) = (\vec{a} \times \lambda \vec{a}) \cdot \vec{c}$

$= (\vec{a} \times \lambda \vec{c}) \cdot \vec{c} = 0$

5) $\vec{a} = (x_1, y_1, z_1)$

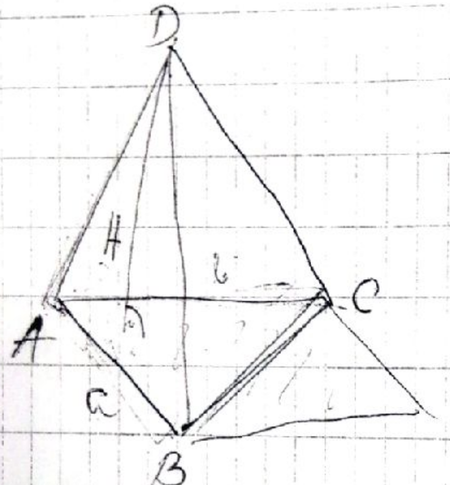
$\vec{b} = (x_2, y_2, z_2)$ координате у

$\vec{c} = (x_3, y_3, z_3)$ ортонормираној бази $(\vec{i}, \vec{j}, \vec{k})$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

① Тјемена тетраедра су $A(1,1,1)$, $B(0,2,1)$, $C(-2,-2,3)$ $D(3,4,-3)$. Одредити дужину висине из тјемена D.

<u>Р</u>



$$V = \frac{1}{6} \left| (\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} \right| \quad (1)$$

$$V_t = \frac{1}{3} B \cdot H$$ B – база површина троуг. ABC

$$V_t = \frac{1}{3} \cdot \frac{1}{2} \left| \overrightarrow{AB} \times \overrightarrow{AC} \right| \cdot H =$$

$$= \frac{1}{6} \left| \overrightarrow{AB} \times \overrightarrow{AC} \right| \cdot H \quad (2)$$

из (1) и (2) слиједи да је ~~H~~

$$\boxed{H = \frac{\left| (\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} \right|}{\left| \overrightarrow{AB} \times \overrightarrow{AC} \right|}}$$

$\overrightarrow{AB} = (-1, 1, 0)$

$\overrightarrow{AC} = (-3, -3, 2)$

$\overrightarrow{AD} = (2, 3, -1)$

$$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = \begin{vmatrix} -1 & 1 & 0 \\ -3 & -3 & 2 \\ 2 & 3 & -4 \end{vmatrix} =$$

$$= (-1) \cdot 6 - 1 \cdot 8 + 0 = -14$$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -3 & -3 & 2 \end{vmatrix} = 2\vec{i} - (-2)\vec{j} + 6\vec{k}$$

$$\vec{AB}\times\vec{AC} = (2,2,6)$$

$$(\vec{AB}\times\vec{AC})\cdot\vec{AB} = 2\cdot 2 + 2\cdot 3 + 6(-4) = -14$$

$$H = \frac{|-14|}{\sqrt{4+4+36}} = \frac{14}{2\sqrt{11}} = \frac{7}{\sqrt{11}}\cdot\frac{\sqrt{11}}{\sqrt{11}} = \frac{7\sqrt{11}}{11}$$

(2) Запремина тетраедра је 5. Три његова темена су тачке $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$. Нађи координате четвртог темена D, ако оно припада Oy-оси

$D(0,b,0)$

$V_t = 5$

$$V_t = \frac{1}{6}\left|(\vec{AB}\times\vec{AC})\cdot\vec{AD}\right|$$

$1-(-1)$

$$\left|(\vec{AB}\times\vec{AC})\cdot\vec{AD}\right| = 30$$

$\vec{AB}(1,-1,2)$

$\vec{AC}(0,-2,4)$

$\vec{AD} = (-2, b-1, 1)$

$$(\vec{AB}\times\vec{AC})\cdot\vec{AD} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & b-1 & 1 \end{vmatrix} = 1\cdot(-2-4(b-1)) - (-1)8 + 2\cdot$$

$$= -2 - 4b + 4 =$$

$$= 2 - 4b$$

$|2-4b| = 30$

$2|1-2b| = 30$

$|1-2b| = 15$

1° $1-2b=15$

$2b = -14$

$b = -7$

$\boxed{D(0,-7,0)}$

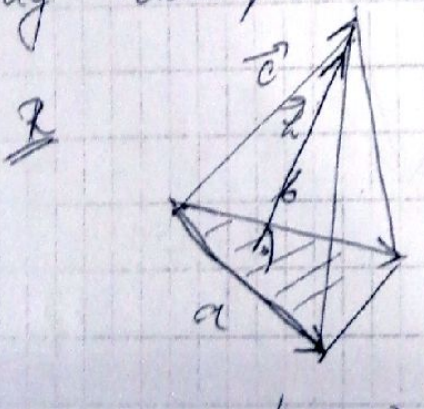
2° $1-2b = -15$

$-2b = -14$

$b = 7$

$\boxed{D(0,8,0)}$

③ Одредити вектор висине тетраедра конструисаног над векторима $\vec{a}, \vec{b}, \vec{c}$.



