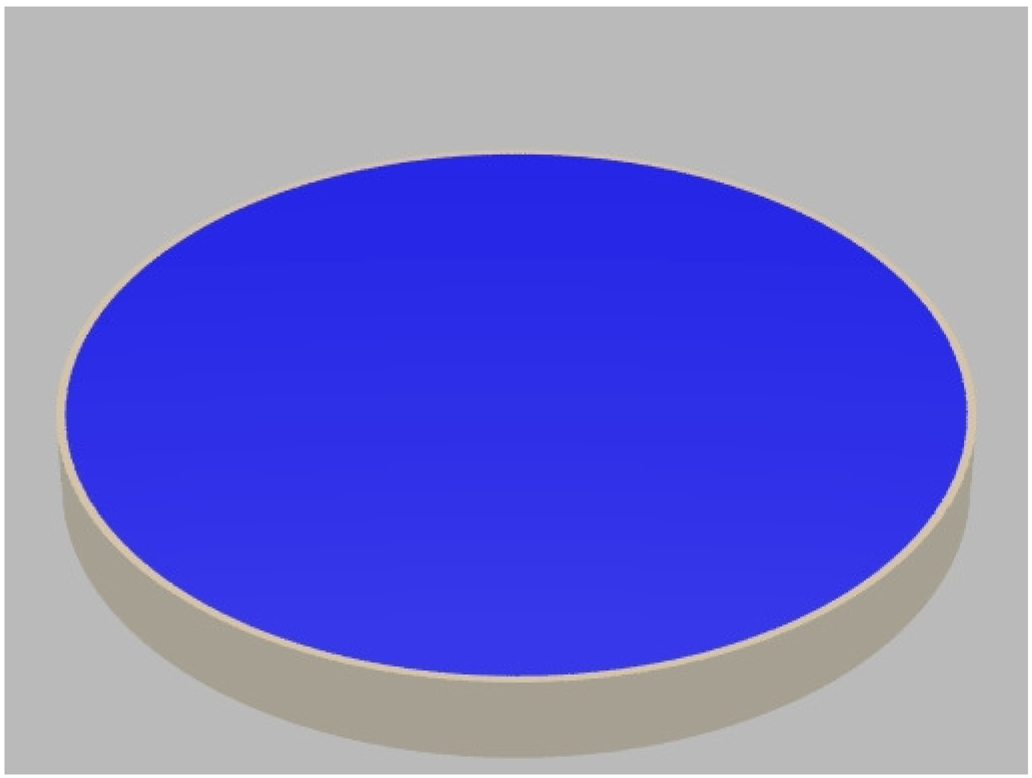
Lekcija V

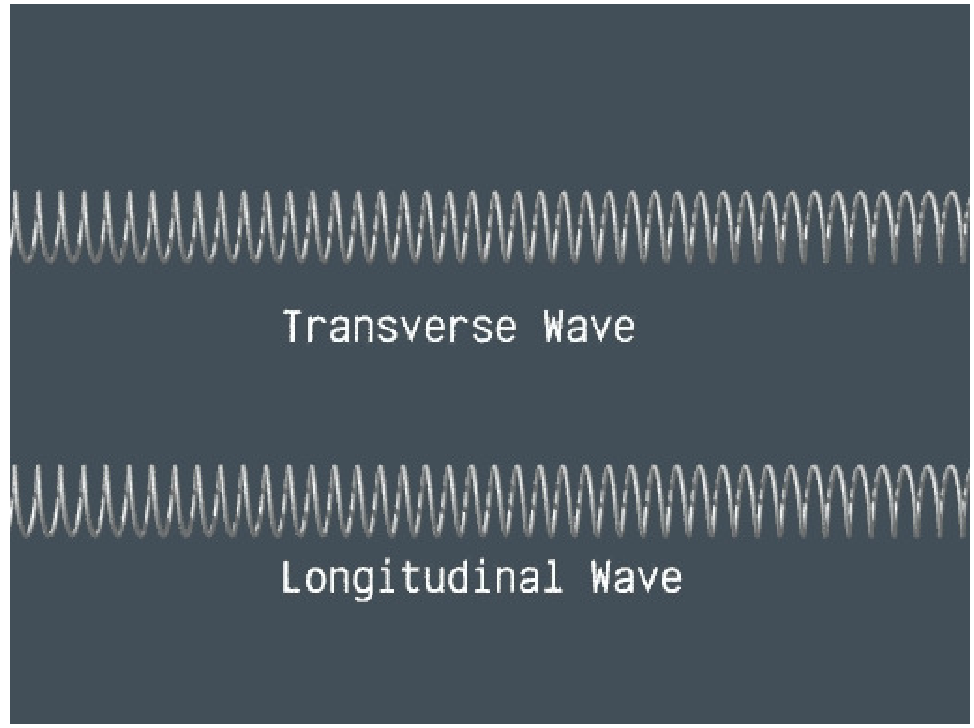
Talasi:

Tals je samoodrživi poremećaj (disturbancija) medija koji se prostire određenom brzinom

