Лопиталово правило

дод. напред ln ln ln

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ \infty^0,\ 1^\infty,\ 0^0$$

① $$\lim_{x\to+\infty}\frac{\frac{\pi}{2}-\operatorname{arctg}x}{\ln\left(1+\frac{1}{x^2}\right)}\overset{\frac{0}{0}}{=}\lim\frac{-\frac{1}{1+x^2}}{\frac{1}{1+\frac{1}{x^2}}\cdot\left(\frac{-2}{x^3}\right)}=$$

x^{-2}

$$=\lim_{x\to+\infty}\frac{\frac{1}{1+x^2}}{\frac{x^2}{1+x^2}\cdot\frac{2}{x^3}}=\lim_{x\to+\infty}\frac{1}{2}\cdot x=+\infty$$

② $$\lim_{x\to 0}\frac{x\cdot\operatorname{ctg}x-1}{x^2}=$$

$$=\lim_{x\to 0}\frac{x\cdot\frac{\cos x}{\sin x}-1}{x^2}=\lim_{x\to 0}\frac{x\cos x-\sin x}{x^2\sin x}\underset{\text{Л.п.}}{\overset{\frac{0}{0}}{=}}$$

$$\lim_{x\to 0}\frac{\cos x+x\cdot(-\sin x)-\cos x}{2x\cdot\sin x+x^2\cdot\cos x}=$$

$$=\lim_{x\to 0}\frac{-x\cdot\sin x}{2x\cdot\sin x+x^2\cdot\cos x}=$$

$$=\lim_{x\to 0}\frac{-\sin x}{2\cdot\sin x+x\cdot\cos x}\overset{\frac{0}{0}}{=}$$

$$= \lim_{x \to 0} \frac{-\cos x}{2\cos x + \cos x + x - (-\sin x)} = -\frac{1}{3}$$

(3.) $\lim\limits_{x \to 0^+} \dfrac{\ln(tg\,\alpha x)}{\ln(ctg\,\beta x)} \overset{\frac{\infty}{\infty}}{\underset{\text{Л.Л}}{=}} \lim \dfrac{\frac{1}{tg\,\alpha x} \cdot \frac{1}{\cos^2 \alpha x} \cdot \alpha}{\frac{1}{ctg\,\beta x} \cdot \frac{-1}{\sin^2 \beta x} \cdot \beta} =$

$$= \lim_{x \to 0^+} \frac{\frac{1}{\sin\alpha x \cos 2\alpha} \cdot \alpha}{\frac{-1}{\sin\beta x \cos\beta x} \cdot \beta} =$$

$$= -\frac{\alpha}{\beta} \cdot \lim_{x \to 0^+} \frac{\sin\beta x \cdot \cos\beta x}{\sin\alpha x \cdot \cos\alpha x} =$$

$$= -\frac{\alpha}{\beta} \cdot \lim_{x \to 0^+} \frac{\frac{\sin\beta x}{\beta x} \cdot \beta x}{\frac{\sin\alpha x}{\alpha x} \cdot \alpha x} \cdot \lim_{x \to 0^+} \frac{\cos\beta x}{\cos\alpha x} =$$

$$= -\frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} \cdot 1 = \underline{\underline{-1}}$$

?

b

(4.) $\lim\limits_{x \to +\infty} \dfrac{2 \cdot \ln x}{x^b}, \quad b > 0$

$$\lim_{x \to +\infty} \frac{2 \cdot \ln x}{x^b} \overset{\frac{\infty}{\infty}}{\underset{\text{Л.Л}}{\Longrightarrow}} \lim_{x \to +\infty} \frac{2 \cdot \frac{1}{x}}{b \cdot x^{b-1}} = \frac{2}{b} \cdot \lim_{x \to +\infty} \frac{1}{x^b} =$$

$$x^{-1}$$

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⑤ $\lim_{x\to 0} x^k \cdot \ln x,\ k>0$

$$\lim_{x\to 0^+} x^k \cdot \ln x \overset{0(\infty)}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{-k}{x^{k+1}}} \underset{\text{l.H.}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-k}{x^{k+1}}} =$$

