S.N. Kruzhkov's lectures on first-order quasilinear PDEs

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Abstract. The present contribution originates from short notes intended to accompany the lectures of Professor Stanislav Nikolaĭevich Kruzhkov given for the students of the Moscow State Lomonosov University during the years 1994–1997. Since then, they were enriched by many exercises which should allow the reader to assimilate more easily the contents of the lectures and to appropriate the fundamental techniques. This text is prepared for graduate students studying PDEs, but the exposition is elementary, and no previous knowledge of PDEs is required. Yet a command of basic analysis and ODE tools is needed. The text can also be used as an exercise book.

The lectures provide an exposition of the nonlocal theory of quasilinear partial differential equations of first order, also called conservation laws. According to S. N. Kruzhkov's "ideology", much attention is paid to the motivation (from both the mathematical viewpoint and the context of applications) of each step in the development of the theory. Also the historical development of the subject is reflected in these notes.

We consider questions of local existence of smooth solutions to Cauchy problems for linear and quasilinear equations. We expose a detailed theory of discontinuous weak solutions to quasilinear equations with one spatial variable. We derive the Rankine–Hugoniot condition, motivate in various ways admissibility conditions for generalized (weak) solutions and relate the admissibility issue to the notions of entropy and of energy. We pay special attention to the resolution of the so-called Riemann problem. The lectures contain many original problems and exercises; many aspects of the theory are explained by means of examples. The text is completed by an afterword showing that the theory of conservation laws is yet full of challenging questions and awaiting for new ideas.[0]

Keywords. PDE, first-order quasilinear PDE, characteristics, generalized solution, shock wave, rarefaction wave, admissibility condition, entropy, Riemann problem.

AMS classification. 35F20, 35F25, 35L65.

[0] *Note added by the translator (NT)* — The authors, the translator and the editors made an effort to produce a readable English text while preserving the flavour of S. N. Kruzhkov's expression and his original way of teaching. The reading of the lectures will surely require some effort (for instance, many comments and precisions are given in parentheses). In some cases, we kept the original "russian" terminology (usually accompanied by footnote remarks), either because it does not have an exact "western" counterpart, or because it was much used in the founding works of the Soviet researchers, including S. N. Kruzhkov himself.

We hope that the reader will be recompensed for her or his effort by the vivacity of the exposition and by the originality of the approach. Indeed, while at the mid-1990th, only few treaties on the subject of conservation laws were available (see [20, 48, 49]), the situation changed completely in the last ten years. The textbooks and monographs [11, 14, 22, 32, 33, 35, 47] are mainly concerned with conservation laws and systems. With respect to the material covered, the present notes can be compared with the introductory chapters of [11, 22, 33] and with the relevant chapters of the already classical PDE textbook [16]. Yet in the present lecture notes the exposition is quite different, with a strong emphasis on examples and motivation of the theory.

Introduction

The study of first-order partial differential equations is almost as ancient as the notion of the partial derivative. PDEs of first order appear in many mechanical and geometrical problems, due to the physical meaning of the notion of derivative (the velocity of motion) and to its geometrical meaning (the tangent of the angle). Local theory of such equations was born in the 18th century.

In many problems of this type one of the variables is the time variable, and processes can last for a sufficiently long time. During this period, some singularities of classical solutions can appear. Among these singularities, we consider only weak discontinuities (which are jumps of derivatives of the solution) and strong discontinuities (which are jumps of the solutions themselves). We do not deal with the “blow up”-type singularities.

It is clear that after the singularities have appeared, in order to give a meaning to the equation under consideration one has to define weak derivatives and weak solutions. These notions were introduced into mathematical language only in the 20th century. The first mathematical realization of this “ideology” was the classical paper of E. Hopf [23] (1950). In this paper, a nonlocal theory for the Cauchy problem was constructed for the equation

$$u_t + \big(u^2/2\big)_x = 0 \tag{0.1}$$

with initial datum

$$u\big|_{t=0} = u_0(x), \tag{0.2}$$

where $u_0(x)$ is an arbitrary bounded measurable function. The equation

$$u_t + (f(u))_x = 0 \tag{0.3}$$

is a natural generalization of equation (0.1). Important results for the nonlocal theory of this equation were obtained (in the chronological order of the papers) by O. A. Oleĭnik [36, 37], A. N. Tikhonov, A. A. Samarskiĭ [50], P. D. Lax [31], O. A. Ladyzhenskaya [29], I. M. Gel’fand [18].[1] The most complete theory of the Cauchy problem (0.3), (0.2) in the space of bounded measurable functions was achieved in the papers by S. N. Kruzhkov [25, 26] (see also [27]).[2]

1 Derivation of the equations

The Hopf equation. Consider a one-dimensional medium consisting of particles moving without interaction in the absence of external forces. Denote by $u(t,x)$ the velocity of the particle located at the point x at the time instant t. If $x = \varphi(t)$ is the

This is a beginner’s course on conservation laws; in a sense, it stops just where the modern theory begins, before advanced analysis techniques enter the stage. For further reading, we refer to any of the above textbooks.

