Вјежбе бр. 10 (недјеља 10)

Интеграција рационалних фја ($\int \frac{P(x)}{Q(x)}dx$, P, Q - полиноми)

① $$\int \frac{x^4}{(x+1)^3}dx = \int \frac{x^4}{x^3+3x^2+3x+1}dx$$

[трострука реална нула]

$$x^4 : (x^3+3x^2+3x+1) = x-3$$
$$\underline{-\,x^4 \pm 3x^3 \pm 3x^2 \pm x}$$
$$-3x^3-3x^2-x$$
$$\underline{-3x^3 - 9x^2 - 9x - 3}$$
$$+\quad\;\; +\quad\;\; +\quad\; +$$
$$6x^2+8x+3$$

$$\int \frac{x^4}{(x+1)^3}dx = \int\left((x-3) + \frac{6x^2+8x+3}{(x+1)^3}\right)dx =$$

$$= \int(x-3)dx + \int \frac{6x^2+8x+3}{(x+1)^3}dx = \frac{x^2}{2} - 3x + 6\ln|x+1| + \frac{4}{x+1} - \frac{1}{2(x+1)^2} + C$$

$$\int(x-3)dx = \frac{x^2}{2} - 3x$$

$$\int \frac{6x^2+8x+3}{(x+1)^3}dx = ?$$

$$\frac{6x^2+8x+3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} = \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3}$$

$$\frac{6x^2+8x+3}{(x+1)^3} = \frac{Ax^2+2Ax+A+Bx+B+C}{(x+1)^3} = \frac{Ax^2+(2A+B)x+A+B+C}{(x+1)^3} \quad /\cdot(x+1)^3$$

$$6x^2+8x+3 = Ax^2+(2A+B)x+A+B+C$$

$$\left.\begin{array}{l} A=6 \\ 2A+B=8 \\ A+B+C=3 \end{array}\right\} \Rightarrow \begin{array}{l} A=6 \\ B=8-12=-4 \\ C=3-6+4=1 \end{array}$$

$$\int \frac{6x^2+8x+3}{(x+1)^3}dx = \int \frac{6}{x+1}dx - 4\int\frac{dx}{(x+1)^2} + \int\frac{dx}{(x+1)^3} =$$

$$\int \frac{6}{t}dt - 4\int\frac{dt}{t^2} + \int\frac{dt}{t^3} =$$

$$= 6\ln|x+1| + \frac{4}{x+1} - \frac{1}{2(x+1)^2}$$

[$x+1=t$
$dx=dt$]

② $\int \frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}dx = ?$

$x^2+2x+2=0$

$x_{1,2} = \frac{-2 \pm \sqrt{4-2\cdot 4}}{2}$

$D<0 \Rightarrow$ комплексне нуле

$$\boxed{\frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}} = \frac{Ax+B}{(x^2+2x+2)} + \frac{Cx+D}{(x^2+2x+2)^2}$$

$$= \frac{(Ax+B)(x^2+2x+2)+Cx+D}{(x^2+2x+2)^2} = \boxed{\frac{Ax^3+2Ax^2+2Ax+Bx^2+2Bx+2B+Cx+D}{(x^2+2x+2)^2}}$$

Множењем са $(x^2+2x+2)^2$,

$2x^3+3x^2+x-1 = Ax^3+(2A+B)x^2+(2A+2B+C)x+2B+D$

$$\left.\begin{array}{l} A=2 \\ 2A+B=3 \\ 2A+2B+C=1 \\ D=-1 \end{array}\right\} \begin{array}{l} A=2 \\ B=-1 \\ C=-1 \\ D=1. \end{array}$$

Сада, $\int \frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}dx = \underbrace{\int \frac{2x-1}{x^2+2x+2}dx}_{I_1} + \underbrace{\int \frac{-x+1}{(x^2+2x+2)^2}dx}_{I_2}$

$I_1 = \int \frac{2x+2-3}{(x^2+2x+2)}dx = \int \frac{(2x+2)dx}{(x^2+2x+2)} - 3\int \frac{dx}{(x^2+2x+2)}$

$\left[\begin{array}{l}(x^2+2x+2)=t \\ (2x+2)dx=dt\end{array}\right]$ $= \int \frac{dt}{t} - 3\cdot\int \frac{dx}{(x+1)^2+1}$

$= \ln|x^2+2x+2|$ $\left[\begin{array}{l}x+1=t \\ dx=dt\end{array}\right]$

$\boxed{I_1 = \ln|x^2+2x+2| - 3\cdot \operatorname{arctg}(x+1)}$

$I_2 = \frac{1}{2}\int \frac{-2x+2}{(x^2+2x+2)^2}dx = \frac{1}{2}\cdot(-1)\int \frac{2x-2}{(x^2+2x+2)^2}dx = -\frac{1}{2}\int \frac{2x+2-4}{(x^2+2x+2)^2}dx$

$$= -\frac{1}{2}\left(\int \frac{2x+2}{(x^2+2x+2)^2} - 4\cdot\int\frac{dx}{(x^2+2x+2)^2}\right) = -\frac{1}{2}\cdot\underbrace{\int\frac{2x+2}{(x^2+2x+2)^2}}_{I_3} + 2\cdot\underbrace{\int\frac{dx}{(x^2+2x+2)^2}}_{I_4}$$

$$I_3: \quad \begin{bmatrix} x^2+2x+2=t \\ (2x+2)dx=dt \end{bmatrix} = \int\frac{dt}{t^2} = -\frac{1}{t} = \frac{-1}{x^2+2x+2}$$

$$I_4: \int\frac{dx}{(x^2+2x+2)^2} = \int\frac{dx}{((x+1)^2+1)^2} = \begin{bmatrix} x+1=t \\ dx=dt \end{bmatrix} = \int\frac{dt}{(t^2+1)^2}$$

$$I_4 = \frac{1}{2}\operatorname{arctg} t + \frac{t}{2(t^2+1)} = \frac{1}{2}\operatorname{arctg}(x+1) + \frac{x+1}{2((x+1)^2+1)}$$