$$= -\frac{1}{k} \cdot \lim_{x\to 0} x^k = 0$$

⑥ $$\lim_{x\to \frac{\pi}{2}} (1-\sin x)\,tg^2 x \overset{0\cdot\infty}{=} \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\frac{1}{tg^2 x}} =$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{\frac{-2}{tg^3 x} \cdot \frac{1}{\cos^2 x}} = \frac{1}{2} \cdot \lim_{x\to\frac{\pi}{2}} \frac{\cos^3 x}{\frac{\cos^3 x}{\sin^3 x}} =$$

$$= \frac{1}{2} \lim_{x\to\frac{\pi}{2}} \sin^3 x = \frac{1}{2}$$

⑦ $$\lim_{x\to+\infty} \left(x \cdot e^{\frac{1}{x}} - x\right) \overset{\infty-\infty}{=} \lim_{x\to+\infty} x \cdot \left(e^{\frac{1}{x}} - 1\right)$$

$$\overset{\infty\cdot 0}{=} \lim_{x\to+\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \underset{\text{l.H.}}{\overset{\frac{0}{0}}{=}} \lim_{x\to+\infty} \frac{e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to+\infty} e^{\frac{1}{x}\to 0} = \underline{1}$$

$$e^{\ln x} = x$$

8. $$\lim_{x \to \frac{\pi}{2}} \left(\frac{x}{ctg\, x} - \frac{\pi}{2\cos x} \right) \overset{\infty-\infty}{=} \lim_{x \to \frac{\pi}{2}} \left(\frac{x \cdot \sin x}{\cos x} - \frac{\pi}{2\cos x} \right) =$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2x \sin x - \pi}{2\cos x} \overset{\frac{0}{0}}{=} (1.4)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2 \cdot \sin x + 2x\cos x}{-2\sin x} = \frac{2}{-2} = -1$$

$$f(x)^{g(x)} = e^{\ln f(x)^{g(x)}}$$

$$\lim_{x \to a} f(x)^{g(x)} = e^{\lim_{x \to a} g(x) \ln f(x)}$$

$$= \lim_{x \to a} g(x) \cdot \ln f(x)$$

9. $$\lim_{x \to 1} x^{\frac{1}{1-x}} \qquad 1^{\infty}$$

$$e^{\lim_{x \to 1} \ln x^{\frac{1}{1-x}}} = e^{\lim_{x \to 1} \frac{1}{1-x} \cdot \ln x} \; (*) =$$

$$(*): \lim_{x \to 1} \frac{1}{1-x} \cdot \ln x = \lim_{x \to 1} \frac{\ln x}{1-x} \overset{\frac{0}{0}}{\underset{1.U}{=}}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{-1} = \lim_{x \to 1} -\frac{1}{x} = -1$$

10. $$\lim_{x \to \frac{\pi}{2}} (tg\, x)^{2x-\pi} \overset{\infty^0}{=} e^{\lim_{x \to \frac{\pi}{2}} \ln (tg\, x)^{2x-\pi}} =$$

$$= e^{\lim_{x \to \frac{\pi}{2}} (2x-\pi) \cdot \ln(tg\, x)} \overset{(*)}{=} e^{0}$$

$$= e^0 = 1$$

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$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{tgx}\cdot\frac{1}{\cos^2 x}}{\frac{-1}{(2x-\pi)^2\cdot 2}} = -\frac{1}{2}\lim_{x\to\frac{\pi}{2}} \frac{(2x-\pi)^2}{\sin x\cdot\cos x} =$$

$$= -\lim_{x\to\frac{\pi}{2}} \frac{(2x-\pi)^2}{\sin 2x} \overset{\frac{0}{0}}{\underset{\text{1.U}}{=}}$$

$$= -\lim_{x\to\frac{\pi}{2}} \frac{2(2x-\pi)\cdot 2}{\cos 2x\cdot 2} = \frac{0}{-2} = 0$$

(11) $\lim_{x\to 0} (1-\cos x)^x \overset{0^0}{=} e^{\lim_{x\to 0} \ln(1-\cos x)^x} = e^{\lim x\ln(1-\cos x)}$