[1] *NT* — Throughout the lectures, no attempt is made to give a complete account on the works on the subject of first-order quasilinear equations; the above references were those that most influenced S. N. Kruzhkov’s work.

[2] *NT* — Also should be mentioned the contribution by A. I. Vol’pert [52], who constructed a complete well-posedness theory in the smaller class BV of all functions of bounded variation. As shown in [52], this class is a convenient generalization of the class of piecewise smooth functions widely used in the present lectures.

trajectory of a fixed particle, then the velocity of this particle is $\dot{\varphi}(t) = u(t, \varphi(t))$, and the acceleration $\ddot{\varphi}(t)$ is equal to zero for all t. Hence,

$$0 = \frac{d^2\varphi}{dt^2} = \frac{d}{dt}u(t, \varphi(t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\dot{\varphi} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u.$$

The obtained equation

$$u_t + uu_x = 0, \tag{1.1}$$

which describes the velocity field u of non-interacting particles, is called the Hopf equation.

The continuity (or mass conservation) equation. This equation, usually presented in a course on the mechanics of solids, describes the movement of a fluid (a liquid or a gas) in $\mathbb{R}^n$ if there are no sinks nor sources. Denote the velocity vector of the fluid by $v(x, t) = (v_1, \ldots, v_n)$ and its density by $\rho(x, t)$. Let us fix a domain $V \subset \mathbb{R}^n$. At the moment t, the mass of the fluid contained in this domain is equal to

$$M_V(t) = \int_V \rho(x, t)\, dx;$$

this mass is changing with the rate dM_V/dt. On the other hand, in the absence of sources and sinks inside V, the change of mass M_V is only due to movements of the fluid through the boundary ∂V of the domain, i.e., the rate of change of the mass $M_V(t)$ is equal to the flux of the fluid through ∂V:

$$\frac{dM_V}{dt} = -\int_{\partial V} (v(x, t), \nu) \cdot \rho(x, t)\, dS_x.$$

Here (v, ν) is the scalar product of the velocity vector v and the outward unit normal vector ν to the boundary ∂V at the point $x \in \partial V$; dS_x is an element of area on ∂V.

Hence, we have

$$\frac{d}{dt}\int_V \rho(x, t)\, dx = -\int_{\partial V} (v(x, t), \nu) \cdot \rho(x, t)\, dS_x. \tag{1.2}$$

Under the assumption that ρ and v are sufficiently smooth, we rewrite the right-hand side of the formula (1.2) with the help of the divergence theorem (the Gauss–Green formula), i.e., using the fact that the integral of the divergence over a domain is equal to the flux through the boundary of this domain:

$$\int_V \frac{\partial \rho}{\partial t}\, dx = -\int_V \operatorname{div}(\rho v)\, dx. \tag{1.3}$$

Here div is the divergence operator with respect to the spatial variables. Let us remind that the divergence of the vector field $a(x) = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is the scalar

$$\operatorname{div} a = (a_1)_{x_1} + \cdots + (a_n)_{x_n}.$$

Since the domain $V \subset \mathbb{R}^n$ is arbitrary, using (1.3) we get the so-called continuity equation, well-known in hydrodynamics:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0. \tag{1.4}$$

Equation of fluid infiltration through sand. For the sake of simplicity, we introduce several natural assumptions. Suppose that the fluid moves under the sole action of the gravity, i.e., the direction of the movement is vertical and there is no dependence on horizontal coordinates. Neither sources nor sinks are present. The speed of infiltration v is a function of the density $\rho \equiv u(t,x)$, i.e., $v = v(u)$.

It is experimentally verified that the dependence $v(u)$ has a form as in Figure 1. On the segment $[0, u_0]$ one can assume that the dependence is almost parabolic, i.e., $v(u) = Cu^2$.

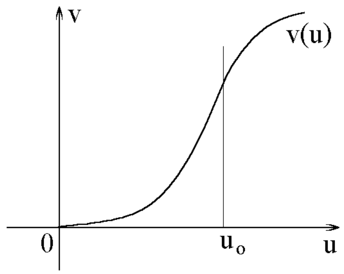

Figure 1. Experimental dependence $v = v(u)$.

In the one-dimensional case under consideration, the equation (1.4) will be rewritten as follows :

$$u_t(t,x) + [u(t,x) \cdot v\,(u(t,x))]_x = 0, \tag{1.5}$$

or

$$u_t + p(u)u_x = 0, \quad \text{where} \quad p(u) = v(u) + v'(u)u.$$

Keeping in mind the experimental dependence of the speed of infiltration on the density, we assume that $v(u) = u^2/3$, and finally we get

$$u_t + u^2 u_x = 0.$$

The traffic equation. This equation can also be derived from the one-dimensional (in x) continuity equation (1.4). In traffic problems, $u(t,x)$ represents the density of cars on the road (at point x at time t); and the dependence of the velocity v of cars on the density u is linear:

$$v(u) = C - ku, \qquad C, k = \text{const} > 0.$$

In this case, equation (1.5) reads as follows:

$$u_t + (Cu - ku^2)_x = 0.$$

2 The local classical theory

First order PDEs can be solved locally by means of methods of the theory of ordinary differential equations, using the so-called *characteristic system*. From the physical point of view this fact can be considered as an expression of the duality of the wave theory and the particle theory of media. The field satisfies a PDE of first order; and the behaviour of the particles constituting the field is described by a system of ODEs. The connection between the first-order PDE and the corresponding system of ODEs allows to study the behaviour of particles instead of studying the evolution of waves.