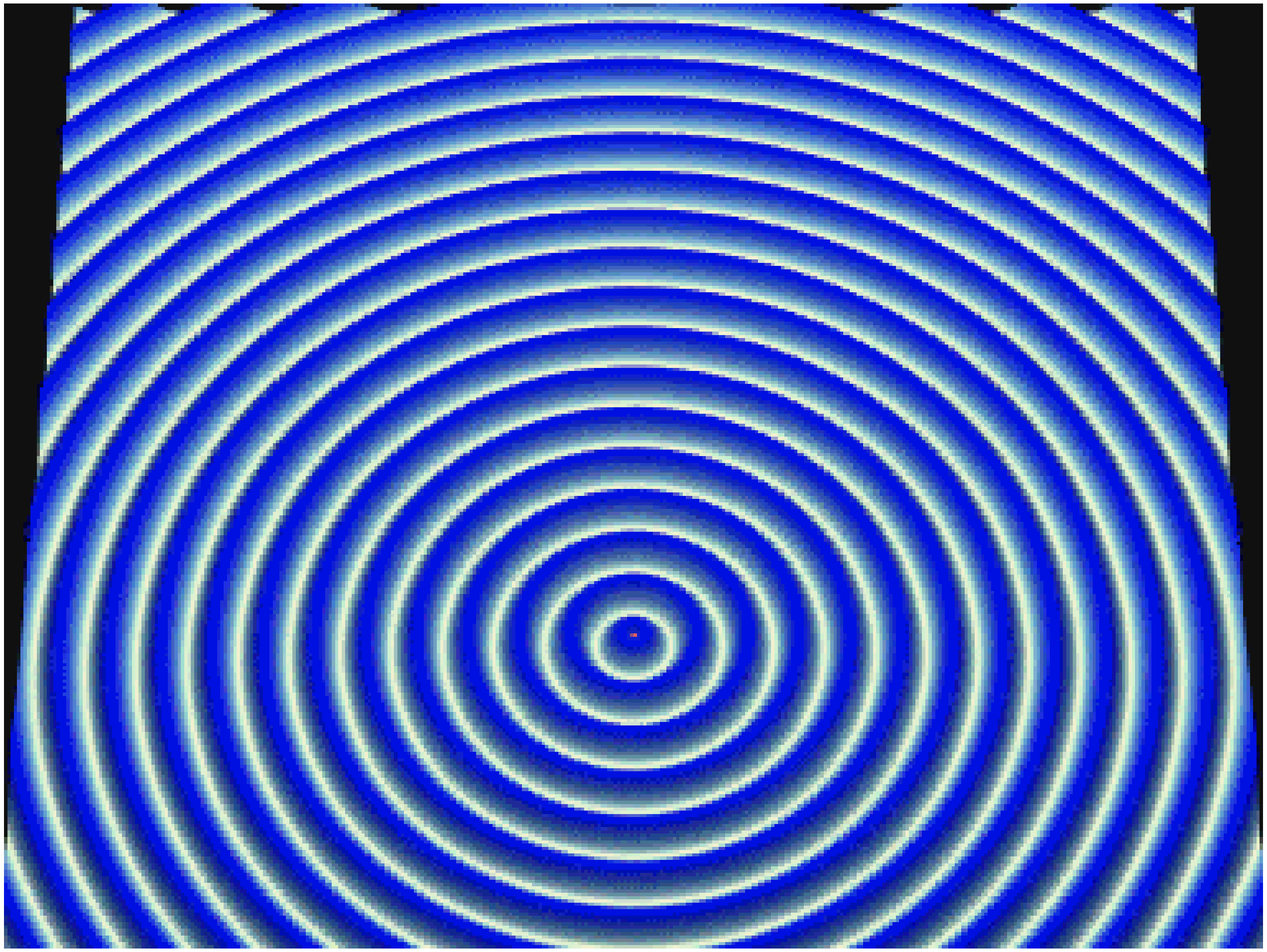


Jedno-dimenzioni talasi

- Poremećaj medija koji čini talas nosi sa sobom određenu energiju ali ne i masu (t.z.v. Progresivni talas. Za vizualizaciju su najpogodniji mehanički talasi kao npr., na struni, površinski talasi na vodi, zvučni talasi u vazduhu itd.
- Elektomagnetski talasi su radio talasi, x-zraci, svetlost. Oni se mogu prostirati i u vakuumu.
- Zvučni talasi su **longitudinalni –** *poremećaj sredine je u pravcu prostiranja talasa.*
- Talas na struni i elektromagnetski talasi su primeri **transverzalnih talasa-***poremećaj sredine je u pravcu normalnom na pravcu prostiranja talasa.*



Primer dvodimenzionog talasa



Poremećaj (disturbancija) sredine se opisuje kao: $\psi(x,t) = f(x,t)$

Oblik poremećaja se može videti u nekom konstantnom trenutku npr., $\psi(x,t)_{t=0} = f(x,0) = f(x)$

Ako vežemo novi sistem referencije *S'* za poremećaj koji se kreće u pozitivnom smeru x'ose brzinom *v*, tada u tom sistemu ψ nije više funkcija vremena