(рађен на претходним вјежбама).

$$I = \ln(x^2+2x+2) - 3\operatorname{arctg}(x+1) + \frac{1}{2}\cdot\frac{1}{x^2+2x+2} + \frac{1}{2}\operatorname{arctg}(x+1) + \frac{x+1}{2(x^2+2x+2)} + C$$

$\left[x^2+2x+2=(x+1)^2+1>0\right.$, па $\left.|x^2+2x+2| = x^2+2x+2\right]$

③ $$\int\frac{\cos x}{1+\sin^4 x}\,dx = \begin{bmatrix} \sin x=t \\ \cos x\,dx=dt \end{bmatrix} = \int\frac{dt}{1+t^4}$$

$$t^4+1 = (t^2+1)^2 - 2t^2 = (t^2+1-\sqrt{2}t)(t^2+1+\sqrt{2}t)$$

$\Big[\; t^2-\sqrt{2}t+1=0$, $\quad t_{1,2} = \frac{\sqrt{2}\pm\sqrt{2-4}}{2}$, $D<0 \Rightarrow$ комплексне нуле

$t^2+\sqrt{2}t+1=0$, $\quad t_{1,2} = \frac{-\sqrt{2}\pm\sqrt{2-4}}{2}$, $D<0 \Rightarrow$ комплексне нуле $\Big]$

$$\frac{1}{1+t^4} = \frac{At+B}{t^2-\sqrt{2}t+1} + \frac{Ct+D}{t^2+\sqrt{2}t+1} = \frac{(At+B)(t^2+\sqrt{2}t+1)+(Ct+D)(t^2-\sqrt{2}t+1)}{(t^2-\sqrt{2}t+1)(t^2+\sqrt{2}t+1)}$$

Сређивањем и изједначавањем коефицијената,

$A = -\frac{1}{2\sqrt{2}}$, $B = \frac{1}{2}$, $C = +\frac{1}{2\sqrt{2}}$, $D = \frac{1}{2}$

$$\int\frac{dt}{t^4+1} = \int\frac{-\frac{1}{2\sqrt{2}}t+\frac{1}{2}}{t^2-\sqrt{2}t+1}\,dt + \int\frac{\frac{1}{2\sqrt{2}}t+\frac{1}{2}}{t^2+\sqrt{2}t+1}\,dt$$

$$= -\frac{1}{2\sqrt{2}}\int\frac{(t-\sqrt{2})\,dt}{t^2-\sqrt{2}t+1} + \frac{1}{2\sqrt{2}}\int\frac{t+\sqrt{2}}{t^2+\sqrt{2}t+1}$$

$$= -\frac{1}{4\sqrt{2}}\Big(\underbrace{\int\frac{2t-2\sqrt{2}}{t^2-\sqrt{2}t+1}\,dt}_{I_1}\Big) + \frac{1}{4\sqrt{2}}\underbrace{\int\frac{2t+2\sqrt{2}}{t^2+\sqrt{2}t+1}\,dt}_{I_2}$$

$$I_1=\int\frac{2t-\sqrt2}{t^2-\sqrt2 t+1}dt-\int\frac{\sqrt2}{t^2-\sqrt2 t+1}dt=\int\frac{du}{u}-\sqrt2\int\frac{dt}{(t-\frac{\sqrt2}{2})^2+\frac12}$$

$$=\ulcorner\ t^2-\sqrt2 t+1=u$$
$$(2t-\sqrt2)dt=du\ \lrcorner$$

$$=\ln|t^2-\sqrt2 t+1|+(-\sqrt2)\cdot\int\frac{dt}{(t-\frac{\sqrt2}{2})^2+\frac12}=$$

$$=\ln|t^2-\sqrt2 t+1|-\sqrt2\cdot\sqrt2\cdot\operatorname{arctg}\frac{t-\frac{\sqrt2}{2}}{\frac{\sqrt2}{2}}+C$$

$$\ulcorner\int\frac{dt}{(t-\frac{\sqrt2}{2})^2+\frac12}=\int\frac{dt}{\frac12\left(\frac{(t-\frac{\sqrt2}{2})^2}{\frac12}+1\right)}=2\cdot\int\frac{dt}{\left(\frac{t-\frac{\sqrt2}{2}}{\frac{\sqrt2}{2}}\right)^2+1}=\ulcorner u=\frac{t-\frac{\sqrt2}{2}}{\frac{\sqrt2}{2}}$$
$$du=dt\cdot\sqrt2\ \lrcorner$$

$$=2\cdot\int\frac{du}{\sqrt2\,(u^2+1)}=\sqrt2\operatorname{arctg}u+C=\sqrt2\operatorname{arctg}\frac{t-\frac{\sqrt2}{2}}{\frac{\sqrt2}{2}}+C$$

На аналотан начин се нађе I_2 (за вјежбу).