It should be noted that the majority of questions in this chapter are considered in the textbooks on ODEs (for instance, [3, Chapter 2]). Different exercises on linear and quasilinear equations of first order can be found in [17, §20].

Below we remind basic notions of the aforementioned local theory for linear and quasilinear equations.

2.1 Linear equations

Let $v = v(x)$ be a smooth vector field in a domain $\Omega \subset \mathbb{R}^n$.

Definition 2.1. The equation

$$L_v[u] \equiv v_1(x)\frac{\partial u}{\partial x_1} + \cdots + v_n(x)\frac{\partial u}{\partial x_n} = 0. \tag{2.1}$$

is said to be a *linear homogeneous* PDE of first-order.

A continuously differentiable function $u = u(x)$ is called *classical* solution of this equation if u satisfies the equation at any point of its domain.

Recall that in the ODE theory, the operator $L_v \equiv v_1\frac{\partial}{\partial x_1} + \cdots + v_n\frac{\partial}{\partial x_n}$ is called the derivation operator along the vector field v. Geometrically, equation (2.1) means that the gradient $\nabla u \equiv \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$ of the unknown function $u = u(x)$ is orthogonal to the vector field v in all points of the domain Ω.

A smooth function $u = u(x)$ is a solution of the equation (2.1) if and only if u is constant along the phase curves of the field v, i.e., it is the first integral of the system of equations

$$\begin{cases} \dot{x}_1 &= v_1(x_1, \ldots, x_n), \\ \dot{x}_2 &= v_2(x_1, \ldots, x_n), \\ & \ldots \\ \dot{x}_n &= v_n(x_1, \ldots, x_n). \end{cases} \tag{2.2}$$

The system (2.2), which can be written in vector form $\dot{x} = v(x)$, is called *the characteristic system of the linear equation* (2.1). A solution of the characteristic system is called *a characteristic*, the vector field $v = v(x)$ over the n-dimensional space of x is called *the characteristic vector field of the linear equation*.

Definition 2.2. A *linear inhomogeneous* first-order PDE is the equation

$$L_v[u] = f(x), \tag{2.3}$$

where $f = f(x)$ is a given function.

Equation (2.3) expresses the fact that if we move along the characteristic $x = x(t)$ (i.e., along the solution $x = x(t)$ of the system (2.2)), then $u(x(t))$ is changing with the given speed $f(x(t))$. Thus, in the case of an inhomogeneous linear equation, the characteristic system (2.2) should be supplemented with the additional equation on u:

$$\dot{u} = f(x_1, \dots, x_n). \tag{2.4}$$

2.2 The Cauchy problem

Definition 2.3. The *Cauchy problem* for a first-order partial differential equation is the problem of finding the solution $u = u(x)$ of this equation satisfying the initial condition

$$u\big|_{\gamma} = u_0(x), \tag{2.5}$$

where $\gamma \subset \mathbb{R}^n$, $\dim \gamma = n - 1$, is a fixed smooth hypersurface in the x-space, and $u_0 = u_0(x)$ is a given smooth function defined on γ.

In order to solve the Cauchy problem (2.1), (2.5) for a linear homogeneous equation, it is sufficient to continue the function $u(x)$ from the surface γ along the characteristics $x(t)$ by a constant. In the case of the problem (2.3), (2.5) for the inhomogeneous equation, the initial data should be extrapolated according to the law (2.4).

Note two important features of the Cauchy problem, specified above.

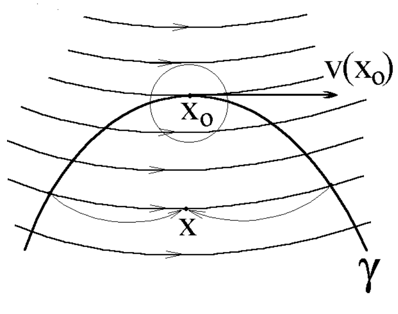

Figure 2. Example of a characteristic point.

Remark 2.4. The Cauchy problem is set locally (i.e., in a neighbourhood of a point x_0 on γ). Otherwise, as it can be seen in Figure 2, characteristics passing through a given point x may cross γ twice (or even several times), carrying different values of u to this point. Thus the solution to the problem (2.1), (2.5) exists only for specially selected initial data u_0.

Moreover, it can happen that the set of all the characteristics which have common points with the initial surface γ do not cover the whole domain where we want to solve the Cauchy problem. In this case, we have no uniqueness of a solution to the Cauchy problem.

