Građevinski fakultet

Vježbe

1) Odrediti jednačinu tangente na krivu $7x^2-2y^2=14$, koja je normalna na pravu $2x+4y-3=0$.

Rj

$l: 2x+4y-3=0$

$4y=-2x+3 \mid \cdot \frac{1}{4}$

$y=-\frac{1}{2}x+\frac{3}{4}$

$k_l=-\frac{1}{2}$

$t \perp l \Rightarrow k_t \cdot k_l=-1$

$k_t=2$

$t: y-y_0=k_t(x-x_0)$

kriva: $7x^2-2y^2=14 \mid '$

$14x-4y\cdot y'=0$

$y'=\frac{14x}{4y}$

$y'=\frac{7x}{2y}$

$y'(x_0,y_0)=2$

$\frac{7x_0}{2y_0}=2$

$7x_0=4y_0$

$x_0=\frac{4y_0}{7}$

$\cancel{7}^{1}\cdot\frac{16y_0^2}{\cancel{49}_7}-2y_0^2=14 \mid \cdot 7$

$16y_0^2-14y_0^2=14\cdot 7$

$2y_0^2=14\cdot 7$

$y_0^2=49$

$\Leftarrow y_0=\pm 7$

$x_0=\frac{4\cdot 7}{7}=4$

$x_0=\frac{4\cdot(-7)}{7}=-4$

$\Rightarrow$ $M(4,7)$

$M_1(-4,-7)$

$t_1: \; y-7=2(x-4)$

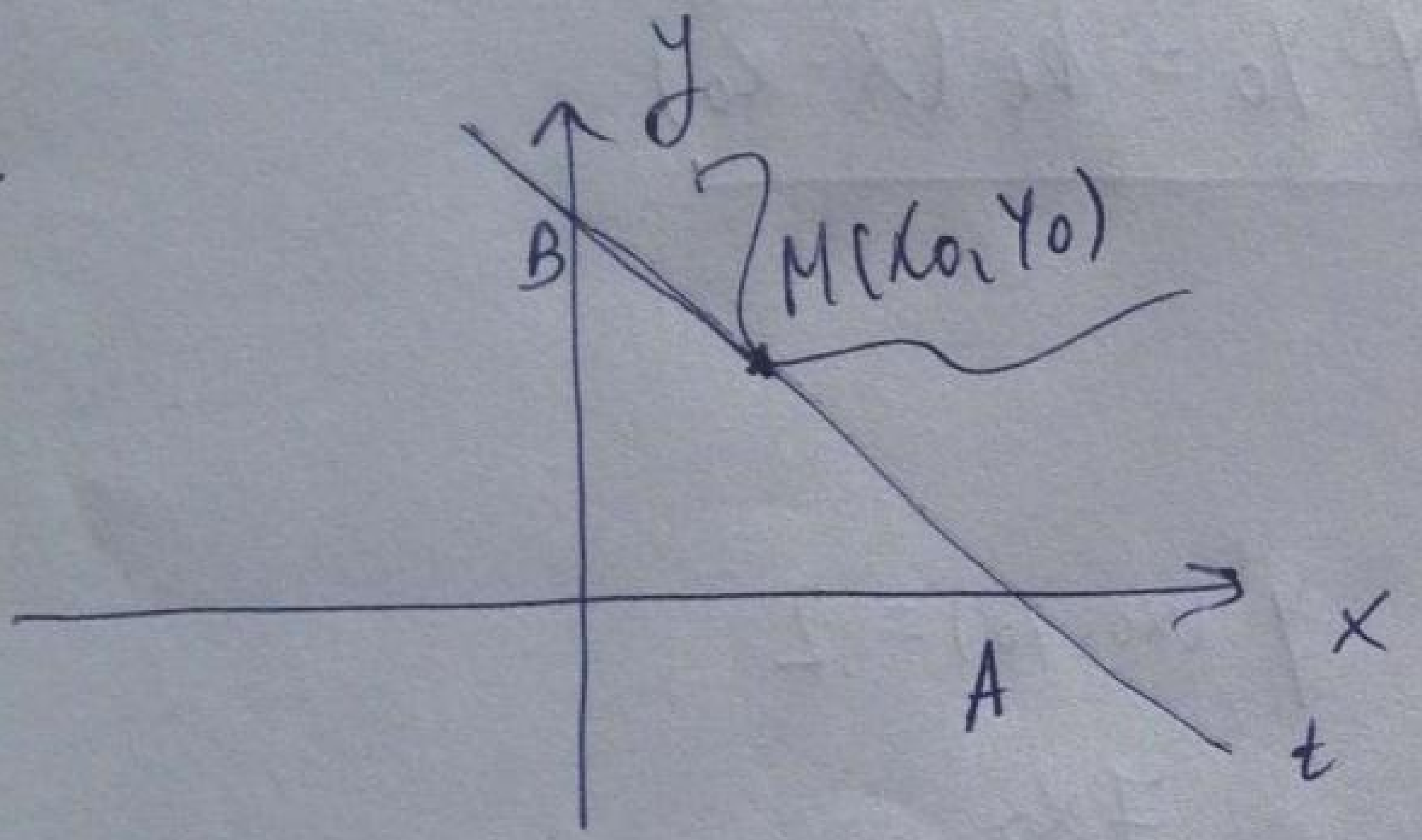
$\underline{y=2x-1}$

$t_2: \; y+7=2(x+4)$

$\underline{y=2x+1}$

2) Data je kriva $y=(\sqrt{a}-\sqrt{x})^2$. Tangenta krive sijece koordinatne ose u tačkama A i B. Dokazati da je $OA+OB=\text{const}$.