$V_t = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}| \cdot (1)$

$V_t = \frac{1}{3} \cdot B \cdot |\vec{h}| = \frac{1}{3} \cdot \frac{1}{2} |\vec{a} \times \vec{b}| \cdot |\vec{h}| = \frac{1}{6} |\vec{a} \times \vec{b}| \cdot |\vec{h}|$ (2)

$(1) \wedge (2) \Rightarrow |\vec{h}| = \frac{|(\vec{a} \times \vec{b}) \cdot \vec{c}|}{|\vec{a} \times \vec{b}|}$

$\vec{h} = |\vec{h}| \cdot \vec{h_0}$ $\vec{h_0}$ - јединични вектор вектора $\vec{h}$

$$\left.\begin{matrix} \vec{h} \perp \vec{a} \\ \vec{h} \perp \vec{b} \end{matrix}\right\} \Rightarrow \vec{h} = \lambda (\vec{a} \times \vec{b})$$

Вектори $\vec{h}$ и $\vec{a} \times \vec{b}$ су исте смјера (у складу са изабраном оријентацијом)

$$\vec{h} = \frac{|(\vec{a} \times \vec{b}) \circ \vec{c}|}{|\vec{a} \times \vec{b}|^2} \cdot (\vec{a} \times \vec{b})$$

4. Дати су вектори $\vec{m} = (1,1,1)$, $\vec{n} = (2,1,1)$. Вектор $\vec{p}$ са осом Ox заклапа угао $\frac{\pi}{4}$, а осом Oy угао $\frac{\pi}{3}$, а са осом Oz изући угао. Запремина тетраедра образованог над векторима $\vec{m}, \vec{n}$ и $\vec{p}$ је ~~V=12~~. Одредити координате вектора $\vec{p}$.

$V = 2$

R: $\alpha = \angle (\vec{p}, Ox) = \frac{\pi}{4}$

$\beta = \angle (\vec{p}, Oy) = \frac{\pi}{3}$

$\gamma = \angle (\vec{p}, Oz) = ?$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \cos^2\gamma = 1$$

$$\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$$

$$\cos^2\gamma = \frac{1}{4}$$

$$\cos\gamma = \pm\frac{1}{2}$$

$\vec{p_0}$ - јединични вектор вектора $\vec{p}$

$\gamma = \sphericalangle(\vec{p}, \vec{k})$

γ - туп

$\cos\gamma < 0$

$\Rightarrow \cos\gamma = -\frac{1}{2}$

$\vec{p_0} = (\cos\alpha, \cos\beta, \cos\gamma)$

$\vec{p_0} = \left(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

$\boxed{\vec{p} = |\vec{p}| \cdot \vec{p_0}}$, $|\vec{p}| = \lambda > 0$. $\vec{p} = \left(\frac{\sqrt{2}}{2}\lambda, \frac{\lambda}{2}, -\frac{\lambda}{2}\right)$

$V_t = \frac{1}{6} \left| (\vec{m} \times \vec{n}) \cdot \vec{p} \right|$

$\left| (\vec{m} \times \vec{n}) \cdot \vec{p} \right| = 12$

$$\vec{m} \times \vec{n} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0 \cdot \vec{i} - (-1) \cdot \vec{j} + (-1) \cdot \vec{k}$$

$\vec{m} \times \vec{n} = (0, 1, -1)$

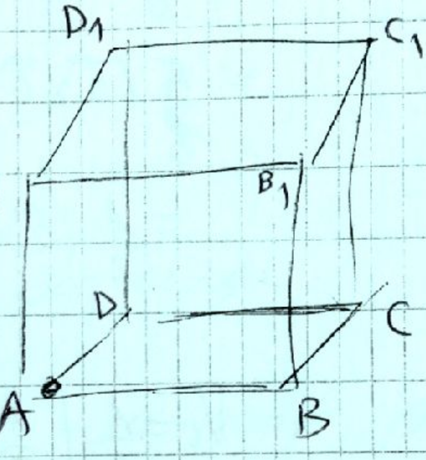
$(\vec{m} \times \vec{n}) \cdot \vec{p} = \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda$

$\left.\begin{array}{l} |\lambda| = 12 \\ \lambda > 0 \end{array}\right\} \Rightarrow \lambda = 12$

$\boxed{\vec{p} = (6\sqrt{2}, 6, -6)}$

⑥ Запремина паралелопипеда је V, а V_t запремина тетраедра чија су темена једно теме паралелопипеда и центри три стране наспрам тог темена. Наћи однос ових запремина:

E, F, G - центри страна BCC_1B_1, DCC_1D_1, $A_1B_1C_1D_1$



$\vec{AB} = \vec{a}$
$\vec{AD} = \vec{b}$
$\vec{AA_1} = \vec{c}$

$$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

$$V_t = \frac{1}{6} |(\vec{AE} \times \vec{AF}) \cdot \vec{AG}|$$

$$\vec{AE} = \vec{AB} + \vec{BE} = \vec{a} + \frac{1}{2}(\vec{b} + \vec{c}) = \vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}$$

$$\vec{AF} = \vec{AD} + \vec{DF} = \vec{b} + \frac{1}{2}(\vec{a} + \vec{c}) = \frac{1}{2}\vec{a} + \vec{b} + \frac{1}{2}\vec{c}$$

$$\vec{AG} = \vec{AA_1} + \vec{A_1G} = \vec{c} + \frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} + \vec{c}$$