Remark 2.5. If in the point $x_0 \in \gamma$ the vector $v(x_0)$ is parallel to the surface γ (such points x_0 are called *characteristic points*, see Figure 2), then, even choosing a very small neighbourhood of this point, we cannot guarantee that we shall not have the same difficulties as we mentioned in Remark 2.4. Hence, the existence and the uniqueness of a solution to a Cauchy problem can be guaranteed only in a neighbourhood of a non-characteristic point on γ.

Linear first-order PDEs can be impossible to solve in a neighbourhood of a characteristic point even in the case when each characteristic has exactly one point of intersection with the initial surface γ.

Example 2.6. Consider the following Cauchy problem:

$$\frac{\partial u}{\partial x} = 0, \qquad u\big|_{y=x^3} = x^2. \tag{2.6}$$

The characteristic vector field is the constant field $(1, 0)$, the characteristics are the straight lines $y = C$; each of them has only one intersection point with the curve $\gamma = \{(x, y) \mid y = x^3\}$. If we extend the initial function $u_0(x) = x^2$ (which is equal to $y^{2/3}$ on γ) so that it is constant along the characteristics, we get the x-independent "solution" $u(x, y) = y^{2/3}$ which is not a classical solution because it is not a continuously differentiable function on the line $y = 0$.

The possible objection that, nevertheless, the function constructed above has a partial derivative with respect to x (and hence satisfies the equation in the classical sense) is easy to remove. It is sufficient to change the variables in problem (2.6) according to the formula $x = x' + y'$, $y = x' - y'$. After this rotation and rescaling on the axes, we obtain the following Cauchy problem:

$$\frac{\partial u}{\partial x'} + \frac{\partial u}{\partial y'} = 0, \qquad u\big|_{\gamma} = (x' + y')^2,$$

the curve γ being defined by the equation $x' - y' = (x' + y')^3$. The transformed "solution" $u(x', y') = (x' - y')^{2/3}$ has no partial derivatives in x' nor in y' on the line $x' - y' = 0$.

2.3 Quasilinear equations

Definition 2.7. The equation

$$L_{v(x,u)}[u] \equiv v_1(x, u)\frac{\partial u}{\partial x_1} + \cdots + v_n(x, u)\frac{\partial u}{\partial x_n} = f(x, u) \tag{2.7}$$

is called a *quasilinear* first-order PDE. If in the equation (2.7) all the coefficients v_i are independent of u, i.e., $v_i = v_i(x)$, then the PDE is called *semilinear*.

As for the linear equation, we write down the system (2.2), (2.4):

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & v_1(x_1, \ldots, x_n, u), \\ & \cdots & \\ \dot{x}_n & = & v_n(x_1, \ldots, x_n, u), \\ \dot{u} & = & f(x_1, \ldots, x_n, u). \end{array} \right. \tag{2.8}$$

This system is called the *characteristic system of the quasilinear equation* (2.7); solutions $(x, u) = (x(t), u(t)) \in \mathbb{R}^{n+1}$ to the system (2.8) are called *characteristics* of this equation; a *characteristic vector field of a quasilinear equation* (2.7) is a smooth vector field with components $(v_1(x, u), \dots, v_n(x, u), f(x, u))$ in the $(n+1)$-dimensional space with coordinates $(x_1, \dots, x_n, u)$.

Remark 2.8. If a linear equation is considered as being quasilinear, and also in the case of a semilinear equation, the projection $(v_1, \dots, v_n)$ on the x-space of the vector $(v_1, \dots, v_n, f)$ in the point (x_0, u_0) does not depend on u_0, since the coefficients v_i do not depend on u. Hence in these cases the projections on the x-space of the characteristics that lie at "different heights" coincide (here we mean that the vertical axis represents the variable u).

If the smooth hypersurface $M \subset \mathbb{R}^{n+1}$ is the graph of a function $u = u(x)$, then the normal vector to this surface in the coordinates (x, u) has the form $(\nabla_x u, -1) = (\partial u/\partial x_1, \dots, \partial u/\partial x_n, -1)$. Therefore, geometrically, the equation (2.7) expresses the orthogonality of the characteristic vector $(v(x, u), f(x, u))$ and the normal vector to M. Thus, we have the following theorem.

Theorem 2.9. *A smooth function $u = u(x)$ is a solution to the equation (2.7) if and only if the graph $M = \{(x, u(x))\}$, which is a hypersurface in the space $\mathbb{R}^{n+1}$, is tangent, in all its points, to the characteristic vector field $(v_1, \dots, v_n, f)$.*

Corollary 2.10. *The graph of any solution $u = u(x)$ to the equation (2.7) is spanned by characteristics.*

Indeed, by definition, the characteristics $(x(t), u(t))$ are tangent to the characteristic vector field (see (2.8)); therefore any characteristics having a point in common with the graph of u lies entirely on this graph. (Here and in the sequel, we always assume that the characteristic system complies with the assumptions of the standard existence and uniqueness theorems of the theory of ODEs.)