$(*) = e$
$e^0 =$

$(*)$: $\lim_{x\to 0} x\cdot\ln(1-\cos x) \overset{0\cdot(-\infty)}{=\!=} \lim_{x\to 0} \frac{\ln(1-\cos x)}{\frac{1}{x}} \overset{\frac{\infty}{\infty}}{\underset{\text{1.U}}{=}}$

$$\lim_{x\to 0} \frac{\frac{1}{1-\cos x}\cdot(\sin x)}{-\frac{1}{x^2}} = -\lim_{x\to 0}$$

$$-\lim_{x\to 0} \frac{x^2\cdot\sin x}{1-\cos x} \overset{\frac{0}{0}}{\underset{\text{1.U}}{=}}$$

$$\lim_{x\to 0} \frac{2x\cdot\sin x + x^2\cdot\cos x}{\sin x} \overset{\frac{0}{0}}{\underset{\text{1.U}}{=}}$$

$$-\frac{\sin x}{\frac{1-\cos x}{x^2}} = -\frac{\frac{\sin x}{x}\cdot x}{\frac{1-\cos x}{x^2}}$$

$$\lim_{x\to 0} \frac{2x\cdot \sin x + x^2\cdot \cos x}{\sin x} \overset{0/0}{\underset{L.H.}{=}} \lim_{x\to 0} \frac{2\sin x + 2x\cos x + 2\cos x + x^2\sin x}{\cos x}$$

$$= \frac{0}{1} = 0$$

Цртање графика функција

1. Област дефинисаности
2. (Не)парност
3. Нуле и знак функције
4. Понашање на крајевима интервала дефинисаности и асимптоте
5. Ток и екстреми
6. Конвексност и конкавност, превојне тачке
7. ~~[illegible]~~ График

1.

1. $y = x \cdot e^{\frac{1}{x-2}}$

1° $D(y): x-2 \neq 0$

$D(y): (-\infty, 2) \cup (2, +\infty)$

sjutra :-)

2° $y(-x) = -x \cdot e^{\frac{1}{-x-2}} \neq -y(x) \Rightarrow$ није непарна

$y(-x) = y \neq y(x) \Rightarrow$ није парна

3° $y=0 \Leftrightarrow x \cdot \underbrace{e^{\frac{1}{x-2}}}_{>0\ (\forall x\in D_y)} = 0$

$\Leftrightarrow x=0$

$y<0 \Leftrightarrow x<0$
$y>0 \Leftrightarrow x>0$

4° $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} x \cdot e^{\frac{1}{x-2}} = +\infty$ $\quad \frac{1}{x-2} \to +\infty$, $x-2 \to 0^+$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} x \cdot e^{\frac{1}{x-2}} = 0$ $\quad \frac{1}{x-2} \to -\infty$, $x-2 \to 0^-$, $e^{\frac{1}{x-2}} \to 0$

$x=2$ је вертикална асимптота

5° $\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} x \cdot e^{\frac{1}{x-2}} = +\infty$ $\quad \frac{1}{x-2} \to 0$

$\lim\limits_{x\to \infty} f(x) = \lim\limits_{x\to \infty} x \cdot e^{\frac{1}{x-2}} = -\infty$ $\quad \frac{1}{x-2} \to 0$

Нема хоризонталне асимптоте.

$k = \lim\limits_{x\to \pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to \pm\infty} e^{\frac{1}{x-2}} = \underline{1}$ $\quad \frac{1}{x-2} \to 0$

$y' = 0 \iff x = 1 \lor x = 4$

y' није дефинисано у $x = 2$ али
$x = 2 \in D(y)$

$$y' = \frac{(x-1)(x-4)}{(x-2)^2} \cdot e^{\frac{1}{x-2}}$$

	$-\infty$	1		4	$+\infty$
$x-1$	$-$	0	$+$		$+$
$x-4$	$-$		$-$	0	$+$
y'	$+$	0	$-$	0	$+$
y	$\nearrow$	max	$\searrow$	min	$\nearrow$

У тачки $x = 1$ функција достиже
максимум

$$y_{max} = f(1) = 1 \cdot e^{\frac{1}{1-2}} = \frac{1}{e}$$

$$M_1\left(1, \frac{1}{e}\right)$$

У тачки $x = 4$ функција достиже
минимум

$$y_{min} = f(4) = 4 \cdot e^{\frac{1}{4-2}} =$$

$$= 4 \cdot e^{\frac{1}{2}} = 4\sqrt{e} \quad M_2(4, 4\sqrt{e})$$

6° $y = \frac{x^2-5x+4}{(x-2)^2} \cdot e^{\frac{1}{x-2}}$

$y' = \frac{(2x-5)(x-2)^2-(x^2-5x+4)\cdot 2(x-2)}{(x-2)^4} \cdot e^{\frac{1}{x-2}}$