$$(\vec{AE} \times \vec{AF}) = \left(\vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}\right) \times \left(\frac{1}{2}\vec{a} + \vec{b} + \frac{1}{2}\vec{c}\right) =$$

$$= \vec{a} \times \vec{b} + \frac{1}{2}\vec{a} \times \vec{c} + \frac{1}{4}\vec{b} \times \vec{a} + \frac{1}{4}\vec{b} \times \vec{c} + \frac{1}{4}\vec{c} \times \vec{a} + \frac{1}{2}\vec{c} \times \vec{b}$$

$$= \frac{3}{4}\vec{a} \times \vec{b} + \frac{1}{4}\vec{a} \times \vec{c} - \frac{1}{4}\vec{b} \times \vec{c}$$

$$\left(\vec{AE}\times\vec{AF}\right)\cdot\vec{AG}=\left(\frac{3}{4}\vec{a}\times\vec{b}\right)+\frac{1}{4}\,\vec{a}\times\vec{c}-\frac{1}{4}\vec{b}\times\vec{c}\Big)\cdot\left(\frac{1}{2}\vec{a}+\frac{1}{2}\vec{b}+\vec{c}\right)=$$

$$=\frac{3}{4}(\vec{a}\times\vec{b})\cdot\vec{c}+\frac{1}{8}(\vec{a}\times\vec{c})\cdot\vec{b}-\frac{1}{8}(\vec{b}\times\vec{c})\cdot\vec{a}=$$

$$=\frac{3}{4}(\vec{a}\times\vec{b})\cdot\vec{c}+\frac{1}{8}(\vec{c}\times\vec{b})\cdot\vec{a}-\frac{1}{8}(\vec{c}\times\vec{a})\cdot\vec{b}=$$

$$\frac{3}{4}(\vec{a}\times\vec{b})\cdot\vec{c}+\frac{1}{8}(\vec{b}\times\vec{a})\cdot\vec{c}-\frac{1}{8}(\vec{a}\times\vec{b})\cdot\vec{c}$$

$$=\frac{5}{8}(\vec{a}\times\vec{b})\cdot\vec{c}-\frac{1}{8}(\vec{a}\times\vec{b})\cdot\vec{c}=$$

$$=\frac{1}{2}(\vec{a}\times\vec{b})\cdot\vec{c}$$

$$V_t=\frac{1}{6}\left|\frac{1}{2}(\vec{a}\times\vec{b})\cdot\vec{c}\right|$$

$$V_t=\frac{1}{12}\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$$

$$\boxed{V_t=\frac{1}{12}V}$$

Математичка индукција

1) Математичком индукцијом доказати да све природне бројеве важи:

$$1+2+3+\dots n = \frac{n(n+1)}{2}$$

R 1° $n=1$

$$1 = \frac{1(1+1)}{2}$$

$1 = 1$ тврђење је тачно за $n=1$

2° Нека је тврђење тачно за $n=k$, нека важи $1+2+\dots k = \frac{k(k+1)}{2}$

Докажимо да је тада тврђење тачно и за $n=k+1$ тј. докажимо да важи $1+2+\dots k+1 = \frac{(k+1)(k+2)}{2}$

$1+2+\dots k+k+1 =$ [на основу индуктивне претпоставке]

$$= \frac{k(k+1)}{2} + k+1 = \frac{(k+1)(k+2)}{2}$$

Из 1 и 2 слиједи да је $1+2+\dots+n = \frac{n(n+1)}{2}$

$; n \geq 1$

(2) $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$, $n\in N$

1° $n=1$

$$1^2=\frac{1\cdot(1+1)\cdot(2\cdot1+1)}{6}$$

$$1=\frac{1\cdot2\cdot3}{6}$$

$\underline{1}=\underline{1}$ Тврђење је тачно за $n=k$, тј. да важи

$$1^2+2^2+\cdots k^2=\frac{k(k+1)(2k+1)}{6}$$

Докажимо да је тада тврђење тачно и за $n=k+1$, тј. да важи $1^2+2^2+\cdots+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$

$$1^2+2^2+\cdots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$=\frac{(k+1)\cdot(2k^2+7k+6)}{6}$$

Из 1° и 2° следи да је тврђење тачно за свако $k\in N$

$$2k^2+7k+6=0$$

$$k_{1/2}=\frac{-7\pm\sqrt{49-48}}{4}$$

$$k_{1/2}=\frac{-7\pm1}{4}$$

$$k_1=-2 \qquad k_2=\frac{3}{2}$$

$$2k^2+7k+6=2(k+2)\left(k+\frac{3}{2}\right)=$$

$$=(k+2)(2k+3)$$

③ Доказати да је $2^n \geq 2\cdot n$ за сваки природан број

Р: 1° $n=1$

$2^1 \geq 2\cdot 1$

$2 \geq 2$ тврђење је тачно за $n=1$

2° Претпоставимо да је тврђење тачно за $n=k$, тј. да важи $2 \geq 2\cdot k$

Докажимо да је тада тврђење тачно и за $n=k+1$, тј. да тада важи $2^{k+1} \geq 2(k+1)$?