For the case of a quasilinear equation, the Cauchy problem (2.7), (2.5) can be solved geometrically as follows. Let

$$\Gamma = \{(x, u_0(x)) \mid x \in \gamma\} \subset \mathbb{R}^{n+1}, \qquad \dim \Gamma = n - 1,$$

be the graph of the initial function $u_0 = u_0(x)$. Issuing a characteristic from each point of Γ, we obtain some surface M of codimension one. Below we show that, whenever the point $(x_0, u_0(x_0))$ is non-characteristic, at least locally (in some neighbourhood of the point $(x_0, u_0(x_0)) \in \Gamma$) the hypersurface M represents the graph of the unknown solution $u = u(x)$.

Definition 2.11. A point $(x_0, u_0) \in \Gamma$ is called *a characteristic point,* if the vector $v(x_0, u_0)$ is tangent to γ at this point.

Remark 2.12. In the case of a quasilinear equation, one does not ask whether a point $x_0 \in \gamma \subset \mathbb{R}^n$ is a characteristic point. Indeed, the characteristic vector field also depends on u. In this case, one should ask whether a point $(x_0, u_0(x_0)) \in \Gamma \subset \mathbb{R}^{n+1}$ is a characteristic point.

If $(x_0, u_0(x_0)) \in \Gamma$ is a non-characteristic point, then the hyperplane T tangent to M at this point projects isomorphically onto the x-space. Indeed, the hyperplane T is spanned by the directions tangent to Γ (their projections span the hyperplane in $\mathbb{R}^n$ tangent to γ) and by the characteristic vector $(v(x_0, u_0(x_0)), f(x_0, u_0(x_0)))$ (its projection is the vector $v(x_0, u_0(x_0))$ transversal to γ). Consequently, locally in a neighbourhood of the point $(x_0, u_0(x_0)) \in \Gamma$, the hypersurface M constructed above represents the graph of a smooth function $u = u(x)$, which is the desired solution.

3 Classical (smooth) solutions of the Cauchy problem and formation of singularities

3.1 Quasilinear equations with one space variable

In the sequel, we will always consider the following equation in the unknown function $u = u(t, x)$ depending on two variables (t has the meaning of time, and $x \in \mathbb{R}^1$ represents the one-dimensional space coordinate):

$$u_t + (f(u))_x \equiv u_t + f'(u)u_x = 0. \tag{3.1}$$

Here $f \in C^2$ is a given function, which will be called the *flux function*. The initial data is prescribed at time $t = 0$:

$$u\big|_{t=0} = u(0, x) = u_0(x). \tag{3.2}$$

In this section, we investigate the possibility to construct solutions of the problem (3.1)–(3.2) within the class of smooth functions defined in the strip

$$\Pi_T \equiv \{(t, x) \mid -\infty < x < +\infty,\ 0 < t < T\}.$$

Let us apply the results of the general theory, as exposed above, to this concrete case.

We see that the equation (3.1) is quasilinear; for this case, the characteristic system (2.8) takes the form

$$\begin{cases} \dot{t} &= 1, \\ \dot{x} &= f'(u), \\ \dot{u} &= 0. \end{cases} \tag{3.3}$$

The first equation in system (3.3) together with the initial condition $t(0) = 0$ (we take this condition because of (3.2)) means exactly the following: the independent variable in system (3.3) (the differentiation with respect to this variable is denoted by a dot ($\dot{}$)) coincides with the time variable t of the equation (3.1). Thus it is natural to exclude the first equation from the characteristic system (3.3) associated with the Cauchy problem (3.1)–(3.2).

In the case considered, the initial curve $\gamma \in \mathbb{R}^2_{t,x}$ is the straight line $t = 0$, i.e., $\gamma = \{(t, x) \mid t = 0\}$, and the curve $\Gamma \in \mathbb{R}^3_{t,x,u}$ is the set of points

$$\Gamma = \{(t, x, u) \mid t = 0,\ x = y,\ u = u_0(y)\},$$

parameterized by the space variable y. Let us stress that in this case, all the points of Γ are non-characteristic, since the vector $(\dot{t}, \dot{x}) = (1, f'(u))$ is transversal to $\gamma = \{t = 0\}$.

Thus in our case, we can rewrite the characteristic system (3.3) (with the initial data corresponding to (3.2)) in the form

$$\begin{cases} \dot{x} = f'(u), & x(0) = y, \\ \dot{u} = 0, & u(0) = u_0(y). \end{cases} \tag{3.4}$$

Solutions of this system (i.e., the characteristics of equation (3.1)) are the straight lines

$$u \equiv u_0(y), \qquad x = y + f'(u_0(y))t \tag{3.5}$$

in the three-dimensional space of points (t, x, u).

