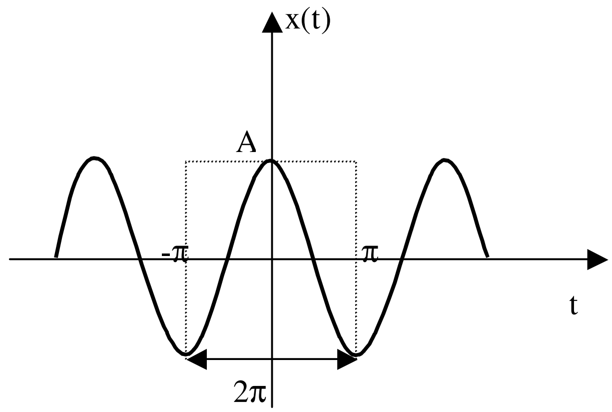
FOURIEROVI REDOVI

Harmonijske oscilacije

Definicija: Funkcija $x(t)$ je periodična sa periodom T (T>0) ako za svako t vrijedi: $x(t+T)=x(t)$. Najmanja vrijednost T za koju funkcija $x(t)$ ima navedenu osobinu naziva se osnovnim periodom funkcije $x(t)$.
Primjer: Osnovne trigonometrijske funkcije $\sin(t)$ i $\cos(t)$ su periodične funkcije sa osnovnim periodom 2π.
Primjer: $x(t) = A \cdot \cos(t)$, $x(t+2\pi) = x(t)$

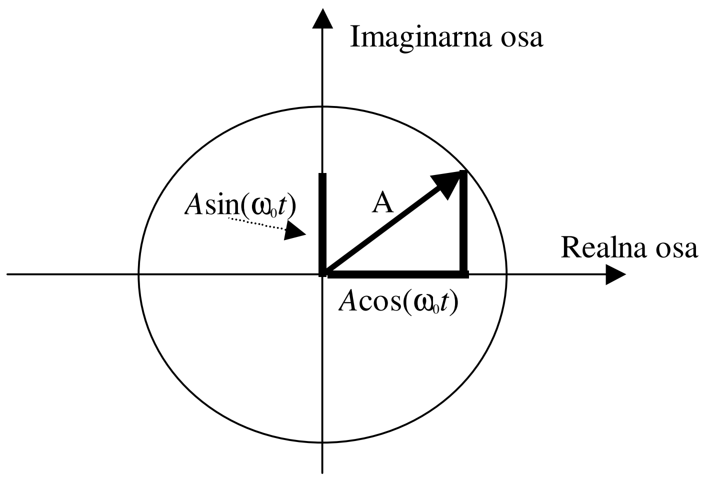


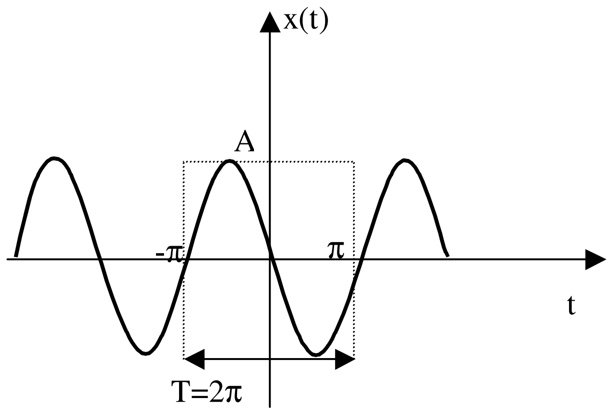
Ako je funkcija periodična sa periodom T onda je ona periodična i sa periodom kT, gdje je k prirodan broj.

Primjer: Ispitati periodičnost i odrediti osnovni period funkcije $x(t) = A \cdot \cos(\omega_0 t)$.
Napišimo uslov periodičnosti za posmatranu funkciju

$$x(t+T) = A \cdot \cos(\omega_0 (t+T)) = A \cdot \cos(\omega_0 t + \omega_0 T) = x(t)$$

Poznavajući osobine funkcije $\cos(t)$ zaključujemo da je navedeni izraz tačan za svako t ukoliko vrijedi: $\omega_0 t = 2\pi k \quad \Rightarrow \quad T = 2\pi k / \omega_0$ gdje je k proizvoljan prirodan broj. Osnovni period posmatrane funkcije je najmanji od svih njenih perioda: $T = 2\pi / \omega_0$.