$+ \frac{x^2-5x+4}{(x-2)^2} \cdot e^{\frac{1}{x-2}} \cdot \left(-\frac{1}{(x-2)^2}\right)$

$y' = \frac{(x-2)\left[(2x-5)(x-2)-2(x^2-5x+4)\right]}{\ } \cdot e^{\frac{1}{x-2}} - \frac{x^2-5x+4}{(x-2)^4} \cdot e^{\frac{1}{x-2}}$

$y'' = \frac{5x-8}{(x-2)^2} \cdot e^{\frac{1}{x-2}} > 0 \ \forall x \in D$

$y'' = 0 \iff x = \frac{8}{5}$

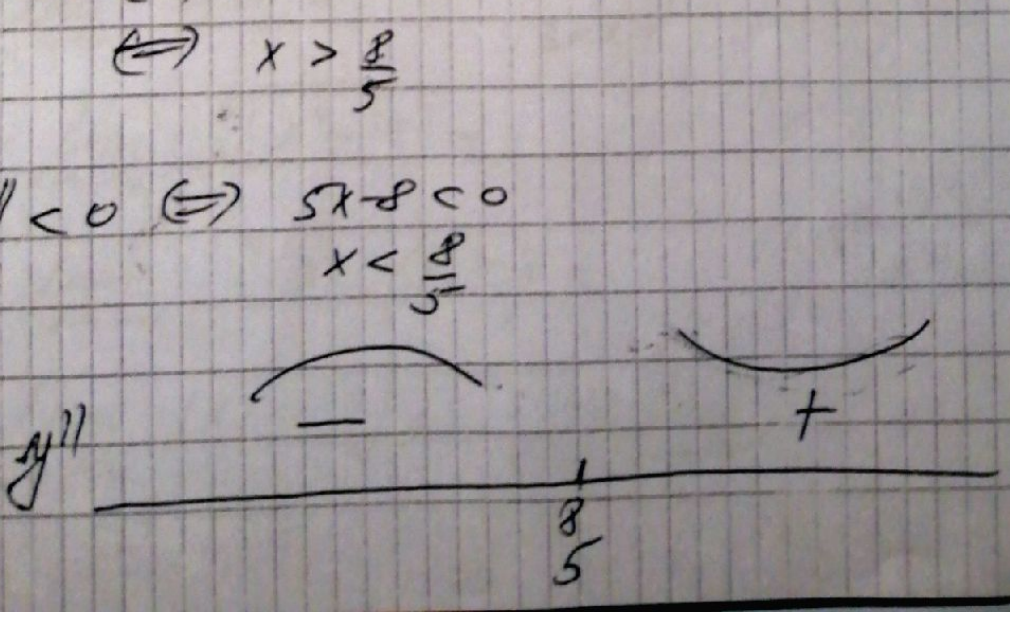
y'' није дефинисано у $x=2$, али $x=2 \notin D(y)$