$\Downarrow$

$2k+2k$

$$2^{k+1} = 2^k\cdot 2 \geq 2k\cdot 2 = 2\cdot 2k = 2k+2k \geq 2k+2 = 2(k+1)$$

Из 1. и 2. следи да је тврђење тачно за $\forall n\in N$ $\quad 2k \geq 2 \ \forall k\in N$

Комплексни бројеви

$z = a_i + bi \qquad a\in R$

$\qquad\qquad\quad b\in R$

$z = ai + bi$ – алгебарски облик комплексног броја

$i = \sqrt{-1}$

$i^2 = -1$

$$i^{4k} = (i^2)^{2k} = (-1)^{2k} = 1$$

$$i^{4k+1} = (i^{4k}) \cdot i = 1 \cdot i = i$$

$$i^{4k+2} = (i^{4k}) \cdot i^2 = 1 \cdot (-1) = -1$$

$$i^{4k+3} = (i^{4k}) \cdot i^3 = 1(-i) = -i$$

$z = a + bi$

a - реални дио

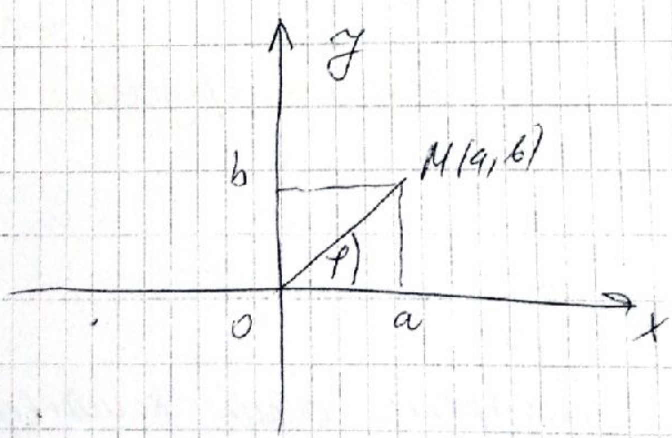
b - имагинарни дио

$Re(z) = a$

$Im(z) = b$

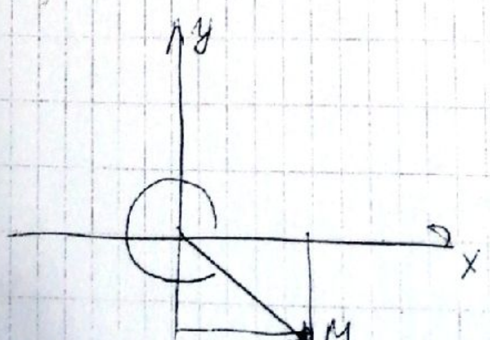
$|z| = \sqrt{a^2+b^2}$

$\rho = \sqrt{a^2+b^2}$



$Z = 2 - 3i$

$M(2, -3)$



$z = |z| \cdot (\cos\varphi + \sin\varphi)$ тригонометријски облик броја z

$tg\varphi = \frac{b}{a}$

$z_1 + z_2 = z_2 + z_1$

$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$z_1 = a_1 + b_1 i$

$z_2 = a_2 + b_2 i$

$z_1 + z_2 = a_1 + a_2 + (b_1 + b_2) i$

$z_1 = z_2$ акко $a_1 = a_2 \wedge b_1 = b_2$

$5 + 0 \cdot i$

? $0 \in Z$ $\quad 0 = 0 + 0 \cdot i$

$z = a + bi$

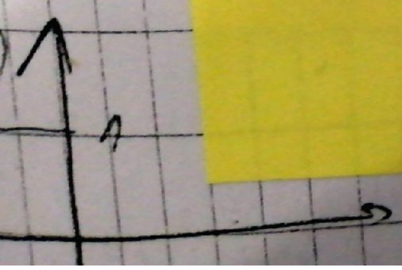
$\bar{z} = a - bi$ - z-конјуговано

$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

$z \cdot \bar{z} = |z|^2$

$$\arg z = \begin{cases} \operatorname{arctg} \frac{b}{a}, & a>0 \wedge b>0 \\ \operatorname{arctg} \frac{b}{a} + \pi, & a<0 \wedge b>0 \\ \operatorname{arctg} \frac{b}{a} + \pi, & a<0 \wedge b<0 \\ \operatorname{arctg} \frac{b}{a} + 2\pi, & \underline{a>0} \quad b<0 \\ \frac{\pi}{2}, & a=0 \wedge b>0 \\ \frac{3\pi}{2}, & a=0 \quad b<0 \end{cases}$$



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општем случају ако је $z_1 = |z_1| \cdot (\cos\theta_1 + i \cdot \sin\theta_1)$

$$\underbrace{z_2 = |z_2| \cdot (\cos\theta_2 + i\sin\theta_2)}$$

$$z_1 \cdot z_2 = |z_1 \cdot z_2| \cdot (\cos(\theta_1 + \theta_2) + i \cdot \sin(\theta_1 + \theta_2))$$

~~Ако је $z = \cos\theta +$~~ i

Ако је $z = |z| (\cos\theta + i \cdot \sin\theta)$

$\boxed{z^n = |z|^n (\cos(n \cdot \theta) + i \sin(n\theta)).}$ Де Моавн

$$= 1$$

$$z = |z| \cdot (\cos \Theta + i \cdot \sin \Theta$$

$$z = |z| \cdot (\cos \varphi + i \cdot \sin \varphi)$$

$$\sqrt[n]{z} = \sqrt[n]{|z|} \cdot \left(\cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1$$