<u>Интеграција тригонометријских ф-ја</u> ($\int R(\sin x,\cos x)dx$, R – рационална ф-ја)

Неке од смјена: $t=tg\frac{x}{2}$, $t=tg x$

① $$\int\frac{1-\sin x+\cos x}{1+\sin x-\cos x}dx=\ulcorner\ t=tg\frac{x}{2}$$
$$\sin x=\frac{2t}{1+t^2}\quad dx=\frac{2dt}{1+t^2}$$
$$\cos x=\frac{1-t^2}{1+t^2}\ \lrcorner$$

$$=\int\frac{1-\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2}=$$

$$\int\frac{\frac{1+t^2-2t+1-t^2}{1+t^2}}{\frac{1+t^2+2t-1+t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2}=2\int\frac{(2-2t)}{(2t^2+2t)\cdot(1+t^2)}dt=2\cdot\int\frac{(1-t)}{t(t+1)(t^2+1)}$$

$$\frac{1-t}{t(t+1)(t^2+1)}=\frac{A}{t}+\frac{B}{t+1}+\frac{Ct+D}{t^2+1}=\frac{A(t+1)(t^2+1)+Bt(t^2+1)+(Ct+D)t(t+1)}{t(t+1)(t^2+1)}$$

Одавде,

$$\frac{1-t}{t(t+1)(t^2+1)}=\frac{(A+B+C)t^3+(A+C+D)t^2+(A+B+D)t+A}{t(t+1)(t^2+1)} \qquad /\,t(t+1)(t^2+1)$$

$$\left.\begin{array}{l} A+B+C=0 \\ A+C+D=0 \\ A+B+D=-1 \\ A=1 \end{array}\right\} \quad \begin{array}{l} A=1 \\ B=-1 \\ D=-1 \\ C=0 \end{array}$$

$$I=2\left(\int\frac{dt}{t}-\int\frac{dt}{t+1}-\int\frac{dt}{t^2+1}\right)=2(\ln|t|-\ln|t+1|-\operatorname{arctg} t)+C$$

$$=2\cdot\ln\left|\operatorname{tg}\frac{x}{2}\right|-2\ln\left|\operatorname{tg}\frac{x}{2}+1\right|-2\operatorname{arctg}\operatorname{tg}\frac{x}{2}+C=$$

$$\underline{2\cdot\ln\left|\frac{\operatorname{tg}\frac{x}{2}}{\operatorname{tg}\frac{x}{2}+1}\right|-x+C}$$

$\ulcorner\; t=\operatorname{tg}\frac{x}{2}$, $\sin x=\frac{\sin x}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}=\frac{2t}{1+t^2}$

$$\cos x=\frac{\cos x}{1}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

$\operatorname{arctg} t=\frac{x}{2}\Rightarrow x=2\operatorname{arctg} t$, па $dx=\frac{2dt}{1+t^2}\;\lrcorner$

② $$\int\frac{\sin^2x\,dx}{\cos^3x(\sin x+\cos x)}=\int\frac{\sin^2x\,dx}{\cos^3x\sin x+\cos^3x\cos x}=$$

$$\int\frac{\sin^2x\,dx}{\cos^4x\left(\frac{\cos^3x\sin x}{\cos^4x}+1\right)}=\int\frac{\sin^2x\,dx}{\cos^2x\cdot\cos^2x\cdot\left(\frac{\sin x}{\cos x}+1\right)}=\int\frac{\operatorname{tg}^2x}{(\operatorname{tg}x+1)}\cdot\frac{1}{\cos^2x}dx$$

$$=\begin{bmatrix}\operatorname{tg}x=t\\ \frac{dx}{\cos^2x}=dt\end{bmatrix}=\int\frac{t^2}{t+1}\cdot dt=\int\frac{(t^2-1+1)}{t+1}dt=\int(t-1)\,dt+\int\frac{dt}{(t+1)}$$

$$=\frac{t^2}{2}-t+\ln|t+1|+C=\frac{\operatorname{tg}^2x}{2}-\operatorname{tg}x+\ln|\operatorname{tg}x+1|+C$$

Интеграција ирационалних ф-ја

④ $\int R\left(x, x^{\frac{m_1}{n_1}}, x^{\frac{m_2}{n_2}}, \ldots x^{\frac{m_s}{n_s}}\right)dx$.