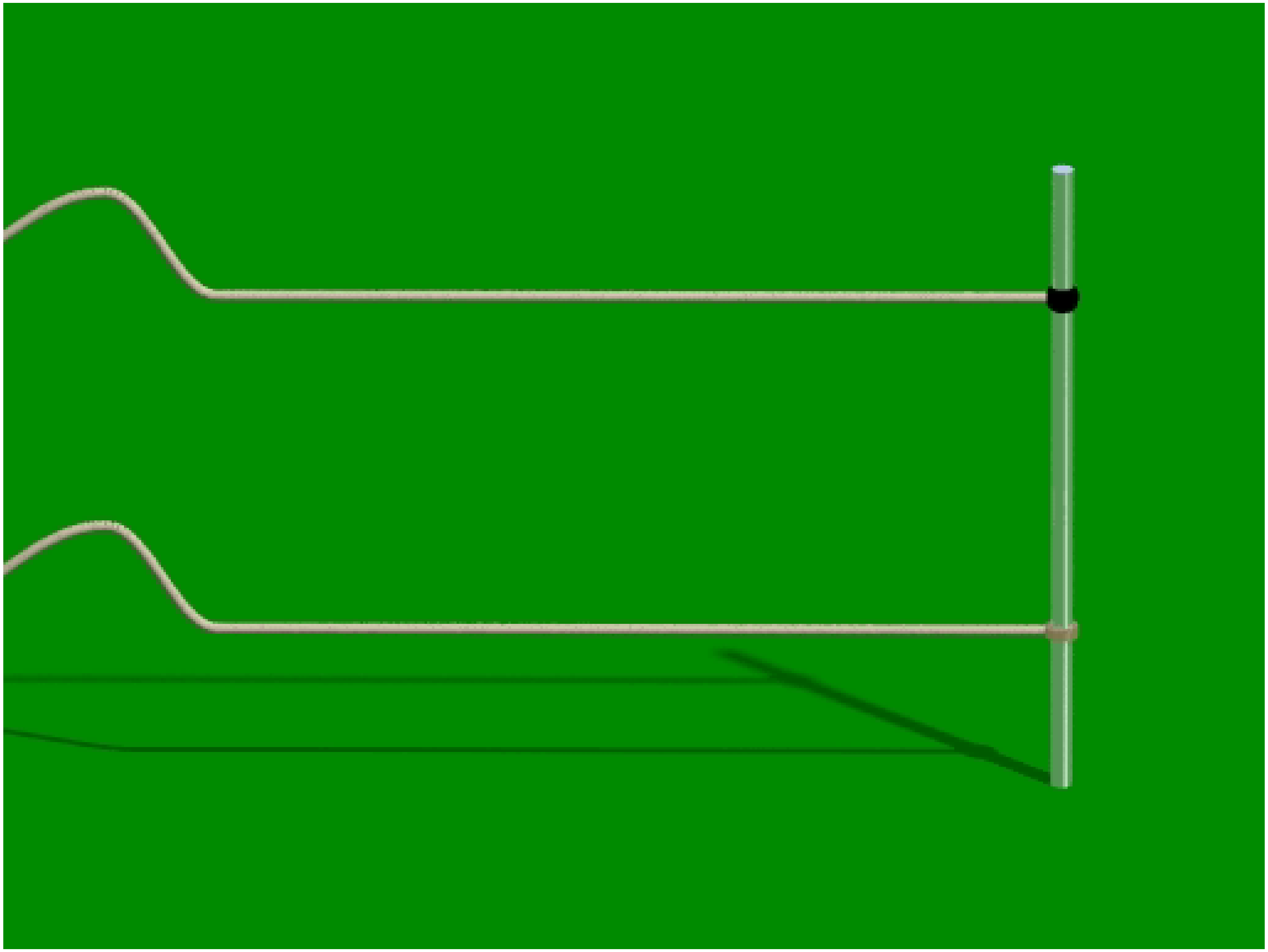
$$\psi = f\left(x'\right)$$

Ako izvršimo transformaciju koordinata kao $x' = x - vt$

Dobijamo izraz za talas u sistemu koji miruje S

$$\psi(x,t) = f(x - vt).$$

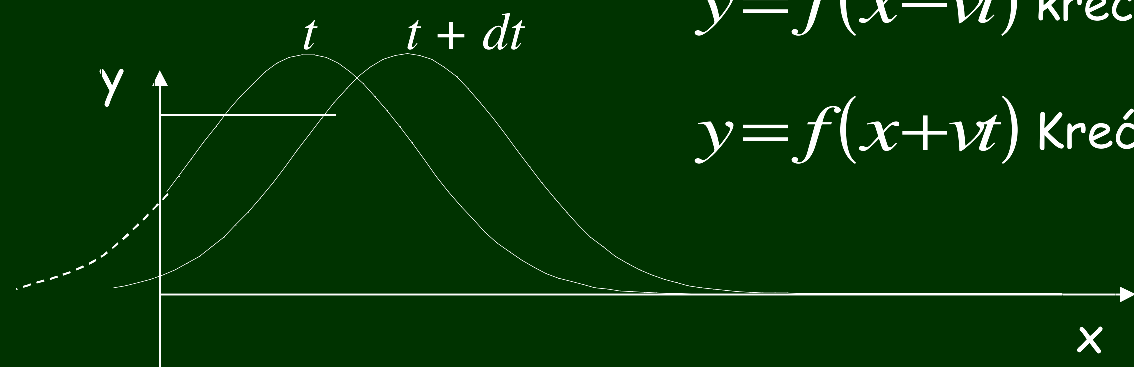
Ovaj izraz reprezentuje u opštoj formi *jedno-dimenzionu* ***talasnu funkciju.***