1. Odrediti sve kompl. br. z takve da brojevi $z, \frac{1}{z}, 1-z$ imaju jednake module. Za tako nađeno z izračunati $\sqrt[3]{(z+\frac{1}{z}+i)^5}$

1) $|z| = |\frac{1}{z}|$

$|z| = \frac{1}{|z|}$

$|z|^2 = 1$

$|z| = 1$

2) $|z| = |1-z| \Rightarrow |1-z| = 1$

$z = x + yi$

$|z| = 1 \Rightarrow \sqrt{x^2+y^2} = 1$

$x^2+y^2 = 1$

$1-z = 1-x-yi$

$|1-z| = 1 \Rightarrow \sqrt{(1-x)^2+(-y)^2} = 1$

$(1-x)^2+y^2 = 1$

3) $\begin{cases} x^2+y^2=1 \\ (1-x^2)+y^2=1 \end{cases} (-1)$

$\begin{cases} x^2+y^2=1 \\ 1-2x=0 \Rightarrow x=\frac{1}{2} \end{cases}$

$y^2 = 1-x^2$

$y^2 = \frac{3}{4}$

$y = \pm\frac{\sqrt{3}}{2}$

$z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

4) $z + \frac{1}{z} + i = z + \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}} + i =$

$= z + \frac{\bar{z}}{|z|^2} + i = z + \bar{z} + i =$

$= x + yi + x - yi + i = 2x + i =$

$= 2\cdot\frac{1}{2} + i = 1 + i$

$t = 1 + i$

$t = |t|(\cos\varphi + i\sin\varphi)$

$a = 1,\ b = 1$

$|t| = \sqrt{a^2+b^2} = \sqrt{2}$

$\operatorname{tg}\varphi = \frac{b}{a}$

$\operatorname{tg}\varphi = 1 \Rightarrow \varphi = \frac{\pi}{4}$

$t = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

5) $t^5 = (\sqrt{2})^5 \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n}\right) \quad k = 0, 1, \dots n-1$$

$$\sqrt[3]{t^5} = \sqrt[3]{(\sqrt{2})^5}\left(\cos \frac{\frac{5\pi}{4} + 2k\pi}{3} + i \sin \frac{\frac{5\pi}{4} + 2k\pi}{3}\right) \quad k = 0, 1, 2$$

$k = 0: \quad z_0 = \sqrt[3]{4\sqrt{2}} \cdot \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}\right)$

$k = 1: \quad z_1 = \sqrt[3]{4\sqrt{2}} \left(\cos \frac{13\pi}{12} + i \sin \frac{13\pi}{12}\right)$

$k = 2: \quad z_2 = \sqrt[3]{4\sqrt{2}} \left(\cos \frac{21\pi}{12} + i \sin \frac{21\pi}{12}\right)$

2. Neka je $\operatorname{Im}\left(\frac{z+2}{z-i}\right) = 1$ i $\operatorname{Re}(z^2+1) = 1$. Naći z

$z = x + yi$

$$\frac{z+2}{z-i} = \frac{x+yi+2}{x+yi-i} = \frac{x+2+yi}{x+(y-1)i} \cdot \frac{x-(y-1)i}{x-(y-1)i} = \frac{(x+2)x - (x+2)(y-1)i + xyi - y(y-1)\dots}{x^2 - (y-1)^2 \cdot (-1)}$$

$$= \frac{x^2+2x+y^2-y+(-\cancel{xy}+x-2y+2+\cancel{xy})i}{x^2+(y-1)^2} =$$

$$\frac{z+2}{z-i} = \frac{x^2+2x+y^2-y}{x^2+y^2-2y+1} + \frac{x-2y+2}{x^2+y^2-2y+1} i$$

$$\operatorname{Im}\left(\frac{z+2}{z-i}\right) = 1 \Rightarrow \frac{x-2y+2}{x^2+y^2-2y+1} = 1$$

$$z^2+1 = (x+yi)^2+1 = x^2+2xyi+y^2\cdot(-1)+1 = x^2-y^2+1+2xyi$$

$$\operatorname{Re}(z^2+1) = 1 \Rightarrow x^2-y^2+1 = 1$$

$$x^2 = y^2$$

$$x = \pm y$$

$x = y$

$2^\circ \quad x = -y \Rightarrow y = -x$

$$\frac{x - 2x + 2}{x^2 + x^2 - 2x + 1} = 1$$

$$\frac{2 - x}{2x^2 - 2x + 1} = 1$$

$$2 - x = 2x^2 - 2x + 1$$

$$2x^2 - x - 1 = 0$$

$$x_{1,2} = \frac{1 \pm 3}{4}$$

$$x_1 = 1 \quad x_2 = -\frac{1}{2}$$

$$z_1 = 1 + i$$

$$z_2 = -\frac{1}{2} - \frac{1}{2} i$$

$$\frac{x + 2x + 2}{x^2 + x^2 + 2x + 1} = 1$$

$$\frac{3x + 2}{2x^2 + 2x + 1} = 1$$

$$3x + 2 = 2x^2 + 2x + 1$$

$$2x^2 - x - 1 = 0$$

$$x_3 = 1 \quad , \quad x_4 = -\frac{1}{2}$$

$$z_3 = 1 - i$$

$$z_4 = -\frac{1}{2} + \frac{1}{2} i$$

17. oktobar

1. Neka je $|z_1| = 2$, $\arg z_1 = \frac{2\pi}{3}$, $\arg z_2 = \frac{\pi}{6}$, $\arg z_3 = -\frac{\pi}{3}$, $z_1 + z_2 + z_3 =$