Смјена: $x=t^k$, $k=NZS(n_1, n_2, \ldots n_s)$

① $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \left[\begin{array}{l} x=t^6,\ 6=NZS(2,3) \\ dx = 6t^5dt \\ \sqrt{x}=x^{\frac{1}{2}}=t^3,\ \sqrt[3]{x}=x^{\frac{1}{3}}=t^2 \end{array}\right] = \int \frac{6t^5dt}{t^3+t^2} = 6\int \frac{\overset{t^3}{\cancel{t^5}}dt}{t^2(t+1)} =$

$6\int \frac{t^3}{t+1}dt = 6\cdot\int \frac{(t^3+1-1)}{t+1}dt = 6\cdot\int(t^2-t+1)dt - 6\int\frac{dt}{t+1} =$

$6\cdot\left(\frac{t^3}{3}-\frac{t^2}{2}+t\right)+6\ln|t+1|+C=$

$= 2\cdot\sqrt{x} - 3\cdot\sqrt[3]{x} + 6\cdot\sqrt[6]{x} + 6\ln(\sqrt[6]{x}+1)+C$

* $\int R\left(x, (ax+b)^{\frac{m_1}{n_1}}, \dots (ax+b)^{\frac{m_s}{n_s}}\right)dx$

Смјена:

$ax+b=t^k,\ k=NZS(n_1, n_2 \dots n_s)$

② $\int \frac{x^2+\sqrt{1+x}}{\sqrt[3]{1+x}}dx = \left[\begin{array}{ll} NZS(2,3)=6 & (1+x)^{\frac{1}{2}}=t^3 \\ 1+x=t^6 & (1+x)^{\frac{1}{3}}=t^2 \\ dx=6t^5dt & t=\sqrt[6]{1+x} \\ x=t^6-1 \Rightarrow x^2=(t^6-1)^2 & \end{array}\right]$

$= \int \frac{((t^6-1)^2+t^3)}{t^2}\cdot 6t^5dt = 6\cdot\int\left(t^3\cdot(t^{12}-2t^6+1)+t^6\right)dt$

$= 6\cdot\int(t^{15}-2t^9+t^3+t^6)dt = \frac{6t^{16}}{16} - 12\cdot\frac{t^{10}}{10} + 6\cdot\frac{t^4}{4} + 6\cdot\frac{t^7}{7} + C$

$= \frac{3}{8}\left(\sqrt[6]{1+x}\right)^{16} - \frac{6}{5}\cdot\left(\sqrt[6]{1+x}\right)^{10} + \frac{3}{2}\cdot\left(\sqrt[6]{1+x}\right)^{4} + \frac{6}{7}\left(\sqrt[6]{1+x}\right)^{7} + C$

* $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \dots \left(\frac{ax+b}{cx+d}\right)^{\frac{m_s}{n_s}}\right)dx$

Смјена:

$\frac{ax+b}{cx+d}=t^k,\ k=NZS(n_1, \dots n_s)$

③ $\int\sqrt{\frac{x+1}{x-1}}dx = \left[\begin{array}{ll} \frac{x+1}{x-1}=t^2 & dx=\frac{2t(t^2-1)-(t^2+1)\cdot 2t}{(t^2-1)^2}dt \\ x+1=(x-1)t^2 & dx=\frac{-4t}{(t^2-1)^2}dt \\ 1+t^2=x(t^2-1) & \\ x=\frac{t^2+1}{t^2-1} & t=\left(\frac{(x+1)}{(x-1)}\right)^{\frac{1}{2}}=\sqrt{\frac{x+1}{x-1}} \end{array}\right]$

Сада, $\int\sqrt{\frac{x+1}{x-1}}dx = \int t\cdot\left(-\frac{4t}{(t^2-1)^2}\right)dt = -4\int\frac{t^2}{(t^2-1)^2}dt$

$\lceil u = t \qquad \frac{t}{(t^2-1)^2}dt = dv$

$du = dt \qquad v = \int \frac{t\,dt}{(t^2-1)^2} = \lceil \begin{matrix} t^2-1=k \\ 2t\,dt = dk \end{matrix} \rfloor = \int \frac{dk}{2k^2} = -\frac{1}{2k} =$

$= -\frac{1}{2(t^2-1)} \rfloor$

$$= -\frac{t}{2(t^2-1)} + \frac{1}{2}\int \frac{dt}{t^2-1} = \frac{-t}{2(t^2-1)} + \frac{1}{4}\int\frac{dt}{t-1} - \frac{1}{4}\int \frac{dt}{t+1} =$$

$$= -\frac{t}{2(t^2-1)} + \frac{1}{4}\ln|t-1| - \frac{1}{4}\ln|t+1| + C$$

$\lceil \int \frac{dt}{t^2-1} = ?$

$$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{At+A+Bt-B}{(t^2-1)} = \frac{(A+B)t+A-B}{t^2-1} \quad /(t^2-1)$$

$$\begin{matrix} A+B=0 \\ A-B=1 \end{matrix} \Rightarrow A = \frac{1}{2};\ B = -\frac{1}{2}$$

Враћамо смјене,

$$I = -4\left(-\frac{\sqrt{\frac{x+1}{x-1}}}{2\left(\frac{x+1}{x-1}-1\right)} + \frac{1}{4}\ln\left|\sqrt{\frac{x+1}{x-1}}-1\right| - \frac{1}{4}\ln\left|\sqrt{\frac{x+1}{x-1}}+1\right|\right) + C$$

$$= \frac{2\cdot\sqrt{\frac{x+1}{x-1}}}{\frac{x+1-x+1}{x-1}} + \ln\left|\sqrt{\frac{x+1}{x-1}}+1\right| - \ln\left|\sqrt{\frac{x+1}{x-1}}-1\right| + C$$

$$= (x-1)\sqrt{\frac{x+1}{x-1}} + \ln\left|\frac{\left(\sqrt{\frac{x+1}{x-1}}+1\right)}{\left(\sqrt{\frac{x+1}{x-1}}-1\right)}\right| + C$$