Na primer, Gaussian: $y = Ae^{-(x-vt)^2/c^2}$

$y = f(x - vt)$ kreće se u +x

$y = f(x + vt)$ Kreće se u -x

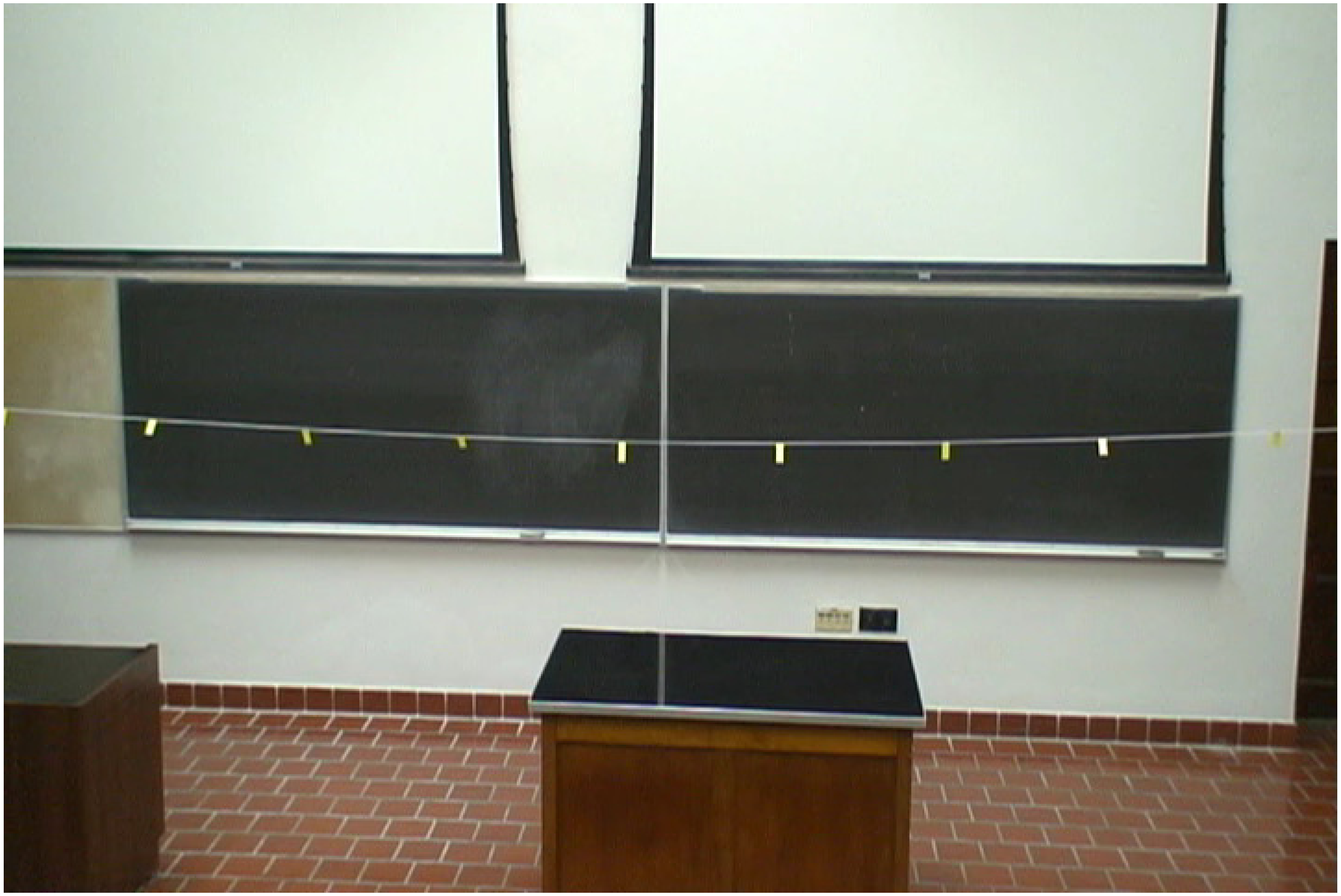


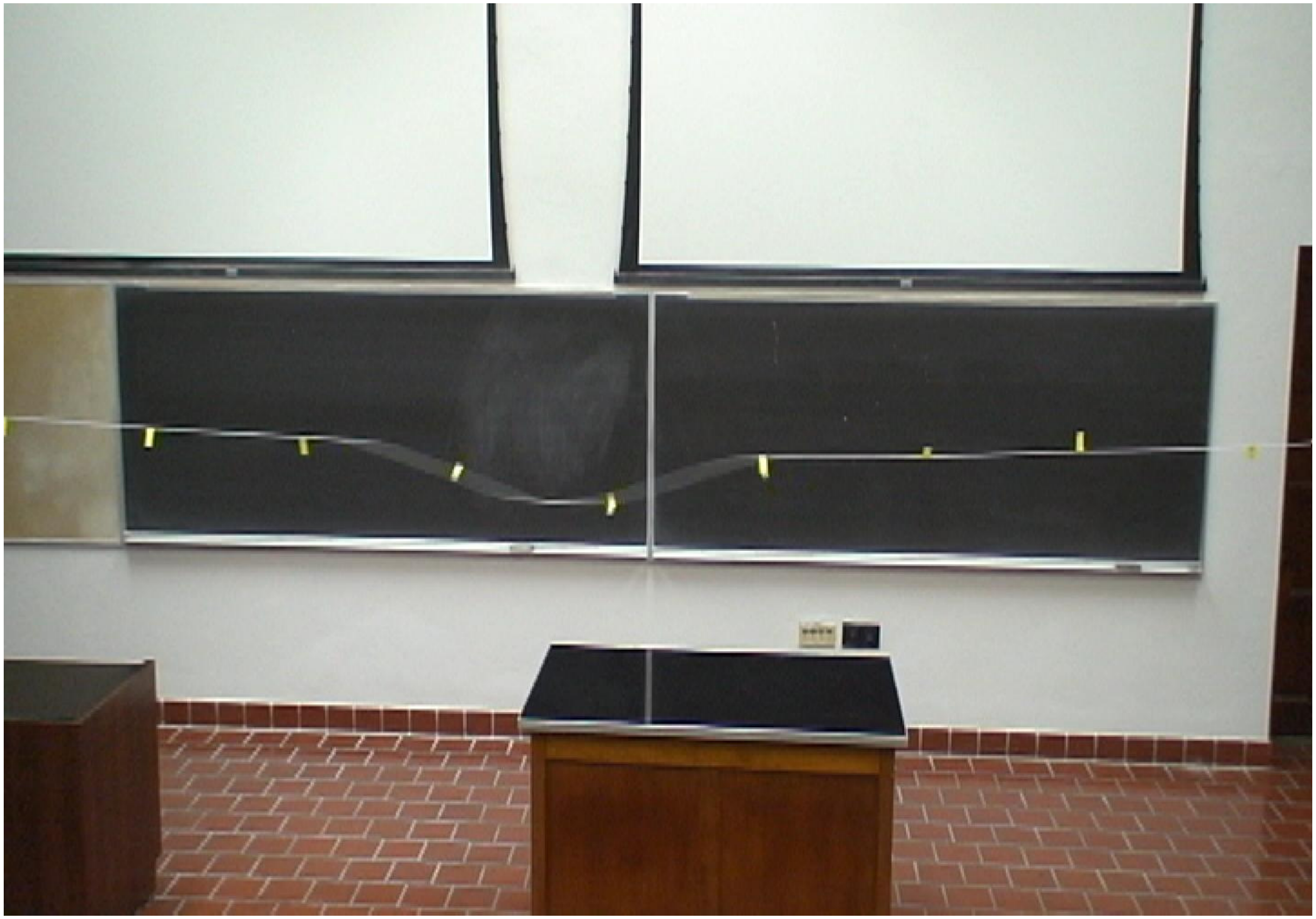
Koliko brzo? $y = f(x,t) = f(x+dx, t+dt)$

$$x - vt = (x + dx) - v(t + dt)$$

$$0 = dx - vdt$$

$$dx/dt = v$$





Diferencijalna talasna jednačina

$$\psi(x,t) = f(x^{'}) \qquad \frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x} \qquad \frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x^{'}}\frac{\partial x^{'}}{\partial x} = \frac{\partial f}{\partial x^{'}}$$ jer je $$\frac{\partial x^{'}}{\partial x} = \frac{\partial(x \mp vt)}{\partial x} = 1$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = \mp v \frac{\partial \psi}{\partial x}$$