Posmatrajmo funkciju $e^{j\omega_0 t} = \cos(\omega_0 t) + j \cdot \sin(\omega_0 t)$, $j = \sqrt{-1}$ (Ojlerova formula)

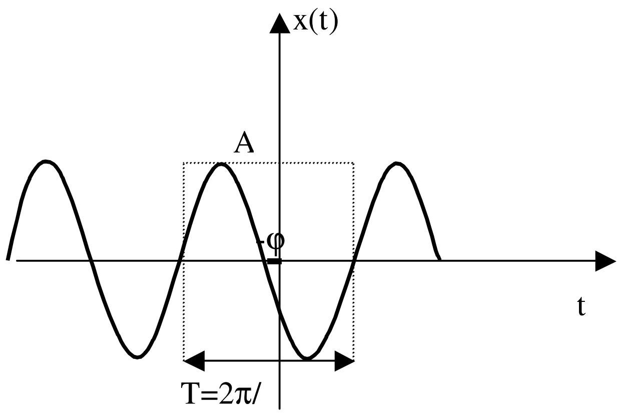
Tvrđenje: Funkcija e^{jt} je periodična sa periodom 2π.

$$e^{j(t+2\pi)} = \cos(t+2\pi) + j \cdot \sin(t+2\pi) = \cos(t) + j \cdot \sin(t) = e^{jt}$$

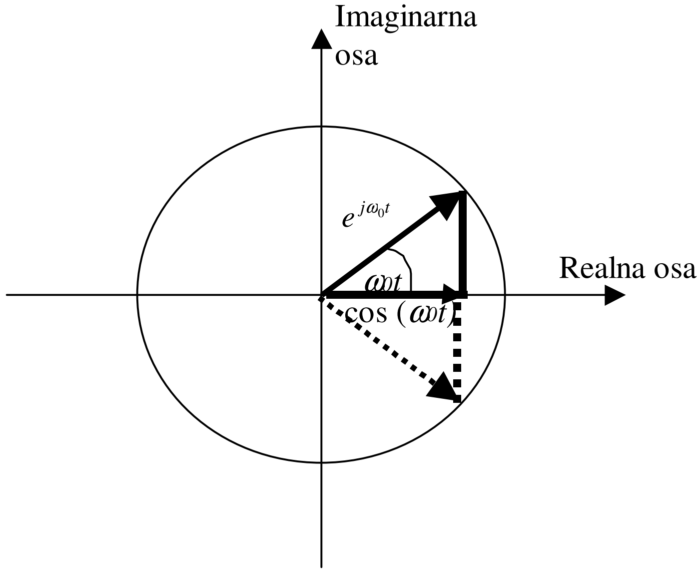
$$e^{j \cdot 2\pi} = 1$$
$$e^{j\pi} = e^{-j\pi} = -1$$
$$e^{j \cdot n\pi} = \left(e^{j\pi}\right)^n = (-1)^n$$

Tvrđenje: Funkcija $e^{j(\omega_0 t+\varphi)}$ je periodična sa periodom $T = 2\pi/\omega_0$.

$$e^{j\omega_0(t+T)+\varphi} = \cos(\omega_0 t + \varphi + \omega_0 T) + j \cdot \sin(\omega_0 t + \varphi + \omega_0 T) = e^{j(\omega_0 t+\varphi)}$$

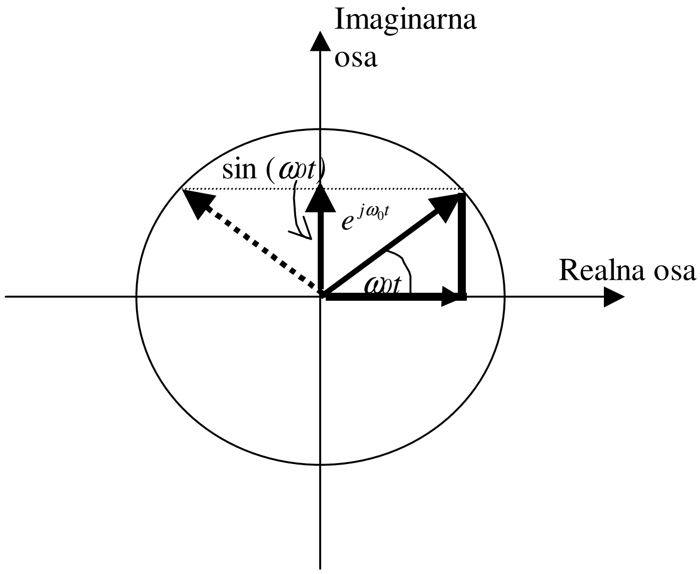


Dobijanje $\cos(\omega_0 t)$ i $\sin(\omega_0 t)$ na osnovu $e^{j\omega_0 t}$:



$$e^{j\omega_0 t} + e^{-j\omega_0 t} = 2\cos(\omega_0 t) \Rightarrow$$

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$



$$e^{j\omega_0 t} - e^{-j\omega_0 t} = 2j\sin(\omega_0 t) \Rightarrow$$

$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

Zbir harmonijskih oscilacija

$$x(t) = A\cdot\cos(\omega_0 t) + B\cdot\sin(\omega_0 t)$$

$$T_1 = 2\pi/\omega_0 \qquad T_2 = 2\pi/\omega_0 \qquad \Rightarrow T = 2\pi/\omega_0$$

jer je $x\left(t + \frac{2\pi}{\omega_0}\right) = x(t)$

$$x(t) = A\cdot\cos(\omega_0 t) + B\cdot\sin(2\omega_0 t)$$

$x(t+T) = A\cdot\cos(\omega_0 (t+T)) + B\cdot\sin(2\omega_0 (t+T)) = A\cdot\cos(\omega_0 t) + B\cdot\sin(2\omega_0 t)$ za

$\omega_0 T_1 = 2n\pi$ i $2\omega_0 T_2 = 2k\pi$

$T = 2n\pi/\omega_0$ i $T = 2k\pi/2\omega_0$

$T = \{2\pi/\omega_0, 4\pi/\omega_0, 6\pi/\omega_0, ...\}$ $T = \{\pi/\omega_0, 2\pi/\omega_0, 3\pi/\omega_0, ...\}$

Najmanji period T za koji su oba elementa periodična je $T = 2\pi/\omega_0$.

Primjer:

$x(t) = Ae^{j(\omega_0 t+\alpha)} + Be^{j\cdot 2\omega_0 t} + Ce^{j(3\omega_0 t+\beta)}$

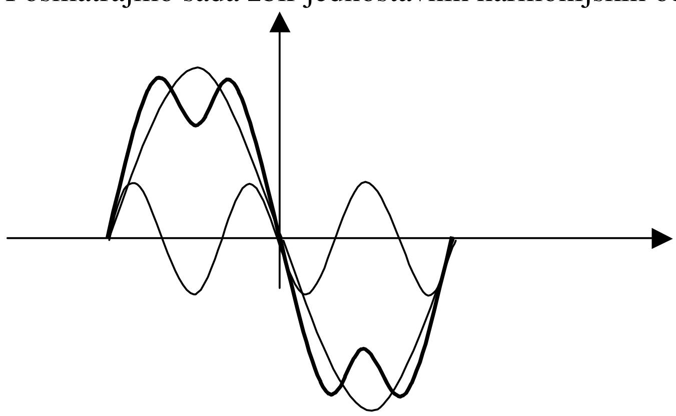
$T_1 = 2\pi n/\omega_0$, $T_2 = 2\pi n/2\omega_0$, $T_3 = 2\pi n/3\omega_0$

$T_1 = \{2\pi/\omega_0, 4\pi/\omega_0, 6\pi/\omega_0, ...\}$, $T_2 = \{\pi/\omega_0, 2\pi/\omega_0, 3\pi/\omega_0, ...\}$,

$T_3 = \{2\pi/3\omega_0, 4\pi/3\omega_0, 6\pi/3\omega_0, ...\}$

$T = 2\pi/\omega_0$

Posmatrajmo sada zbir jednostavnih harmonijskih oblika:



$x(t) = \sin(t) + \frac{1}{3}\cdot\sin(3t)$

Ako funkcija f(t) ima samo konačan broj maksimuma i minimuma u intervalu [-π, π] i ako je kontinualna u tom intervalu, osim na konačnom broju tačaka sa prekidima prvog reda, onda se ona može napisati u obliku Fourierovog reda.

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + a_3\cos(3t) + b_3\sin(3t) + ...$$

$$f(t) = \frac{a_0}{2} + \sum_{n=-\infty}^{\infty}(a_n\cos(nt) + b_n\sin(nt))$$

ili , u kompleksnom obliku:

$$f(n) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega t}$$

Veza između koeficijenata a_n, b_n i F_n je sledeća:

$F_0 = a_0/2$

$$\left(\frac{a_n}{2}+\frac{b_n}{2j}\right)e^{jnt}+\left(\frac{a_n}{2}-\frac{b_n}{2j}\right)e^{-jnt}=F_n e^{jnt}+F_n e^{-jnt}$$

$$F_n=\frac{1}{2}\left(a_n-jb_n\right)$$

$$F_{-n}=\frac{1}{2}\left(a_n+jb_n\right)$$

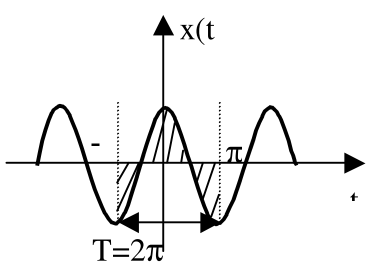
$$\left|F_n\right|=\frac{1}{2}\sqrt{{a_n}^2+{b_n}^2}$$ amplitudski spektar