Rj:



$t: \; y-y_0=k_t(x-x_0)$

$y'=\cancel{2}(\sqrt{a}-\sqrt{x})\cdot\dfrac{-1}{\cancel{2}\sqrt{x}}$

$y'=\dfrac{\sqrt{x}-\sqrt{a}}{\sqrt{x}}$

$\underline{y'(M)=\dfrac{\sqrt{x_0}-\sqrt{a}}{\sqrt{x_0}}=k_t.}$

$t: \; y-y_0=\dfrac{\sqrt{x_0}-\sqrt{a}}{\sqrt{x_0}}(x-x_0)$

$\underline{y_0=(\sqrt{a}-\sqrt{x_0})^2}$

$\left.\begin{array}{l} A\in Ox\text{-osi}; \; A(x_A,0) \\ A\in \text{tangenti} \end{array}\right\} \Rightarrow$

$0-y_0=\dfrac{\sqrt{x_0}-\sqrt{a}}{\sqrt{x_0}}(x_A-x_0)$

$y_0=\dfrac{\sqrt{a}-\sqrt{x_0}}{\sqrt{x_0}}(x_A-x_0)$

$x_A-x_0=\dfrac{y_0\sqrt{x_0}}{\sqrt{a}-\sqrt{x_0}}=\dfrac{(\sqrt{a}-\sqrt{x_0})^{\cancel{2}}\cdot\sqrt{x_0}}{\cancel{\sqrt{a}-\sqrt{x_0}}}$

$$X_A - X_0 = \frac{(\sqrt{a}-\sqrt{x_0})\sqrt{x_0}}{1}$$

$$\underline{\underline{X_A}} = \cancel{x_0} + \sqrt{a}\sqrt{x_0} - \cancel{x_0} = \underline{\underline{\sqrt{ax_0}}}$$ (ovo je dužina OA)

$$\left.\begin{array}{l} B \in O_y\text{-osi}; \quad B(0, y_B) \\ B \in \text{tangenti} \end{array}\right\} \Rightarrow$$

$$y_B - y_0 = \frac{\sqrt{x_0}-\sqrt{a}}{\sqrt{x_0}}(0 - x_0)$$

$$y_B - y_0 = \frac{\sqrt{a}-\sqrt{x_0}}{\cancel{\sqrt{x_0}}} \cdot \cancel{x_0}^{\sqrt{x_0}}$$

$$y_B = y_0 + \sqrt{a}\sqrt{x_0} - x_0$$

$$y_B = (\sqrt{a}-\sqrt{x_0})^2 + \sqrt{ax_0} - x_0$$

$$y_B = a - 2\sqrt{ax_0} + \cancel{x_0} - \cancel{x_0} + \sqrt{ax_0}$$

$$\underline{y_B = a - \sqrt{ax_0}}$$ (ovo je dužina OB)

$$OA + OB = \cancel{\sqrt{ax_0}} + a - \cancel{\sqrt{ax_0}} = a = \text{const}.$$

Scanned by TapScanner

3) Pokazati da funkcija $f(x) = x^3 - 3x^2 + 2x$ zadovoljava uslove Rolove teoreme na segmentu $[0,2]$ i naći odgovarajuće c iz Teoreme.

Rj:

$$f(x) = x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x-1)(x-2)$$

Fja $f(x) = y$ je neprekidna i diferencijabilna na $\mathbb{R}$ jer je data fja polinom (trećeg stepena). $\Rightarrow$

$$\left.\begin{array}{l} ① \ f \in C[0,2] \ \text{ i } ② \ f \in D(0,2). \\ ③ \ f(0) = 0(0-1)(0-2) = 0 \\ \quad\ f(2) = 2(2-1)(2-2) = 0 \end{array}\right\} \xRightarrow{\text{Rolova}} \text{t.}$$

$$\exists\, c \in (0,2) \text{ t.d. } f'(c) = 0.$$

$$f'(x) = 3x^2 - 6x + 2$$

$$f'(c) = 3c^2 - 6c + 2 \qquad 3c^2 - 6c + 2 = 0$$

$$c_{1,2} = \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{6 \pm 2\sqrt{3}}{6}$$

$$\underbrace{c_1 = 1 + \frac{\sqrt{3}}{3}} \qquad \underbrace{c_2 = 1 - \frac{\sqrt{3}}{3}}$$

Postoje dva c iz Teoreme jer $f(1) = 0$.