Ako još jedanput diferenciramo $\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 f}{\partial x^{'2}}$ i $\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\mp v \frac{\partial f}{\partial x^{'}}\right) = \mp v \frac{\partial}{\partial x^{'}}\left(\frac{\partial \psi}{\partial t}\right)$ jer je

$$\frac{\partial \psi}{\partial t} = \frac{\partial f}{\partial t}$$

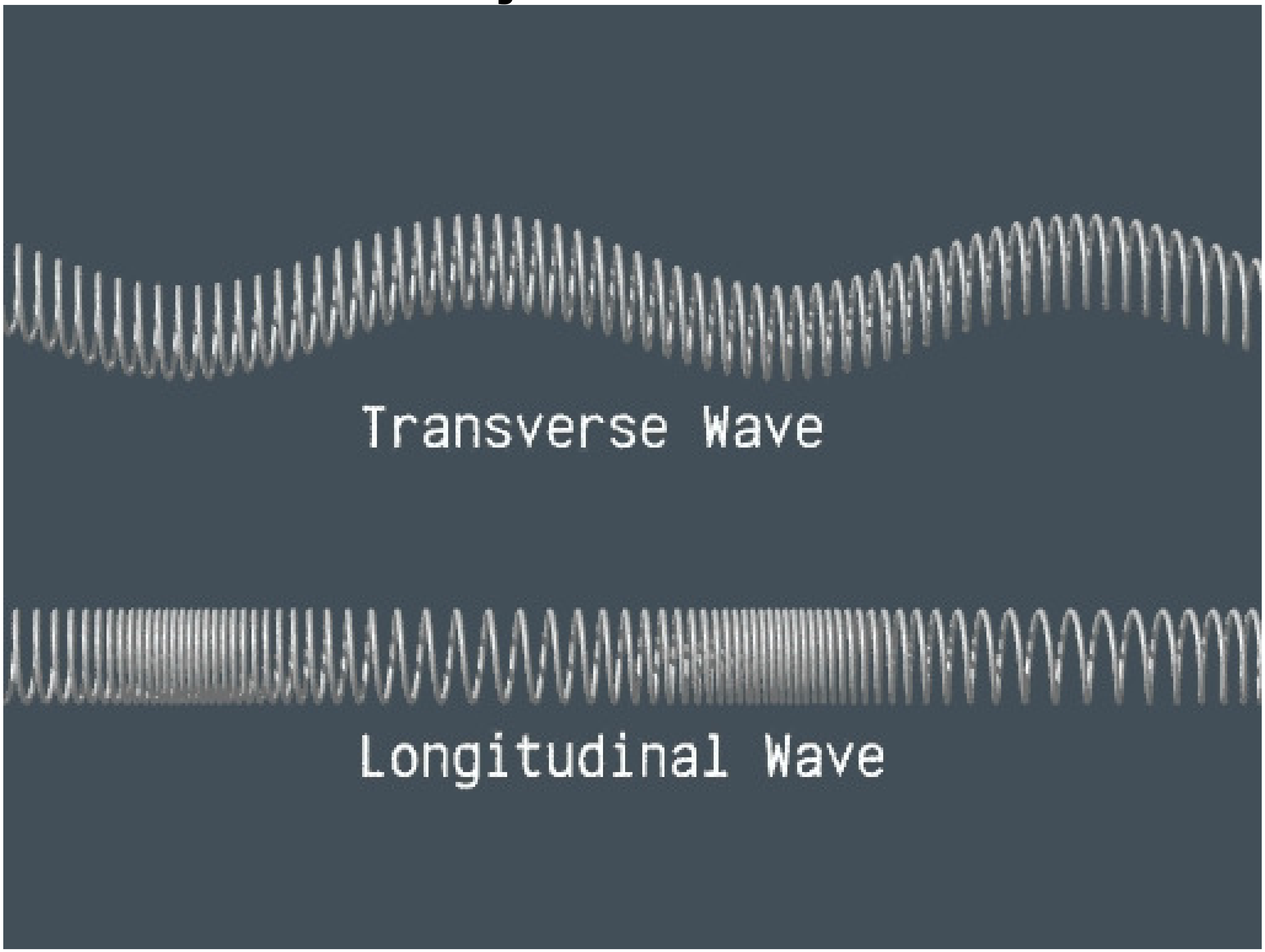
Ako još jednom zamenimo gornje izraze

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 f}{dx^{'2}}$$ i konačno

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

Ovo je *jedno-dimenziona* ***diferencijalna talasna jednačina.***

Harmonijski talasi



Harmonijski talasi

$\psi(x,t)_{t=0} = \psi(x) = A\sin kx = f(x)$ gde je k pozitivna konstanta i zove se **propagacioni broj** a **A-** je **amplituda talasa.**

$$\psi(x,t) = A\sin k(x - vt) = f(x - vt)$$

Prostorni period je poznat kao **talasna dužina** i označava se sa λ.

$$\psi(x,t) = \psi(x \pm \lambda, t)$$

$$\sin k(x - vt) = \sin k[(x \pm \lambda) - vt] = \sin[k(x - vt) \pm 2\pi] \quad \Rightarrow |k\lambda = 2\pi|$$

$$k = 2\pi / \lambda$$

Vremenski period T se dobija iz uslova

$$\sin k(x - vt) = \sin k[x - v(t + T)] = \sin[k(x - vt) \pm 2\pi]$$

Odavde sledi $|kvT| = 2\pi \Rightarrow \frac{2\pi}{\lambda} vT = 2\pi \Rightarrow \quad T = \lambda / v$

$\nu \equiv \frac{1}{T}$ -frekvencija

$$v = \lambda \nu$$

Ugaona vremenska frekvencija $\omega \equiv 2\pi / T = 2\pi\nu$

Prostorna frekvencija ili **talasni broj** $k \equiv 1/\lambda$

Ekvivalenti izrazi za harmonijske talase

$$\psi = A \sin k(x \mp vt)$$

$$\psi = A \sin 2\pi\left(\frac{x}{\lambda} \mp \frac{t}{T}\right)$$

Monohromatski talas
Kvazimonohromatski

$$\psi = A \sin(kx \mp \omega t)$$

Faza i fazna brzina

$\psi(x,t) = A \sin(kx - \omega t + \varphi_0)$ $\varphi = kx - \omega t - \varphi_0$ -**faza talasa;** $\varphi_0 -$ **početna faza.**