(*) Интеграли облика: $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx$, P_n – полином степена n.

$$\int \frac{P_n(x)\,dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + k\cdot\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

(4) $\int \frac{x^2}{\sqrt{x^2+x+1}}\,dx$

$$\int \frac{x^2}{\sqrt{x^2+x+1}}\,dx = (Ax+B)\sqrt{x^2+x+1} + k\int\frac{dx}{\sqrt{x^2+x+1}} \quad /'$$

$$\frac{x^2}{\sqrt{x^2+x+1}} = A\sqrt{x^2+x+1} + (Ax+B)\cdot\frac{1}{2\sqrt{x^2+x+1}}\cdot(2x+1) + C\cdot\frac{1}{\sqrt{x^2+x+1}} \quad /\sqrt{x^2+x+1}$$

$$x^2 = A(x^2+x+1) + \frac{(Ax+B)(2x+1)}{2} + C$$

$$x^2 = 2Ax^2 + \left(\frac{3A}{2}+B\right)x + A + \frac{B}{2} + C$$

$$\begin{aligned} 2A &= 1 \\ \frac{3A}{2}+B &= 0 \\ A+\frac{B}{2}+C &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= \frac{1}{2} \\ B &= -\frac{3}{4} \\ C &= -\frac{1}{8} \end{aligned}$$

$$\int \frac{x^2}{\sqrt{x^2+x+1}}dx = \left(\frac{1}{2}x - \frac{3}{4}\right)\sqrt{x^2+x+1} - \frac{1}{8}\cdot\int\frac{dx}{\sqrt{x^2+x+1}} =$$

$$= \left(\frac{1}{2}x - \frac{3}{4}\right)\sqrt{x^2+x+1} - \frac{1}{8}\cdot\int\frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$$

$$\ulcorner\int\frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}} = \begin{bmatrix} x+\frac{1}{2} = t \\ dx = dt \end{bmatrix} = \int\frac{dt}{\sqrt{t^2+\frac{3}{4}}} = \ln\left|t+\sqrt{t^2+\frac{3}{4}}\right| =$$

$$= \ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| \lrcorner$$

Dakle, $\int\frac{x^2}{\sqrt{x^2+x+1}}dx = \left(\frac{1}{2}x-\frac{3}{4}\right)\sqrt{x^2+x+1} - \frac{1}{8}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + C$

(*) $\int\frac{Mx+N}{(x-\alpha)^k\sqrt{ax^2+bx+c}}dx$ Smjena: $x-\alpha = \frac{1}{t}$

(5) $$\int\frac{x}{(x-1)^2\sqrt{-x^2+2x+1}}dx = \begin{bmatrix} x-1 = \frac{1}{t} \Rightarrow t = \frac{1}{x-1} \\ dx = -\frac{1}{t^2}dt \\ x = \frac{1}{t}+1 = \frac{t+1}{t} \end{bmatrix} =$$

$$\int\frac{\frac{t+1}{t}\cdot dt}{\frac{1}{t^2}\cdot\sqrt{-(1+\frac{1}{t})^2+2\cdot(1+\frac{1}{t})+1}} = \int\frac{\frac{t+1}{t}\cdot dt}{\frac{1}{t^2}\sqrt{-\frac{(t+1)^2}{t^2}+2\cdot\frac{t+1}{t}+1}} = \int\frac{\frac{t+1}{t}\cdot dt}{\frac{\sqrt{-t^2-2t-1+2t^2+2t+t^2}}{\sqrt{t^2}}\cdot\frac{1}{t^2}}$$

$$= \int\frac{(t+1)\cancel{t}\cdot t^2}{\cancel{t}\sqrt{2t^2-1}}\cdot dt = \int\frac{t^3+t^2}{\sqrt{2t^2-1}}dt$$

$$\int \frac{t^3+t^2}{\sqrt{2t^2-1}}dt = (A\cancel{t^2}+B\cancel{t}+C)\sqrt{2t^2-1} + k\cdot\int \frac{dt}{\sqrt{2t^2-1}} \quad /'$$

$$\frac{t^3+t^2}{\sqrt{2t^2-1}} = (2At+B)\sqrt{2t^2-1} + (At^2+Bt+C)\cdot\frac{\overset{2}{\cancel{4}}t}{2\sqrt{2t^2-1}} + k\cdot\frac{1}{\sqrt{2t^2-1}} \quad /\sqrt{2t^2-1}$$

$$t^3+t^2 = (2At+B)(2t^2-1) + (At^2+Bt+C)\cdot 2t + k$$

$$t^3+t^2 = 4At^3-2At+2Bt^2-B+2At^3+2Bt^2+2Ct+k$$

$$t^3+t^2 = 6At^3+4Bt^2+(2C-2A)t+k-B$$

$$\left.\begin{array}{l} 6A=1 \\ 4B=1 \\ 2C-2A=0 \\ k-B=0 \end{array}\right\} \Rightarrow \begin{array}{l} A=\frac{1}{6} \\ B=\frac{1}{4} \\ C=A=\frac{1}{6} \\ k=\frac{1}{4} \end{array}$$