Naći z_1, z_2, z_3 i izračunati $\frac{z_2^3 \cdot z_3^4}{z_1^2}$.

$$z_1 = |z_1| \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$z_1 = 2 \left(-\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \right)$$

$$z_1 = -1 + i\sqrt{3}$$

$$z_2 = |z_2| \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$z_2 = |z_2| \left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2} \right)$$

$$z_2 = |z_2| \cdot \frac{\sqrt{3}}{2} + \frac{|z_2|}{2} \cdot i$$

$$z_3 = |z_3| \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right)$$

$$z_3 = |z_3| \left(\frac{1}{2} + i \cdot \left(-\frac{\sqrt{3}}{2}\right) \right)$$

$$z_3 = \frac{|z_3|}{2} - \frac{\sqrt{3}}{2} |z_3| \cdot i$$

$$-1+i\sqrt{3}+|z_2|\cdot\frac{\sqrt{3}}{2}-\frac{|z_2|}{2}i+\frac{|z_3|}{2}-\frac{\sqrt{3}}{2}|z_3|\cdot i = 1$$

$$-1+\frac{|z_2|}{2}\cdot\sqrt{3}+\frac{|z_3|}{2}+\left(\sqrt{3}-\frac{|z_2|}{2}-\frac{\sqrt{3}}{2}|z_3|\right)i = 1+0\cdot i$$

$$\begin{cases} -1+\frac{|z_2|}{2}\sqrt{3}+\frac{|z_3|}{2}=1 \\ \sqrt{3}-\frac{|z_2|}{2}-\frac{\sqrt{3}}{2}|z_3|=0 \end{cases} \qquad |z_2|=\sqrt{3}, \ |z_3|=1$$

$$z_2 = \frac{3}{2}+\frac{\sqrt{3}}{2}i$$

$$z_3 = \frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$z_1 = 2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) \qquad z_2=\sqrt{3}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) \qquad z_3 = 1\cdot\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z_1^2 = 4\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right) \qquad z_2^3 = 3\sqrt{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) \qquad z_3^4 = 1\cdot\left(\cos\left(-\frac{4\pi}{3}\right)+i\sin\left(-\frac{4\pi}{3}\right)\right)$$

$$\frac{z_2^3\cdot z_3^4}{z_1^2} = \frac{3\sqrt{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\cdot 1\cdot\left(\cos\left(-\frac{4\pi}{3}\right)+i\sin\left(-\frac{4\pi}{3}\right)\right)}{4\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)} = \frac{3\sqrt{3}\left(\cos\left(\frac{\pi}{2}-\frac{4\pi}{3}\right)+i\sin\left(\frac{\pi}{2}-\frac{4\pi}{3}\right)\right)}{4\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)}$$

$$= \frac{3\sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)}{4\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)} = \frac{3\sqrt{3}}{4}\left(\cos\left(-\frac{5\pi}{6}-\frac{4\pi}{3}\right)+i\sin\left(-\frac{5\pi}{6}-\frac{4\pi}{3}\right)\right)=$$

$$= \frac{3\sqrt{3}}{4}\left(\cos\left(-\frac{13\pi}{6}\right)+i\sin\left(-\frac{13\pi}{6}\right)\right) = \frac{3\sqrt{3}}{4}\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)=$$

$$= \frac{3\sqrt{3}}{4}\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right) = \frac{3\sqrt{3}}{4}\cdot\left(\frac{\sqrt{3}}{2}+i\left(-\frac{1}{2}\right)\right) = \frac{9}{8}-\frac{3\sqrt{3}}{8}i$$

2\. Izračunati $(z-2i)^5$ ako je $|\bar{z}+i|=\sqrt{3}$ i $\arg(\bar{z}+i)=\pi$.

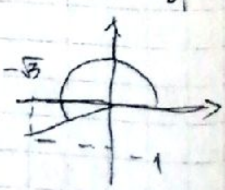
$$\bar{z}+i=\sqrt{3}\left(\cos\pi+i\sin\pi\right) \qquad t = z-2i = -\sqrt{3}+i-2i = -\sqrt{3}-i$$

$$\bar{z}+i = \sqrt{3}\left(-1+i\cdot 0\right) \qquad t=|t|\left(\cos\varphi+i\sin\varphi\right)$$

$$\bar{z}+i = -\sqrt{3} \qquad |t| = \sqrt{3+1} = 2$$

$$\bar{z} = -\sqrt{3}-i$$

$$z = -\sqrt{3}+i \qquad \operatorname{tg}\varphi = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$



$$\varphi = \frac{7\pi}{6}$$

$$t = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$t^5 = 2^5\left(\cos\frac{35\pi}{6} + i\sin\frac{35\pi}{6}\right)$$

$$t^5 = 32\left(\cos\left(4\pi + \frac{11\pi}{6}\right) + i\sin\left(4\pi + \frac{11\pi}{6}\right)\right)$$

$$t^5 = 32\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

$$t^5 = 32\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$$

$$t^5 = 16\sqrt{3} - 16i$$

3. Odrediti kompleksne brojeve z i $\sqrt{w}$ ako je $\arg z = \frac{\pi}{4}$, $\arg \bar{w} = \frac{\pi}{6}$, $z + \bar{w} = \sqrt{3} + i$.