$y'' > 0 \iff 5x-8>0$

$\iff x > \frac{8}{5}$

$y'' < 0 \iff 5x-8<0$

$x < \frac{8}{5}$



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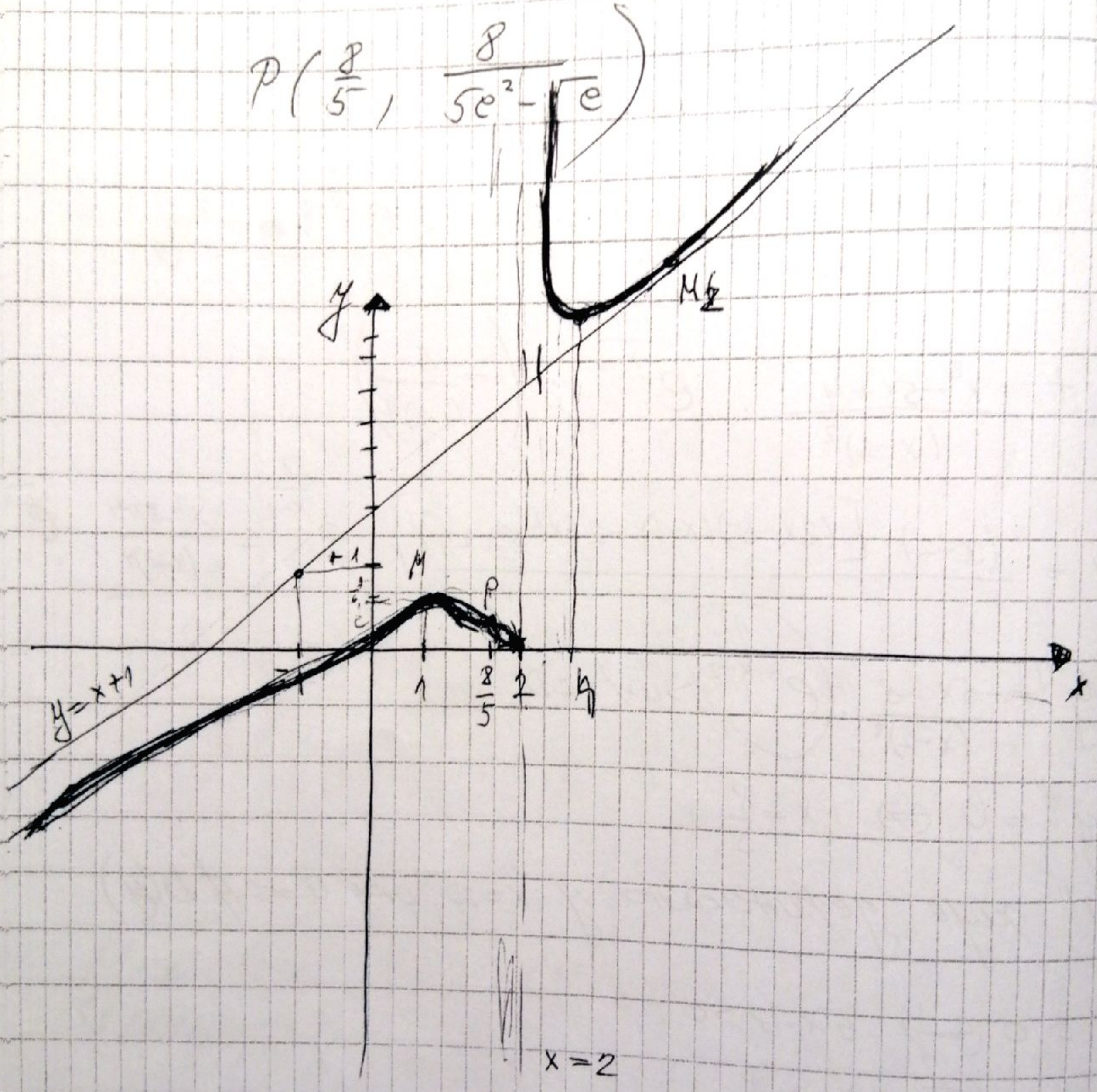
$P\left(\frac{8}{5}, f\left(\frac{8}{5}\right)\right)$

$P\left(\frac{8}{5}, \frac{8}{5e^2\sqrt{e}}\right)$



$\sqrt{x^2} = |x|$

$\sqrt[4]{x^4} = |x|$

(2)

$y = \sqrt[4]{x^4 - 4x^3}$

1° $D(y)$: $x^4 - 4x^3 \geq 0$

$x^3(x-4) \geq 0$

	$-\infty$ 0	0 4	4 $+\infty$
x	$-$	$+$	$+$
$x-4$	$-$	$-$	$+$
$x^3(x-4)$	$+$	$-$	$+$

$D(y)$: $(-\infty, 0] \cup [4, +\infty)$

2° ні парна ні непарна

3° $y = 0 \iff x^4 - 4x^3 = 0$

$\iff x^3 \cdot (x-4) = 0$

$\iff x = 0 \lor x = 4$

$y \geq 0, \ \forall x \in D(y)$

$\lim\limits_{x \to +\infty} f(x) =$

$= \lim\limits_{x \to +\infty} \sqrt[4]{x^4 - 4x^3} =$

$= \lim\limits_{x \to +\infty} \sqrt[4]{x^4 \cdot \left(1 - \frac{4}{x}\right)} =$

$= \lim\limits_{x \to +\infty} |x| \cdot \sqrt[4]{1 - \frac{4}{x}} = \lim\limits_{x \to +\infty} x \cdot \sqrt[4]{1 - \overset{0}{\frac{4}{x}}}$