4) Dokazati nejednakost: ③

$$|\cos 2x - \cos 2y| \leq 2|x-y| \quad \forall x, y \in \mathbb{R}.$$

Rj:

Posmatrajmo f-ju: $f(t) = \cos 2t$, $t \in \mathbb{R}$.

Fja $f(t) = \cos 2t$ je neprekidna i diferencijabilna na $\mathbb{R}$ $\Rightarrow$

$\exists$ proizvoljan segment $[x, y] \subseteq \mathbb{R}$ t.d.

$$f \in C[x,y] \text{ i } f \in D(x,y).$$

$f'(t) = -2\sin 2t$. $\Rightarrow$ Na osnovu Lagr. teoreme

$$f(y) - f(x) = f'(c)(y-x)$$

$$\cos 2y - \cos 2x = -2\sin 2c\,(y-x) \Rightarrow$$

$$|\cos 2y - \cos 2x| = |-2\sin 2c\,(y-x)| \leq \underbrace{|-2\sin 2c|}_{\leq 2}\,|y-x| \leq 2|y-x|.$$

$x \to +\infty$ $\quad = e^{\lim\limits_{x\to+\infty}} \quad = e = 1.$

⊛ Rok $\lim\limits_{x\to\frac{\pi}{4}} \left(x-\frac{\pi}{4}\right) tg\, 2x = \ulcorner 0\cdot\infty \lrcorner = \lim\limits_{x\to\frac{\pi}{4}} \frac{tg\, 2x}{\frac{1}{x-\frac{\pi}{4}}} = \ulcorner \frac{\infty}{\infty} \lrcorner$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\frac{1}{\cos^2 2x}\cdot 2}{-\frac{1}{\left(x-\frac{\pi}{4}\right)^2}} = \lim_{x\to\frac{\pi}{4}} \frac{-2\left(x-\frac{\pi}{4}\right)^2}{\cos^2 2x} = -2\lim_{x\to\frac{\pi}{4}} \frac{\left(x-\frac{\pi}{4}\right)^2}{\cos^2 2x} =$$

$$= -2\lim_{x\to\frac{\pi}{4}} \frac{2\left(x-\frac{\pi}{4}\right)\cdot 1}{-\cancel{2}_{4}\cos 2x\, \sin 2x} = 1 \lim_{x\to\pi/4} \frac{x-\frac{\pi}{4}}{\cos 2x\, \sin 2x} =$$

$$= \lim_{x\to\pi/4} \frac{1}{\underbrace{-2\sin 2x}_{1}\,\underbrace{\sin 2x}_{1} + \underbrace{2\cos 2x\cdot\cos 2x}_{=0}} = -\frac{1}{2}$$

⊛ hon

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x} = \ulcorner 1^\infty \lrcorner =$$ ②

$$= e^{\lim\limits_{x\to 0}\frac{1}{x}\ln\left(\frac{a^x+b^x}{2}\right)} = e^{\lim\limits_{x\to 0}\frac{\ln\left(\frac{a^x+b^x}{2}\right)}{x}} =$$

$$= e^{\lim\limits_{x\to 0}\frac{\frac{2}{a^x+b^x}\cdot(a^x\ln a+b^x\ln b)}{1}} =$$

$$= e^{2\lim\limits_{x\to 0}\frac{a^x\ln a+b^x\ln b}{a^x+b^x}} = e^{2\left(\frac{\ln a+\ln b}{2}\right)} =$$

$$= e^{\ln ab} = \underline{\underline{ab}}$$

---

4) a) $$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \ulcorner \infty-\infty \lrcorner = \lim_{x\to 0}\frac{\sin x - x}{x\sin x} = \ulcorner \tfrac{0}{0} \lrcorner =$$

$$= \lim_{x\to 0}\frac{\cos x - 1}{\sin x + x\cos x} = \ulcorner \tfrac{0}{0} \lrcorner = \lim_{x\to 0}\frac{-\sin x}{\cos x+\cos x - x\sin x} = 0$$

b) $$\lim_{x\to 0}\left(\frac{1}{\operatorname{arctg} x}-\frac{1}{x}\right) = \ulcorner \infty-\infty \lrcorner = \lim_{x\to 0}\frac{x-\operatorname{arctg} x}{x\operatorname{arctg} x} =$$

$$= \lim_{x\to 0}\frac{1-\frac{1}{1+x^2}}{\operatorname{arctg} x + x\frac{1}{1+x^2}} = \lim_{x\to 0}\frac{x^2}{x+\operatorname{arctg} x(1+x^2)} = \ulcorner \tfrac{0}{0} \lrcorner =$$

$$= \lim_{x\to 0}\frac{2x}{1+\frac{1}{1+x^2}(1+x^2)+\operatorname{arctg} x(2x)} = \frac{0}{2} = 0$$