$\varphi = kx - \omega t - \varphi_0 = const$ Brzina prostiranja konstantne faze je data kao

$$\left(\frac{\partial x}{\partial t}\right)_{\varphi} = \frac{\omega}{k} = v$$

Princip superpozicije

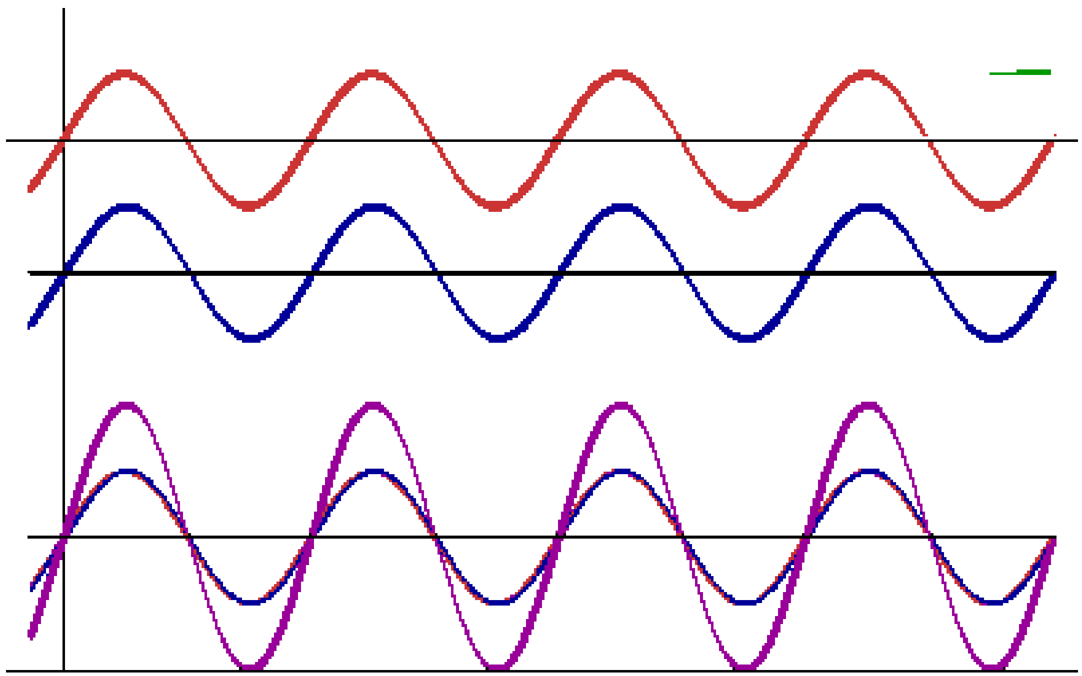
Ako su ψ_1 *i* ψ_2 *rešenja talasne jednačine tada je i* $(\psi_1 + \psi_2)$ *takođe rešenje.*

$$\frac{\partial^2 \psi_1}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi_1}{\partial t^2}$$ *i* $$\frac{\partial^2 \psi_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi_2}{\partial t^2}$$ i ako ih saberemo $$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi_1}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2 \psi_2}{\partial t^2}$$

grupisanjem dobijamo

$$\frac{\partial^2}{\partial x^2}(\psi_1 + \psi_2) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(\psi_1 + \psi_2)$$

Koje nam pokazuje da je zaista $(\psi_1 + \psi_2)$ rešenje.



Kompleksna reprezentacija

“kompleksna ravan”

“kompleksni broj”

$$z = x + iy$$

“realni” $x = r\cos\theta$

“imaginarni” $y = r\sin\theta$

$$\tilde{z} = x + iy = r(\cos\theta + i\sin\theta)$$

Ojlerova formula (Euler)

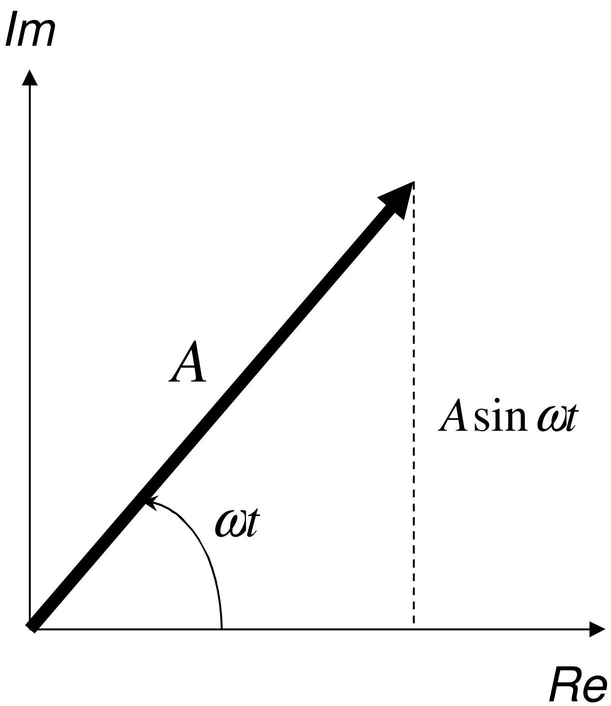
$$e^{i\theta} = \cos\theta + i\sin\theta$$