As was pointed out in Section 2.3, the graph of the solution $u = u(t, x)$ of problem (3.1)–(3.2) is the union of the characteristics issued from the points of the initial curve Γ; thus, the graph of u consists of the straight lines (3.5). Therefore, the solution of problem (3.1)–(3.2) at different time instants $t > 0$ (i.e., the sections of the graph of the solution $u = u(t, x)$ of this problem by different hyperplanes $t =$ const) can be constructed as follows. The graph of the initial function $u = u_0(x)$ should be transformed by displacing each point (x, u) of this graph horizontally (i.e., in the direction of the x-axis) with the speed $f'(u)$. If $f'(u) = 0$ then the point (x, u) does not move. If $f'(u) > 0$, then the point moves to the right; and, the greater $f'(u)$ is, the quicker it moves. Similarly, in the case $f'(u) < 0$, the point (x, u) moves to the left (see Fig. 3).

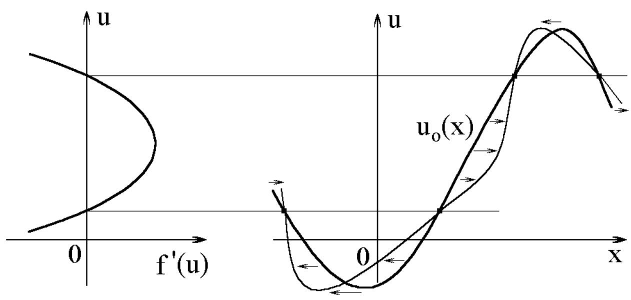

Figure 3. Evolution from initial graph.

Remark 3.1. Assume that the graph of the initial function $u_0 = u_0(x)$ delimits a finite area (this is the case, for instance, when u_0 has finite support). Then the aforementioned transformation of the graph leaves the area invariant. Indeed, all the points of the graph of u_0 lying on the same horizonal line move with the same speed; consequently, the lengths of the horizontal segments joining the points of the graph remain unchanged.

The fact that the area under the graph remains constant can also be obtained by a direct calculation. Let $S(t) = \int_{-\infty}^{+\infty} u(t, x)\, dx$ be the area in question, i.e., the area delimited by the graph of $u = u(t,x)$ of problem (3.1)–(3.2) (here $t \geqslant 0$ is fixed). Then

$$\frac{d}{dt}S(t) = \int_{-\infty}^{+\infty} u_t(t,x)\,dx = -\int_{-\infty}^{+\infty} \left(f(u(t,x))\right)_x\,dx = -f\left(u(t,x)\right)\Big|_{x=-\infty}^{x=+\infty}$$
$$= f(0) - f(0) = 0,$$

which means that $S(t) \equiv$ const.

While the graph of the solution evolves as described above, at a certain moment $T > 0$ it may happen that the transformed curve ceases to represent the graph of a smooth function $u(T,x)$ of variable x.

Consider, for instance, the Hopf equation, i.e., the equation (3.1) with $f(u) = u^2/2$. This equation describes the evolution of the velocity field of a medium consisting of non-interacting particles (see Section 1). Each particle moves in absence of forces and thus conserves its initial speed.

Consider two particles located, at the initial instant $t = 0$, at points x_1 and x_2 with $x_1 < x_2$. If the initial velocity distribution $u_0 = u_0(x)$ is a monotone non-decreasing function, then the initial velocity $u_0(x_1)$ of the first particle (which is its velocity for all subsequent instants of time) is less than or equal to the velocity $u_0(x_2)$ of the second particle: $u_0(x_1) \leqslant u_0(x_2)$. Since also the initial locations of the two particles obey the inequality $x_1 < x_2$, at any time instant $t > 0$ the two particles will never occupy the same space location; i.e., no particle collision happens in this case.

On the contrary, if the initial velocity distribution $u_0 = u_0(x)$ is not a monotone non-decreasing function, then the quicker particles will overtake the slower ones (or, possibly, particles can move towards each other), and at some instant $T > 0$ collisions should occur. Starting from this time instant T, our model does not reflect the physical reality any more, because the particles “passing through each other” should interact (collide) in one way or another. Mathematically, such interaction is usually accounted for by adding a term of the form εu_{xx} onto the right-hand side of equation (3.1), where $\varepsilon > 0$ has the meaning of a viscosity coefficient. We will encounter this model in Section 5.2.

Exercise 3.1. *For the Hopf equation, represent approximatively the velocity distribution $u = u(t,x)$ at different time instants $t > 0$, if the initial velocity distribution is given by the function*

(i) $u_0(x) = \arctan x$,

(ii) $u_0(x) = -\arctan x$,

(iii) $u_0(x) = \sin x$,

(iv) $u_0(x) = -\sin x$,

(v) $u_0(x) = x^3$,

(vi) $u_0(x) = -x^3$.

For the initial data prescribed above, find the maximal time instant $T > 0$ *such that a smooth solution of the Cauchy problem (for the Hopf equation)*

$$u_t + uu_x = 0, \qquad u\big|_{t=0} = u_0(x),$$

exists in the strip $\Pi_T = \{(t,x) \mid 0 < t < T,\ x \in \mathbb{R}\}$.