Сада

$$\int \frac{t^3+t^2}{\sqrt{2t^2-1}}dt = \left(\frac{1}{6}t^2+\frac{1}{4}t+\frac{1}{6}\right)\sqrt{2t^2-1} + \frac{1}{4}\int\frac{dt}{\sqrt{2t^2-1}}$$

$$\int\frac{dt}{\sqrt{2t^2-1}} = \int\frac{1}{\sqrt{2}}\cdot\frac{dt}{\sqrt{t^2-\frac{1}{2}}} = \frac{1}{\sqrt{2}}\ln\left|t+\sqrt{t^2-\frac{1}{2}}\right|,$$ враћамо у смјену

$$I = \left(\frac{1}{6}\cdot\frac{1}{(x-1)^2}+\frac{1}{4}\cdot\frac{1}{x-1}+\frac{1}{6}\right)\sqrt{2\left(\frac{1}{x-1}\right)^2-1} + \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\cdot\ln\left|\frac{1}{x-1}+\sqrt{\left(\frac{1}{x-1}\right)^2-\frac{1}{2}}\right|+C$$

$$t=\frac{1}{x-1},$$

⊛ Интеграли облика $\int R(x, \sqrt{ax^2+bx+c})\,dx$, R–рационална ф-ја.

1° За $a>0$, смјена: $\sqrt{ax^2+bx+c} = t\pm\sqrt{a}\,x$

2° За $c>0$, смјена: $\sqrt{ax^2+bx+c} = xt\pm\sqrt{c}$

3° Ако су x_1, x_2 рјешења квадратне j-не $ax^2+bx+c=0$, онда: $\sqrt{ax^2+bx+c} = a(x-x_1)t$ или $\sqrt{ax^2+bx+c} = a(x-x_2)t$

6) $$\int\frac{dx}{x+\sqrt{x^2+x+1}} = ?$$

Провјерамо услове: $a=1>0 \Rightarrow$ први случај

Смјена: $\sqrt{x^2+x+1} = t \pm \sqrt{1}\cdot x$. Бирамо: $\sqrt{x^2+x+1} = t - x$.

$$\sqrt{x^2+x+1} = t - x \quad /^2$$

$$x^2+x+1 = (t-x)^2$$

$$\cancel{x^2}+x+1 = t^2-2tx+\cancel{x^2}$$

$$x(1+2t) = t^2-1$$

$$x = \frac{t^2-1}{2t+1}$$

$$dx = \frac{2t(2t+1)-(t^2-1)\cdot 2}{(2t+1)^2} = \frac{2(t^2+t+1)}{(2t+1)^2}dt.$$

$$I = \int \frac{\frac{2(t^2+t+1)}{(2t+1)^2}}{\cancel{x}+t-\cancel{x}}\cdot dt = 2\int \frac{t^2+t+1}{(2t+1)^2\cdot t}dt$$

Интеграл рационалне ф-је:

$$\frac{t^2+t+1}{(2t+1)^2 t} = \frac{A}{(2t+1)} + \frac{B}{(2t+1)^2} + \frac{C}{t} = \dots$$

Добијамо: $A=1,\ B=-\frac{3}{2},\ C=-\frac{3}{2}$

$$I = \int \frac{1}{t}dt - \frac{3}{2}\int\frac{dt}{2t+1} - \frac{3}{2}\int\frac{dt}{(2t+1)^2} =$$

$\ulcorner 2t+1 = k$ $\qquad \int \frac{dk}{2\cdot k} \qquad \int \frac{dk}{2\cdot k^2}$

$2dt = dk \lrcorner$

$$= \ln|t| - \frac{3}{2}\cdot\frac{1}{2}\ln|k| - \frac{3}{2}\cdot\left(\frac{-1}{2k}\right) + C =$$

$$= \ln|x+\sqrt{x^2+x+1}| - \frac{3}{4}\ln|2(x+\sqrt{x^2+x+1})+1| + \frac{3}{4}\cdot\frac{1}{(2(x+\sqrt{x^2+x+1})+1)} + C$$

$\ulcorner t = x+\sqrt{x^2+x+1} \lrcorner$

7) $$\int \frac{dx}{1+\sqrt{1-2x-x^2}} = ?$$

$a=-1<0$

$c=1>0 \Rightarrow$ смјена бр. 2

$$\sqrt{1-2x-x^2} = xt-1 \quad /^2$$

$$\cancel{1}-2x-x^2 = x^2t^2-2xt+\cancel{1}$$

$$\cancel{1}-2x-x^2=x^2t^2-2tx+\cancel{1} \quad /:x$$

$$-2-x=xt^2-2t$$

$$-2-x-xt^2+2t=0$$

$$2t-2=x(1+t^2)$$

$$x=\frac{2t-2}{t^2+1} \Rightarrow dx=\frac{2(t^2+1)-(2t-2)\cdot 2t}{(t^2+1)^2}=\frac{-2t^2+4t+2}{(t^2+1)^2}$$

$$I=2\cdot\int\frac{-t^2+2t+1}{(1+t^2)^2}\cdot\frac{dt}{\left(1+\frac{2t-2}{1+t^2}\cdot t-1\right)}=$$

$$=2\cdot\int\frac{-t^2+2t+1}{(1+t^2)^2}\cdot\frac{dt\cdot(1+t^2)}{(2t-2)t}=\int\frac{(-t^2+2t+1)\,dt}{t(t-1)(t^2+1)}$$

$$\frac{-t^2+2t+1}{t(t-1)(t^2+1)}=\frac{A}{t}+\frac{B}{t-1}+\frac{Ct+D}{t^2+1}=\dots$$