$= +\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (-x)\cdot\sqrt[4]{1-\frac{4}{x}} = +\infty$$

$$k = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\cancel{x}\cdot\sqrt[4]{1-\frac{4}{x}}}{\cancel{x}} = 1$$

$$n_1 = \lim_{x\to+\infty} (f(x) - k_1 x) = \lim_{x\to+\infty} \left(x\cdot\sqrt[4]{1-\frac{4}{x}} - x\right) \stackrel{\infty-\infty}{=}$$

$$= \lim_{x\to+\infty} x\cdot\left(\sqrt[4]{1-\frac{4}{x}} - 1\right) \stackrel{\infty\cdot 0}{=} \lim_{x\to+\infty} \frac{\left(1-\frac{4}{x}\right)^{\frac{1}{4}}}{\frac{1}{x}} =$$

$$= \lim_{x\to+\infty} \frac{\frac{1}{4}\left(1-\frac{4}{x}\right)^{-\frac{3}{4}}\cdot\left(\frac{4}{x^2}\right)}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to+\infty} -\frac{1}{\sqrt[4]{\left(1-\frac{4}{x}\right)^3}} = -1$$

$y = k_1 x + n_1$

$y = x - 1$ K.A, $x \to +\infty$

$$k_2 = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{-x\cdot\sqrt[4]{1-\frac{4}{x}}}{x} = -1$$

$$n_2 = \lim_{x\to\infty} (f(x) - k_2 x) =$$

$$= \lim_{x \to -\infty} \left(-x \cdot \sqrt[4]{1-\frac{4}{x}} + x\right) =$$

$$= \lim_{x \to -\infty} x \cdot \left(1 - \sqrt[4]{1-\frac{4}{x}}\right) \stackrel{\infty \cdot 0}{=\!=} \cdots = 1$$