Exercise 3.2. *Represent approximatively the sections of the graph of the solution of the Cauchy problem*

$$u_t + (f(u))_x = 0, \qquad u\big|_{t=0} = u_0(x),$$

at different time instants $t > 0$ *for*

(i) $f(u) = \cos u, \quad u_0(x) = x,$

(ii) $f(u) = \cos u, \quad u_0(x) = \sin x,$

(iii) $f(u) = u^3/3, \quad u_0(x) = \sin x.$

3.2 Reduction of the Cauchy problem to an implicit functional equation

One can solve the Cauchy problem for the quasilinear equation (3.1) directly, making no reference to the local theory of first-order quasilinear PDEs exposed above. This is the goal of the present section.

Assume that we already have a smooth solution $u = u(t,x)$ of the problem (3.1)–(3.2) under consideration.

Proposition 3.2. *The function* $u = u(t,x)$ *is constant along the integral curves of the ordinary differential equation*

$$\frac{dx}{dt} = f'\left(u(t,x)\right). \tag{3.6}$$

Proof. Differentiate the function $u = u(t,x)$ in the direction of the integral curves $(t, x(t))$ of equation (3.6):

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} = u_t + u_x \cdot f'(u) = u_t + (f(u))_x = 0. \qquad \square$$

As u remains constant along these integral curves, it follows that the solutions of (3.6) are the linear functions $x = f'(u)t + C_1$. (The straight lines $x - f'(u)t = C_1$, lying in the hyperplanes $u = C_2$, are exactly the characteristics of the quasilinear equation (3.1).)

Consequently, the value $u(t_0, x_0)$ of the solution $u = u(t,x)$ at the point (t_0, x_0) is conserved along the whole line

$$x - f'\left(u(t_0,x_0)\right) \cdot t = C = x_0 - f'\left(u(t_0,x_0)\right) \cdot t_0. \tag{3.7}$$

Extending this line until it intersects the x-axis at some point $(0, y_0)$, we take the value $u_0(y_0)$ at this point. Since the point $(0, y_0)$ lies on the straight line (3.7), we have $y_0 = x_0 - f'\left(u(t_0,x_0)\right) \cdot t_0$. Thus,

$$u(t_0,x_0) = u_0(y_0) = u_0\left(x_0 - f'\left(u(t_0,x_0)\right) \cdot t_0\right).$$

As the point (t_0, x_0) is arbitrary, we obtain the following identity for the solution u of the Cauchy problem (3.1)–(3.2):

$$u = u_0\left(x - f'(u)t\right). \tag{3.8}$$

Thus, the problem of finding the domain into which the solution $u = u(t,x)$ of (3.1)–(3.2) can be extended amounts to finding the domain where equation (3.8) with the unknown u has one and only one solution.

Remark 3.3. Formula (3.8) can also be obtained while solving practically the Cauchy problem for the quasilinear equation, according to [17, §20]. The characteristic system

$$\frac{dt}{1} = \frac{dx}{f'(u)} = \frac{du}{0}$$

associated with the equation (3.1) possesses two first integrals:

$$I_1(t,x,u) \equiv u, \qquad I_2(t,x,u) \equiv x - f'(u)t. \tag{3.9}$$

On the initial curve $\Gamma = \{(0, y, u_0(y))\} \in \mathbb{R}^3_{t,x,u}$, these two first integrals take the values

$$I_1\big|_\Gamma = u_0(y), \qquad I_2\big|_\Gamma = y.$$

Consequently, I_1 and I_2 are linked on Γ by the relation

$$I_1 = u_0(I_2). \tag{3.10}$$

The first integrals remain constant on the characteristics (i.e., on the integral curves of the characteristic system). Thus, relation (3.10) remains valid on all characteristics issued from the surface Γ. It remains to notice that, upon substituting (3.9) into (3.10), we get exactly the equation (3.8).

On the other hand, the Cauchy problem (3.1)–(3.2) can be solved by extending the solution $u = u(t,x)$ from the initial point $(0, y)$ by the constant value (the value $u_0(y)$ of the solution at this initial point) along the line

$$x - f'\left(u_0(y)\right) \cdot t = C = y - f'\left(u_0(y)\right) \cdot 0 = y, \tag{3.11}$$

that is, by setting $u(t,x) = u_0(y)$ for all x and t which satisfy (3.11). Expressing the variable y in equation (3.11) through x and t, we get a function $y = y(t,x)$; consequently,

$$u(t,x) = u_0\left(y(t,x)\right). \tag{3.12}$$

In this case, extending the solution is reduced to the problem of finding the domain in which equation (3.11), with y for the unknown, can be solved in a unique way.

3.3 Condition for existence of a smooth solution in a strip

Let us find the maximal value among all time instants $T > 0$ for which equation (3.8) determines a smooth solution $u = u(t,x)$ in the strip Π_T. In fact, we have to determine the greatest possible value of T such that the equation

$$\Phi(t,x,u) \equiv u - u_0(x - f'(u)t) = 0, \tag{3.13}$$

with unknown u, has a unique solution for all fixed t in the interval $[0,T)$ and all $x \in \mathbb{R}$. For $t = 0$, the function $\Phi = \Phi(0,x,u)$ is monotone increasing in u. Thus, by the implicit function theorem the time instant T in question is restricted by the relation

$$\Phi_u(u,x,t) = 1 + u_0'(x - f'(u)t) \cdot f''(u) \cdot t > 0 \tag{3.14}$$

for all points (t,x,u) such that $\Phi(t,x,u) = 0$ and $t \in [0,T)$.