$A=-1$
$B=1$
$C=0$
$D=-2$

Odakle, $I=-\int\frac{dt}{t}+\int\frac{dt}{t-1}-2\cdot\int\frac{dt}{t^2+1}=$

$$=-\ln|t|+\ln|t-1|-2\operatorname{arctg}t+C=$$

$$=\ln\frac{|t-1|}{|t|}-2\operatorname{arctg}t+C$$

$$\ulcorner xt=1+\sqrt{1-2x-x^2}$$

$$t=\frac{1+\sqrt{1-2x-x^2}}{x}\to$$

$$=\ln\frac{\left|\frac{1+\sqrt{1-2x-x^2}}{x}-1\right|}{\left|\frac{1+\sqrt{1-2x-x^2}}{x}\right|}-2\operatorname{arctg}\left|\frac{1+\sqrt{1-2x-x^2}}{x}\right|+C$$

8 $\int\frac{x\,dx}{\sqrt{7x-10-x^2}^{\,3}}=?$

$a=-1<0$
$c=-10<0$

$7x-10-x^2=0$
$-(x^2-7x+10)=0$
$-(x-2)(x-5)=0$

Смјена:

$\sqrt{7x-10-x^2}=+(x-2)t \quad /^2$

$7x-10-x^2=(x-2)^2t^2$

$-(\cancel{x-2})(x-5)=(x-2)^{\cancel{2}}t^2$

$5-x=(x-2)t^2$

$5-x=xt^2-2t^2$

$x=\boxed{\frac{2t^2+5}{t^2+1}} \Rightarrow dx=\frac{4t(t^2+1)-(2t^2+5)\cdot 2t}{(t^2+1)^2}=\frac{-6t\cdot dt}{(t^2+1)^2}$

$(x-2)t=+\sqrt{7x-10-x^2}$

$(x-2)t=\left(\frac{2t^2+5}{t^2+1}-2\right)\cdot t=\boxed{\frac{3t}{t^2+1}}$

$$I=\int\frac{\frac{2t^2+5}{t^2+1}}{\left(\frac{3t}{t^2+1}\right)^3}\cdot\left(\frac{-6t}{(1+t^2)^2}\right)dt=-\frac{6}{27}\int\frac{2t^2+5}{t^2}\,dt=$$

$$=-\frac{6}{27}\left(\int 2dt+\int\frac{5}{t^2}dt\right)=-\frac{6}{27}\cdot 2t+\left(-\frac{6}{27}\right)\cdot 5\cdot\left(-\frac{1}{t}\right)+C$$

$$=-\frac{12}{27}\cdot\frac{\sqrt{7x-10-x^2}}{x-2}+\frac{30}{27}\cdot\frac{x-2}{\sqrt{7x-10-x^2}}+C$$

(*) Интеграл диференцијалног бинома

$\int x^m(a+bx^n)^p dx$, $m,n,p\in\mathbb{Q}$ своди се на интеграл рационалне ф-је у три случаја:

1) $p\in\mathbb{Z}$, $x=t^N$, $N=$ НЗС (имениоци разломака m и n)

2) $\frac{m+1}{n}\in\mathbb{Z}$, $a+bx^n=t^N$, N - именилац разломка p

3) $\frac{m+1}{n}+p\in\mathbb{Z}$, $a\cdot x^{-n}+b=t^N$, N - именилац разломка p

⑨ $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx = \int x^{\frac{1}{2}}(1+x^{\frac{1}{3}})^{-2} dx = \ulcorner m=\frac{1}{2}, n=\frac{1}{3}, p=-2 \lrcorner$

$= \ulcorner x = t^{NZS(2,3)} = t^6, \; dx = 6t^5 dt \lrcorner = 6\cdot\int \frac{t^3\cdot t^5}{(1+t^2)^2}\cdot dt = 6\cdot\int \frac{t^8}{(1+t^2)^2}dt$

$t^8 : (t^4+2t^2+1) = t^4-2t^2+3$

$\underline{-t^8 \pm 2t^6 \pm t^4}$

$-2t^6-t^4$

$\underline{-2t^6-4t^4-2t^2}$

$\quad + \quad + \quad +$

$3t^4+2t^2$

$\underline{-3t^4 - 6t^2 - 3}$

$-4t^2-3$

$I = 6\cdot\int\left((t^4-2t^2+3)+\frac{-4t^2-3}{(t^2+1)^2}\right)dt =$

$6\left(\frac{t^5}{5}-2\cdot\frac{t^3}{3}+3t\right) + (-6)\cdot\int\underbrace{\frac{+4t^2+3}{(t^2+1)^2}dt}_{I_1}$

$I_1 = ?$

$\frac{4t^2+3}{(t^2+1)^2} = \frac{At+B}{t^2+1}+\frac{Ct+D}{(t^2+1)^2} = \dots$

$A=0,\ B=4,\ C=0,\ D=3$

$I_1 = \int\frac{4dt}{t^2+1}+\int\frac{3dt}{(t^2+1)^2} = 4\,\mathrm{arctg}\,t + 3\cdot\left(\frac{1}{2}\mathrm{arctg}\,t+\frac{t}{2(t^2+1)}\right)$