$$\left|F_n\right|=-arctg\,\frac{b_n}{a_n}$$ fazni spektar

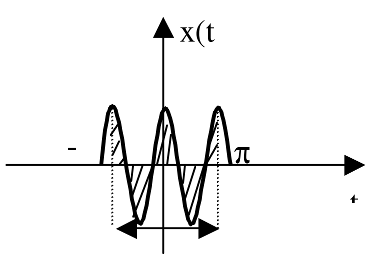
$$a_n=F_n+F_{-n}$$

$$b_n=j\left(F_n-F_{-n}\right)$$

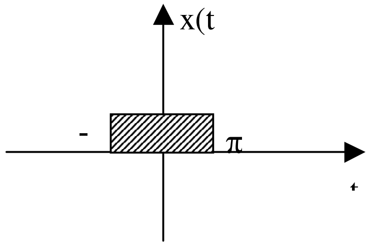
$$\int_{-\pi}^{\pi}\cos\left(t\right)dt=0$$



$$\int_{-\pi}^{\pi}\sin\left(t\right)dt=0$$



$\int_{-\pi}^{\pi}\cos\left(2t\right)dt=0$ $\int_{-\pi}^{\pi}\cos\left(nt\right)dt=0$ Osim ako je n=0, kada je $\int_{-\pi}^{\pi}1\cdot dt=2\pi$.



$$\int_{-\pi}^{\pi} e^{jnt}dt = \begin{cases} 0, & n \neq 0, \\ 2\pi, & n = 0 \end{cases}$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jnt}$$

$$\int_{-\pi}^{\pi} f(t)dt = \sum_{n=-\infty}^{\infty} F_n \int_{-\pi}^{\pi} e^{jnt}dt = 2\pi F_0$$

$$\int_{-\pi}^{\pi} f(t)e^{-2jnt}dt = 2\pi F_2$$

....

$$F_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-jnt}dt$$

Koeficijenti Fourierovog reda

Primjer:

$$x(n) = 1 + \frac{1}{2}\cos(t) = 1 + \frac{1}{2}\cdot\frac{e^{jt} + e^{-jt}}{2} = 1 + \frac{1}{4}e^{jt} + \frac{1}{4}e^{-jt}$$

$$\Rightarrow \quad F_0 = 1, \quad F_1 = \frac{1}{4}, \quad F_{-1} = \frac{1}{4}$$

$$x(n) = \frac{1}{3} + \sin(t) + \cos(3t) = \frac{1}{3} + \frac{1}{2j}e^{jt} - \frac{1}{2j}e^{-jt} + \frac{1}{2}e^{j3t} + \frac{1}{2}e^{-j3t}$$

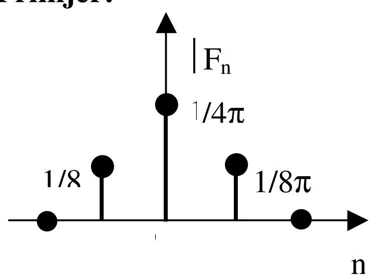
$$\Rightarrow \quad F_0 = \frac{1}{3}, \quad F_1 = \frac{1}{2j}, \quad F_{-1} = -\frac{1}{2j}, \quad F_3 = \frac{1}{2}, \quad F_{-3} = \frac{1}{2},$$

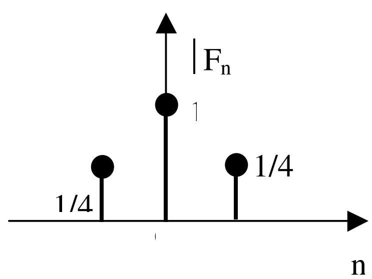
Ako je funkcija f(t) periodična sa periodom T, tj. f(t+T)=f(t) i T=2π/ω_0, tj. ω_0=2π/T, onda je:

$$f(t) = \sum_{-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

Primjer:





$$f(t) = \cos^2(t/2) \Rightarrow$$

$$f(t) = \frac{1}{2}(1 + \cos t)$$

$$F_n = \frac{1}{2\pi}\int_{-o}^{\pi} f(t)e^{-jnt}dt = \frac{1}{2\pi}\int_{-o}^{\pi}\left(\frac{1}{2} + \frac{1}{4}e^{jt} + \frac{1}{4}e^{-jt}\right)e^{-jnt}dt$$

$$F_0 = \frac{1}{4\pi}, \quad F_1 = \frac{1}{8\pi} \quad F_{-1} = \frac{1}{8\pi}$$