$y = k_2 x + n_2$

$y = -x + 1$ К.А, $x \to -\infty$

5+ $y = (x^4 - 4x^3)^{\frac{1}{4}}$

$$y' = \frac{1}{4}(x^4 - 4x^3)^{-\frac{3}{4}} \cdot (4x^3 - 12x^2)$$

$$y' = \frac{x-3}{\sqrt[4]{x(x-4)^3}}$$

$y' = 0 \iff x = 3 \notin D(y)$

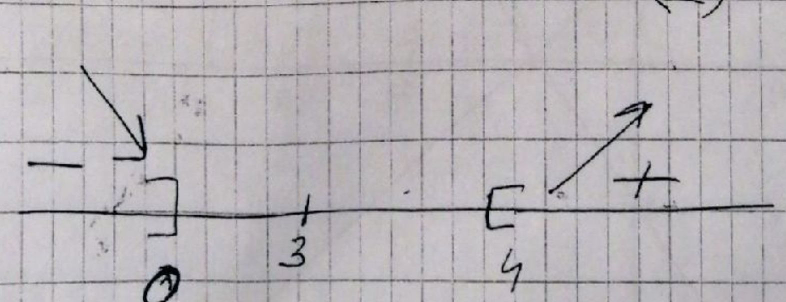
$y \in D(y)$

$y' > 0 \iff x - 3 > 0$

$\iff x > 3 \iff x \in [4, +\infty)$

$y' < 0 \iff x - 3 < 0 \iff$

$\iff x < 3 \iff x \in (-\infty, 0]$



Нема екстрема

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6°

$$y'' = \left(\frac{x-3}{\left(x(x-4)^3\right)^{\frac{1}{4}}}\right)'$$

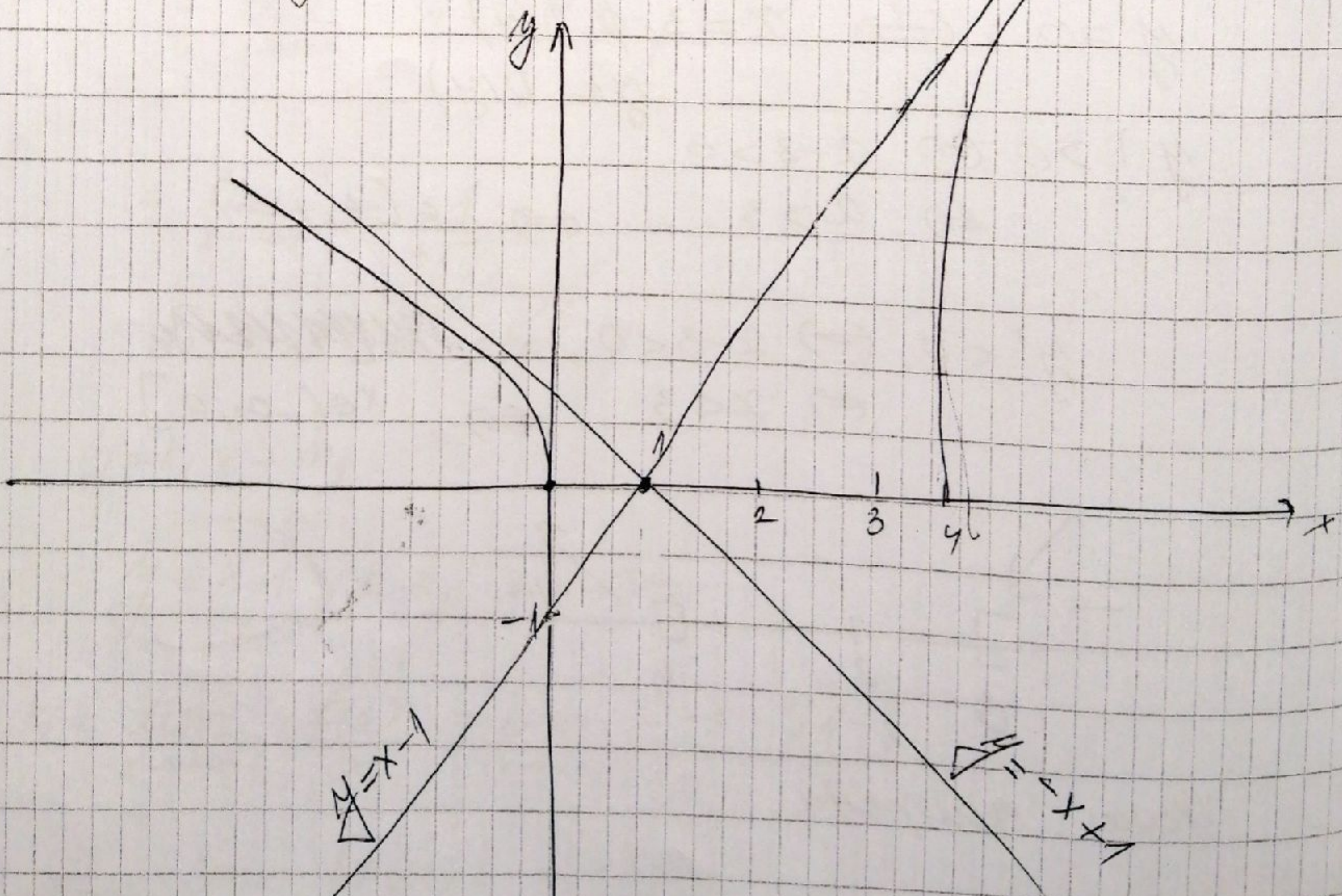
$$y'' = \frac{-3}{\sqrt[4]{x^5 \cdot (x-4)^7}}$$

y'' нема нуле, y'' није дефинисано у $x=0$;

$x=4$ $\quad$ $x=0$, $x=4$ $\in D(y)$

$y'' < 0, \ \forall x \in D(y)$

$\cap$ Функција је стално конкавна, нема превојних тачака



(3.) $y = \frac{\ln|x|+1}{x}$

1° $x \neq 0 \quad \wedge \quad |x| > 0$

$x \neq 0 \quad \wedge \quad x \neq 0$

$D(y) = (-\infty, 0) \cup (0, +\infty)$

2° $y(-x) = \frac{\ln|-x|+1}{-x} = \frac{\ln|x|+1}{-x} =$

$= -\frac{\ln|x|+1}{x} = -y(x)$

y је непарна функција па је њен график симетричан у односу на координатни почетак

Испитајмо $f(x)$ на интервалу $(0, +\infty)$

$$y = \begin{cases} \frac{\ln x + 1}{x}, & x > 0 \\ \frac{\ln(-x)+1}{x}, & x < 0 \end{cases}$$