$$z = |z|\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad \bar{w} = |\bar{w}|\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$z = |z|\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \qquad \bar{w} = |\bar{w}|\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right)$$

$$z = |z|\frac{\sqrt{2}}{2} + |z|\frac{\sqrt{2}}{2}\, i \qquad \bar{w} = \frac{|\bar{w}|\sqrt{3}}{2} + \frac{|\bar{w}|}{2}\, i$$

$$z + \bar{w} = \sqrt{3} + i$$

$$\frac{|z|\sqrt{2}}{2} + \frac{|z|\sqrt{2}}{2}\, i + \frac{|\bar{w}|\sqrt{3}}{2} + \frac{|\bar{w}|}{2}\, i = \sqrt{3} + i$$

$$\begin{cases} \frac{|z|\sqrt{2}}{2} + \frac{|\bar{w}|\sqrt{3}}{2} = \sqrt{3} \\ \frac{|z|\sqrt{2}}{2} + \frac{|\bar{w}|}{2} = 1 \end{cases} \Rightarrow |z| = 0 \quad \boxed{|\bar{w}| = 2} \Rightarrow \bar{w} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$|z| = 0 \Downarrow \boxed{z = 0} \qquad |\bar{w}| = 2 \Downarrow |w| = 2$$

$$\bar{w} = 2\cdot\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$$

$$\underbrace{\bar{w} = \sqrt{3} + i}$$

$$\underbrace{w = \sqrt{3} - i}$$

$$\boxed{\arg w = 2\pi - \arg \bar{w}}$$

$$\arg w = 2\pi - \frac{\pi}{6}$$

$$\arg w = \frac{11\pi}{6}$$

$$w = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

$$\sqrt{w} = \sqrt{2}\left(\cos\frac{\frac{11\pi}{6} + 2k\pi}{2} + i\sin\frac{\frac{11\pi}{6} + 2k\pi}{2}\right)$$

$$\boxed{\begin{aligned} k = 0,\quad & w_0 = \sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) \\ k = 1,\quad & w_1 = \sqrt{2}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) \end{aligned}}$$

4. Dat je broj $z = \left[\frac{i^{34} - i^{71}}{2} + \frac{2}{(1-i)^3} \right]^6$. Odrediti w ako važe jednako

$\arg \bar{w} = \frac{7\pi}{4}$ i $\operatorname{Im}(z - w) = 4$

$\arg w = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$

$i^{34} = i^{4 \cdot 8 + 2} = -1$

$i^{71} = i^{4 \cdot 17 + 3} = -i$

$(1-i)^3 = 1 - 3 \cdot i + 3 \cdot 1 \cdot i^2 - i^3 =$

$= 1 - 3i - 3 + i = -2 - 2i$

$\frac{i^{34} - i^{71}}{2} + \frac{2}{(1-i)^3} = \frac{-1-(-i)}{2} + \frac{2}{-2-2i} = \frac{i-1}{2} - \frac{1}{1+i} \cdot \frac{1-i}{1-i} =$

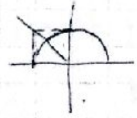
$= \frac{i-1}{2} + \frac{i-1}{2} = i - 1$

$\underline{(i-1)^6}$ $\quad t = i - 1$

$t = -1 + i$

$t = |t|(\cos\varphi + i \sin\varphi)$

$\operatorname{tg}\varphi = \frac{1}{-1} = -1 \Rightarrow \varphi = \frac{3\pi}{4}$



$|t| = \sqrt{1+1} = \sqrt{2}$

$t = \sqrt{2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$

$t^6 = 8\left(\cos\frac{18\pi}{4} + i \sin\frac{18\pi}{4}\right)$

$t^6 = 8\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$

$t^6 = 8 \cdot (0 + i \cdot 1)$

$t^6 = 8i$

$z = 8i$

$w = |w|\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$

$w = \frac{|w|\sqrt{2}}{2} + \frac{|w|\sqrt{2}}{2} \cdot i$

$z - w = 8i - \frac{|w|\sqrt{2}}{2} - \frac{|w|\sqrt{2}}{2} i =$

$= -\frac{|w|\sqrt{2}}{2} + \left(8 - \frac{|w|\sqrt{2}}{2}\right) i$

$\operatorname{Im}(z - w) = 4$

$8 - \frac{|w|\sqrt{2}}{2} = 4$

$\frac{|w|\sqrt{2}}{2} = 4$

$|w| = 4\sqrt{2}$

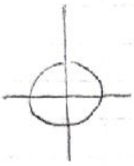
$\boxed{w = 4 + 4i}$

nakost

Naći kompleksni broj z koji je zadat sa jednačinama $Re\left(z^2+1-Re\left(\frac{1-i}{i}\right)\right)=1$ i $arg\left(z^3\cdot(1+i\sqrt{3})\right)=\frac{5\pi}{6}$

$t = 1+\sqrt{3}\,i$

$|t| = \sqrt{1+3} = 2$



$tg\,\varphi = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\varphi = \frac{\pi}{3}$

$t = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z = |z|(\cos\alpha + i\sin\alpha)$

!!