Sada, $I = 6\cdot\left(\frac{\sqrt[6]{x}^5}{5}-2\cdot\frac{(\sqrt[6]{x})^3}{3}+3\cdot\sqrt[6]{x}\right)-24\,\mathrm{arctg}\,t - 18\left(\frac{1}{2}\mathrm{arctg}\sqrt[6]{x}+\frac{\sqrt[6]{x}}{2(\sqrt[3]{x}+1)}\right) + C$

⑩ $\int\frac{x\,dx}{\sqrt{1+\sqrt[3]{x^2}}} = \int x\cdot(1+x^{\frac{2}{3}})^{-\frac{1}{2}}dx = \ulcorner m=1,\ n=\frac{2}{3},\ p=-\frac{1}{2}\lrcorner$

$p\notin \mathbb{Z}$

$\frac{1+1}{\frac{2}{3}} = 3\in\mathbb{Z} \Rightarrow$ Smjena: $1+x^{\frac{2}{3}} = t^2 \Rightarrow \frac{2}{3}\cdot x^{-\frac{1}{3}}dx = 2t\,dt$

Сада, $\frac{2}{3}x^{-\frac{1}{3}}\cdot dx = 2t\,dt$

$\ulcorner \frac{x\cdot 3t\cdot\sqrt[3]{x}\cdot dt}{t} = 3\cdot x\cdot\sqrt[3]{x}\cdot dt = 3\cdot x^{\frac{4}{3}}\,dt \lrcorner$, $3t\cdot\sqrt[3]{x}\cdot dt = dx$

$$I = \int 3\cdot(t^2-1)^2\,dt =$$

$x^{\frac{2}{3}} = t^2-1 \Rightarrow x^{\frac{4}{3}} = (t^2-1)^2 \lrcorner$

$$= 3\cdot\int(t^4-2t^2+1)\,dt$$

$$= 3\cdot\left(\frac{t^5}{5} - 2\cdot\frac{t^3}{3} + t\right) + C = 3\cdot\left(\frac{\left(\sqrt{1+\sqrt[3]{x^2}}\right)^5}{5} - \frac{2}{3}\cdot\left(\sqrt{1+\sqrt[3]{x^2}}\right)^3 + \sqrt{1+\sqrt[3]{x^2}}\right) + C$$

$\ulcorner t = \sqrt{1+\sqrt[3]{x^2}} \lrcorner$

(11) $\int \sqrt[3]{3x-x^3}\,dx = \int (3x-x^3)^{\frac{1}{3}}\cdot dx = \int x^{\frac{1}{3}}\cdot(3-x^2)^{\frac{1}{3}}\cdot dx = ?$

$m = \frac{1}{3},\ n = 2,\ p = \frac{1}{3}$

$p = \frac{1}{3} \notin \mathbb{Z}$

$\frac{m+1}{n} = \frac{\frac{1}{3}+1}{2} = \frac{2}{3} \notin \mathbb{Z}$

$\frac{m+1}{n} + p = \frac{2}{3} + \frac{1}{3} = 1 \in \mathbb{Z}$,

Смена: $\ulcorner 3\cdot x^{-2} - 1 = t^3$

$-6x^{-3}dx = 3t^2dt$

$-\frac{6}{x^3}dx = 3t^2dt$

$dx = \frac{3t^2\cdot x^3dt}{-6} = \frac{t^2x^3dt}{-2}$

$\frac{3}{x^2} - 1 = t^3 \Rightarrow x^2 = \frac{3}{t^3+1} \lrcorner$

$$\ulcorner \frac{t^2\cdot x^3dt}{-2}\cdot x^{\frac{1}{3}}\cdot(3-x^2)^{\frac{1}{3}} = \frac{t^2\cdot x^{\frac{10}{3}}\cdot(3-x^2)^{\frac{1}{3}}}{-2} = \frac{t^2\cdot\left(\frac{3}{t^3+1}\right)^{\frac{5}{3}}\cdot\left(3-\frac{3}{t^3+1}\right)^{\frac{1}{3}}}{-2} \lrcorner$$

Даље, $I = -\frac{1}{2}\cdot\int t^2\cdot\left(\frac{3}{t^3+1}\right)^{\frac{5}{3}}\cdot\left(\frac{3t^3}{t^3+1}\right)^{\frac{1}{3}}dt =$

$= -\frac{1}{2}\cdot 9\int\frac{t^3}{(t^3+1)^2}dt = \ulcorner u = t,\ du = dt$; $\frac{t^2}{(t^3+1)^2}dt = dv$

$v = \int\frac{t^2}{(t^3+1)^2}dt = \ulcorner t^3+1 = u,\ 3t^2dt = du \lrcorner =$

$\int\frac{du}{3\cdot u^2} = \frac{1}{3}\cdot\left(-\frac{1}{u}\right) = -\frac{1}{3}\cdot\frac{1}{t^3+1} \lrcorner$

$$= t\left(-\frac{1}{3}\cdot\frac{1}{t^3+1}\right) + \frac{1}{3}\underbrace{\int\frac{dt}{t^3+1}}_{=I_1}$$

$\frac{1}{t^3+1} = \frac{1}{(t+1)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1} = \ldots =$ (завршити)