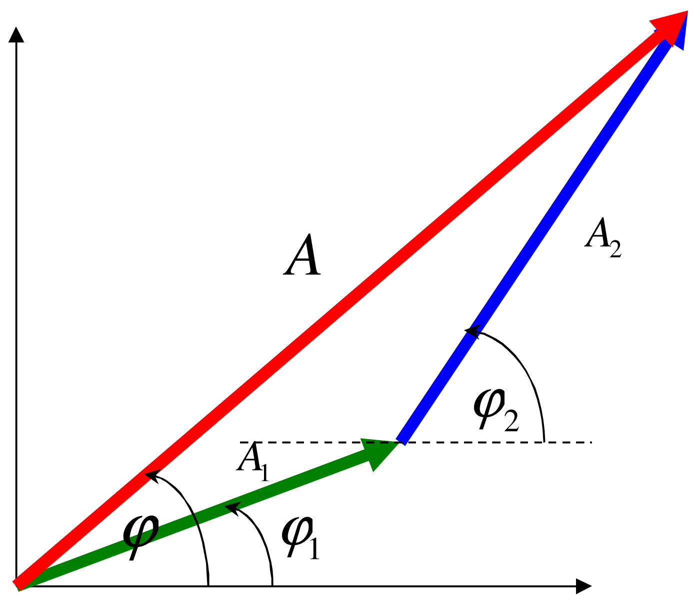
$$\psi(x,t) = \mathrm{Re}\left[Ae^{i(wt - kx + \varphi_0)}\right]$$

$$\psi(x,t) = A\cos(\omega t - kx + \varphi_0)$$

$$\psi(x,t) = Ae^{i(\omega t - kx + \varphi_0)} = Ae^{i\varphi}$$

Fazori i sabiranje talasa





$$A^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\varphi_2 - \varphi_1)$$

Ravanski talasi

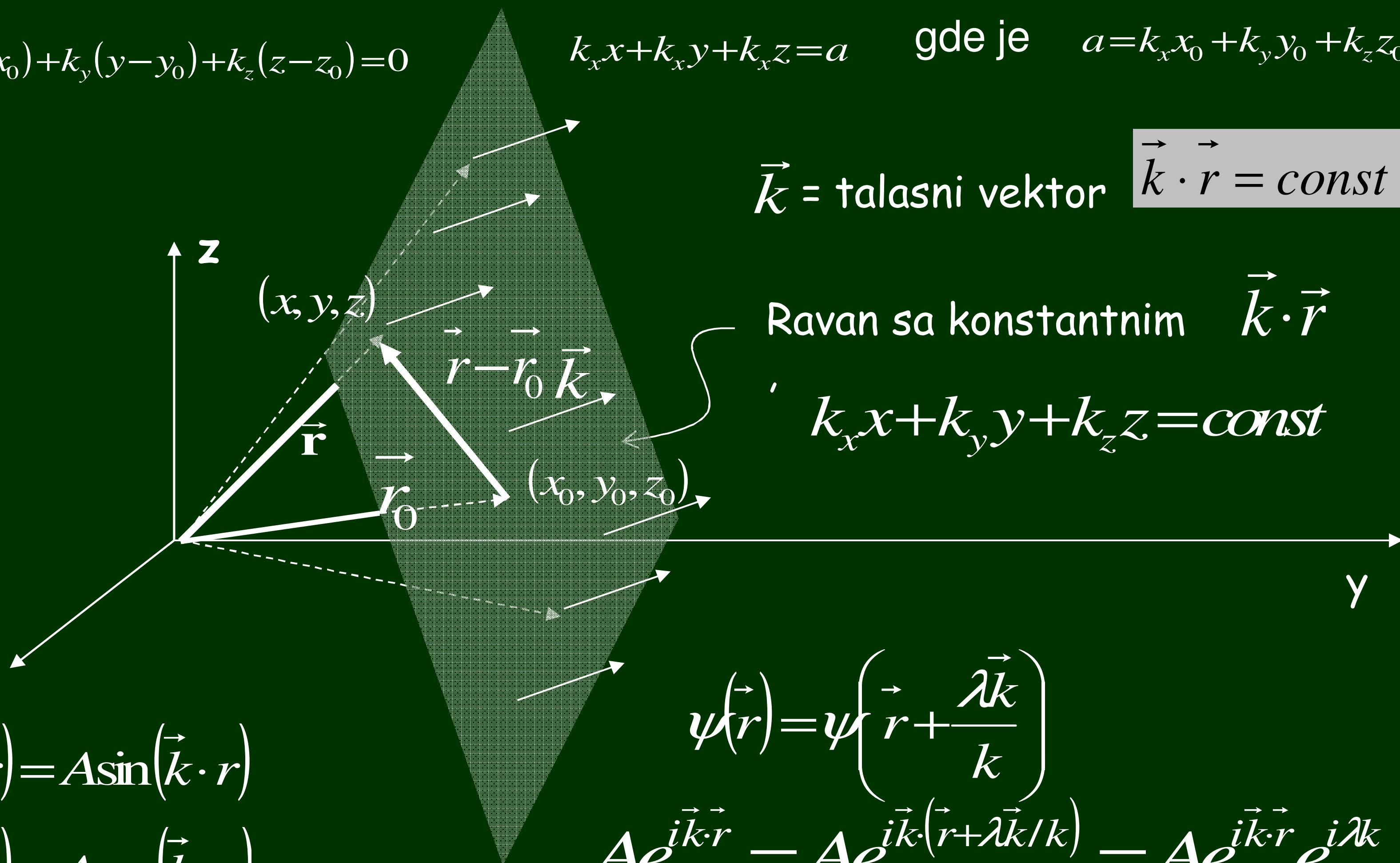
$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ $\quad (\vec{r}-\vec{r}_0) = (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}$ $\quad (\vec{r}-\vec{r}_0)\cdot\vec{k} = 0$ $\quad \vec{k} = k_x\vec{i} + k_y\vec{j} + k_z\vec{z}$

$k_x(x-x_0) + k_y(y-y_0) + k_z(z-z_0) = 0$ $\quad k_x x + k_x y + k_x z = a$ gde je $a = k_x x_0 + k_y y_0 + k_z z_0 = const.$