$y = \frac{\ln x + 1}{x}, \quad x > 0$

3° $y = 0 \iff \ln x + 1 = 0$

$\iff \ln x = -1$

$x = e^{-1} = \frac{1}{e}$

$y = \frac{\ln x + 1}{x}$, $x > 0 \ \forall x \in (0, +\infty)$

$y > 0 \iff \ln x + 1 > 0$

$\Leftrightarrow \ln x > -1$

$\Leftrightarrow \ln x > \ln e^{-1}$

$\Leftrightarrow x > \frac{1}{e}$

$y < 0 \Leftrightarrow \ln x + 1 < 0$

$\Leftrightarrow \ln x < -1$

$\Leftrightarrow \ln x < \ln e^{-1}$

$\Leftrightarrow x < \frac{1}{e}$

$x \in (0, \frac{1}{e})$

4°

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln x + 1}{x} = \lim_{x\to 0} \underbrace{(\ln x + 1)}_{\to -\infty} \cdot \underbrace{\frac{1}{x}}_{\to +\infty} = \underline{\underline{-\infty}}$$

$x = 0$ вертикална асимптота

$x = 0$ $\underline{\underline{в.а}}$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{\ln x + 1}{x} \overset{\frac{\infty}{\infty}}{\underset{Л.П}{=}} \lim_{x\to +\infty} \frac{\overset{\to 0}{\frac{1}{x}}}{1} = 0$$

$y = 0$. х.а., $x \to +\infty$

нема косе асимптоте

5°

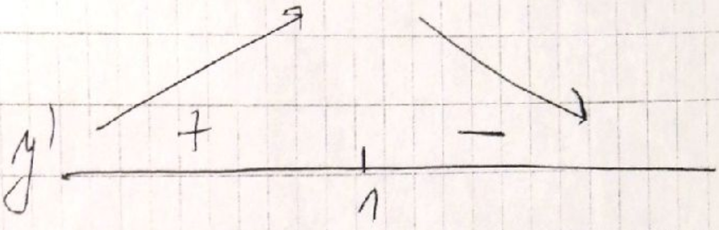
$$y = \frac{\ln x + 1}{x}$$

$$y' = \frac{\frac{1}{x} \cdot x - (\ln x + 1) \cdot 1}{x^2}$$

$$y' = \frac{1 - \ln x - 1}{x^2} = -\frac{\ln x}{x^2} > 0$$

$$y' = 0 \iff \ln x = 0$$
$$\iff x = 1$$

$$y' > 0 \iff \ln x < 0$$
$$\iff x < 1$$
$$\iff x \in (0,1)$$



У $x=1$ функција достиже максимум

$$y_{max} = f(1) = \frac{\ln 1 + 1}{1} = \frac{0+1}{1} = 1$$

$M(1,1)$

6°

$$y'' = \left(-\frac{\ln x}{x^2}\right)' = -\frac{\frac{1}{x}\cdot x^2 - \ln x \cdot 2x}{x^4} = -\frac{x - 2x \cdot \ln x}{x^4} =$$

$$= \frac{2\ln x - 1}{x^3}$$

$$y'' = 0 \iff 2\ln x - 1 = 0$$
$$\iff \ln x = \frac{1}{2}$$
$$\iff x = e^{\frac{1}{2}}$$

y'' није дефинисано у $x=0$, $x = 0 \in D(y)$

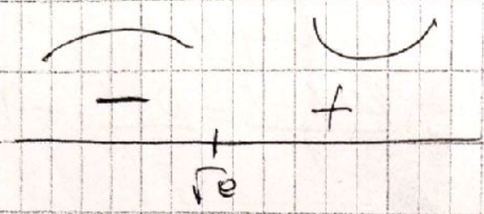
$$y'' > 0 \iff 2\ln x - 1 > 0$$
$$\iff \ln x > \frac{1}{2}$$
$$\iff \ln x > \ln e^{\frac{1}{2}}$$
$$\iff x > e^{\frac{1}{2}}$$
$$\iff x > \sqrt{e}$$

$$y'' < 0 \iff 2\ln x - 1 < 0$$

$$\iff \ln x < \frac{1}{2}$$

$$\iff x < \sqrt{e}$$

$$\iff x \in (0, \sqrt{e})$$



Превојна тачка је

$P(\sqrt{e}, f(\sqrt{e}))$

$$f(\sqrt{e}) = \frac{\ln\sqrt{e}+1}{\sqrt{e}} = \frac{\frac{1}{2}+1}{\sqrt{e}} = \frac{3}{2\sqrt{e}}$$

7° $P\left(\sqrt{e}, \frac{3}{2\sqrt{e}}\right)$