$z^3 = |z|^3(\cos 3\alpha + i\sin 3\alpha)$

$z^3\cdot t = 2|z|^3\left(\cos\left(\frac{\pi}{3}+3\alpha\right) + i\sin\left(\frac{\pi}{3}+3\alpha\right)\right)$

$arg(z^3\cdot t) = \frac{5\pi}{6} \implies \frac{\pi}{3}+3\alpha = 2k\pi + \frac{5\pi}{6}$

$3\alpha = \frac{5\pi}{6} - \frac{\pi}{3} + 2k\pi$

$3\alpha = \frac{\pi}{2} + 2k\pi$

$\alpha = \frac{\pi}{6} + \frac{2k\pi}{3}$

$k=0 \Rightarrow \alpha = \frac{\pi}{6}$

$k=1 \Rightarrow \alpha = \frac{5\pi}{6}$

$k=2 \Rightarrow \alpha = \frac{9\pi}{6} = \frac{3\pi}{2}$

$k=3 \Rightarrow \alpha = \frac{13\pi}{6} > 2\pi$

$\frac{1-i}{i}\cdot\frac{i}{i} = \frac{i-(-1)}{-1} = \frac{i+1}{-1} = -1-i$

$Re\left(\frac{1-i}{i}\right) = -1$

$z^2 = |z|^2(\cos 2\alpha + i\sin 2\alpha)$

$z^2 = |z|^2\cos 2\alpha + |z|^2\cdot\sin 2\alpha\cdot i$

$Re\left(|z|^2\cos 2\alpha + |z|^2\sin 2\alpha\cdot i + 1-(-1)\right) = 1$

$Re\left(2 + |z|^2\cos 2\alpha + |z|^2\sin 2\alpha\cdot i\right) = 1$

$2 + |z|^2\cos 2\alpha = 1$

$|z|^2 = \frac{-1}{\cos 2\alpha} \Rightarrow \cos 2\alpha < 0$

$\alpha = \frac{\pi}{6} \quad |z|^2 = \frac{-1}{\cos\frac{\pi}{3}} = \frac{-1}{\frac{1}{2}}$

$\alpha = \frac{5\pi}{6} \quad |z|^2 = \frac{-1}{\cos\frac{5\pi}{3}} = \frac{-1}{\frac{1}{2}}$

$\alpha = \frac{3\pi}{2} \quad |z|^2 = \frac{-1}{\cos 3\pi} = \frac{-1}{-1} = 1$

$|z| = 1$

$z = \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) \quad \boxed{z = -i}$

6. Izračunati $\sqrt[4]{z}$ ako je $\frac{z}{\bar{z}} + \frac{\bar{z}}{z} + 4 \cdot \left(\frac{1}{z} + \frac{1}{\bar{z}}\right) = 2$; $\arg(z-2) = -\frac{\pi}{2}$

$z = x + yi$

$$\left.\begin{array}{l} z - 2 = x + yi - 2 = x - 2 + yi \\ \arg(z-2) = -\frac{\pi}{2} \end{array}\right\} \begin{array}{l} x - 2 = 0 \Rightarrow x = 2 \\ y < 0 \end{array}$$

$Re(z-2) = 0$

$Im(z-2) < 0$

$$\frac{z}{\bar{z}} + \frac{\bar{z}}{z} + 4\left(\frac{1}{z} + \frac{1}{\bar{z}}\right) = 2$$

$$\frac{z^2 + \bar{z}^2}{\bar{z} \cdot z} + 4 \frac{\bar{z} + z}{z \cdot \bar{z}} = 2$$

$$\frac{z^2 + \bar{z}^2 + 4(\bar{z} + z)}{|z|^2} = 2$$

$$\frac{x^2 + \cancel{2xyi} - y^2 + x^2 - \cancel{2xyi} - y^2 + 4(x - \cancel{yi} + x + \cancel{yi})}{x^2 + y^2} = 2$$

$$\cancel{2x^2} - 2y^2 + 8x = \cancel{2x^2} + 2y^2$$

$8x = 4y^2$

$y^2 = 2x$

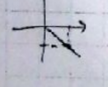
$y^2 = 4$

$y = -2$ jer je $y < 0$

$z = 2 - 2i$

$|z| = \sqrt{4+4} = 2\sqrt{2}$

$\operatorname{tg} \varphi = \frac{-2}{2} = -1$



$\varphi = \frac{7\pi}{4}$

$z = 2\sqrt{2}\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$

$$\sqrt[4]{z} = \sqrt[4]{2\sqrt{2}}\left(\cos \frac{\frac{7\pi}{4} + 2k\pi}{4} + i \sin \frac{\frac{7\pi}{4} + 2k\pi}{4}\right)$$

$k = 0, 1, 2, 3$

$k = 0 \Rightarrow z_0 =$

$z_1 =$

$z_2 =$

$z_3 =$