$\vec{k}$ = talasni vektor $\quad \vec{k}\cdot\vec{r} = const = a$

Ravan sa konstantnim $\vec{k}\cdot\vec{r}$

, $k_x x + k_y y + k_z z = const$



$$\psi(\vec{r}) = A\sin(\vec{k}\cdot r)$$

$$\psi(\vec{r}) = A\cos(\vec{k}\cdot r)$$

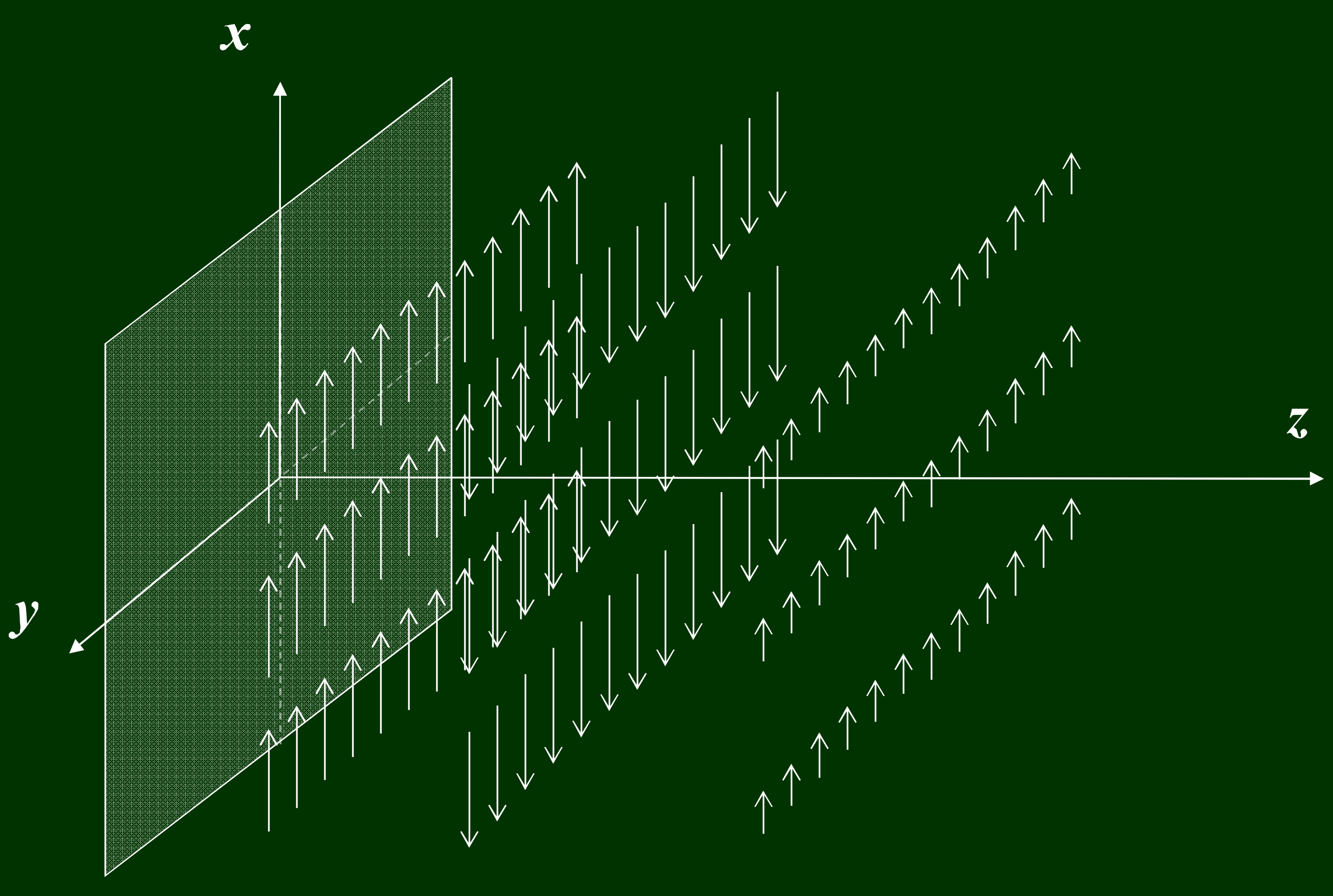
$$\psi(\vec{r}) = Ae^{i(\vec{k}\cdot\vec{r})}$$

$$\psi(\vec{r}) = \psi\left(\vec{r} + \frac{\lambda\vec{k}}{k}\right)$$

$$Ae^{i\vec{k}\cdot\vec{r}} = Ae^{i\vec{k}\cdot(\vec{r}+\lambda\vec{k}/k)} = Ae^{i\vec{k}\cdot\vec{r}}e^{i\lambda k}$$

$e^{i\lambda k} = 1 = e^{i2\pi}$ $\quad \lambda k = 2\pi$ $\quad k = 2\pi/\lambda$ $\quad \vec{k}-$ propagacioni vektor

$$\psi(z,t) = A_{0x}\cos(kz - \omega t)$$



Fazna brzina je ekvivalentna brzini propagacije talasnog fronta

$$\psi(\vec{r},t)=\psi(r_k+dr_k,t+dt)=\psi(r_k,t)$$

U eksponencijalnom obliku

$$Ae^{i(\vec{k}\cdot\vec{r}\mp\omega t)}=Ae^{i(kr_k++kdr_k\mp\omega t\mp\omega dt)}=Ae^{i(kr_k\mp\omega t)}$$

$$kdr_k=\pm\omega dt$$

$$\frac{dr_k}{dt}=\pm\frac{\omega}{k}=\pm v$$

$$\psi(x,y,z)=Ae^{i(k_x x+k_y y+k_z z\mp\omega t)}$$

$$\psi(x,y,z)=Ae^{i(k(\alpha x+\beta y+\gamma z)\mp\omega t)}$$

$$|\vec{k}|=k=\left(k_x^2+k_y^2+k_z^2\right)^{1/2}$$

$$\alpha^2+\beta^2+\gamma^2=1$$

Svaki trodimenzioni talas se može izraziti kao kombinacija ravanskih talasa.

Trodimenziona diferencijalna talasna jednačina

$$\frac{\partial^2\psi}{\partial x^2} = -\alpha^2 k^2 \psi \qquad \frac{\partial^2\psi}{\partial y^2} = -\beta^2 k^2 \psi \qquad \frac{\partial^2\psi}{\partial z^2} = -\gamma^2 k^2 \psi$$

i

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2 \psi$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = -k^2\psi$$

Imajući u vidu da je $v = \omega / k$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Ili uzimajući skraćeniju verziju

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Laplasov operator imamo

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Sferni talasi

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$ gde su r, θ i ϕ

definisani kao $x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$

$$\psi(\vec{r}) = \psi(r,\theta,\phi) = \psi(r)$$