If $|f''(u)| \leqslant L$ on the range of the function $u_0 = u_0(x)$, and if, in addition, $|u_0'| \leqslant K$, then (3.14) is certainly satisfied whenever $1 - KL \cdot t > 0$. Therefore, there exists a smooth solution of problem (3.1)–(3.2) in the strip

$$0 < t < \frac{1}{KL}\ .$$

Problem 3.1. *Show that if the functions u_0' and f'' keep constant signs* (*i.e., the function u_0 is monotone, and the function f is either convex or concave*) *and if the two signs coincide, then a smooth solution $u = u(t,x)$ exists in the whole half-space $t > 0$.*

Starting from inequality (3.14), we can also obtain the exact value of the maximal time instant T which delimits the time interval of existence of a smooth solution. To do this, denote $y = x - f'(u)t$ and notice that $u = u_0(y)$ because of (3.13). Then (3.14) is rewritten as

$$1 + u_0'(y) \cdot f''(u_0(y)) \cdot t > 0.$$

Hence,

$$T = \frac{1}{-\inf\limits_{y\in\mathbb{R}} \left[u_0'(y) f''(u_0(y))\right]} = \frac{1}{-\inf\limits_{y\in\mathbb{R}} \left[\frac{d}{dy} f'(u_0(y))\right]} \tag{3.15}$$

if only the above infimum is negative. Otherwise, if $\inf_{y\in\mathbb{R}} \left[u_0'(y) f''(u_0(y))\right] \geqslant 0$, then $T = +\infty$ (see Problem 3.1).

Problem 3.2. *Check that a function $u = u(t,x)$, which is smooth in a strip Π_T and which satisfies* (3.8)*, is a solution of the Cauchy problem* (3.1)–(3.2).

Problem 3.3. *Show that the function $u = u(t,x)$ given by* (3.12)*, where $y = y(t,x)$ is a smooth function in Π_T such that* (3.11) *holds, is a solution to the Cauchy problem* (3.1)–(3.2).

Problem 3.4. *Show that the formulas* (3.8) *and* (3.12) *define the same solution of the Cauchy problem* (3.1)–(3.2).

Problem 3.5. *Show that, whenever* $\inf_{y\in\mathbb{R}}[u_0'(y)f''(u_0(y))] = -\infty$*, there is no strip* $\Pi_T = \{(t,x) \mid 0 < t < T, x \in \mathbb{R}\}$*,* $T > 0$*, such that a smooth solution to problem* (3.1)–(3.2) *exists.*

Exercise 3.3. *Find the maximal value* $T > 0$ *for which there exists a smooth solution to the Cauchy problem*

$$u_t + f'(u)u_x = 0, \qquad u\big|_{t=0} = u_0(x), \tag{3.16}$$

in the strip $\Pi_T = \{(t,x) \mid 0 < t < T, x \in \mathbb{R}\}$*, for*

(i) $f(u) = u^2/2, \quad u_0(x) = \arctan x,$

(ii) $f(u) = u^2/2, \quad u_0(x) = -\arctan x,$

(iii) $f(u) = \cos u, \quad u_0(x) = x,$

(iv) $f(u) = \cos u, \quad u_0(x) = \sin x,$

(v) $f(u) = u^3/3, \quad u_0(x) = \sin x.$

Exercise 3.4. *Which of the Cauchy problems of the form* (3.16)*, with the data prescribed below, admit a smooth solution* $u = u(t,x)$ *in the whole half-space* $t > 0$*, and, in contrast, which of them do not possess a smooth solution in any strip* Π_T*,* $T > 0$*:*

(i) $f(u) = u^2/2, \quad u_0(x) = x^3,$

(ii) $f(u) = u^2/2, \quad u_0(x) = -x^3,$

(iii) $f(u) = u^4, \quad u_0(x) = x,$

(iv) $f(u) = u^4, \quad u_0(x) = -x$?

3.4 Formation of singularities

To fix the ideas, consider the following Cauchy problem for the Hopf equation (1.1), i.e., for the equation of the form (3.1) with $f(u) = u^2/2$:

$$u_t + uu_x = 0, \qquad u\big|_{t=0} = u_0(x), \tag{3.17}$$

the initial datum u_0 being the smooth function given by

$$u_0(x) = \begin{cases} 2 & \text{for } x \leqslant -3, \\ \psi_1(x) & \text{for } -3 < x < -1, \\ -x & \text{for } -1 \leqslant x \leqslant 1, \\ \psi_2(x) & \text{for } 1 < x < 3, \\ -2 & \text{for } x \geqslant 3 \end{cases}$$

(see Fig. 4a). Here the functions ψ_1 and ψ_2 connect, in a smooth way, the two constant values taken by u_0 as $|x| \geqslant 3$ with the linear function representing u_0 as $|x| \leqslant 1$. While

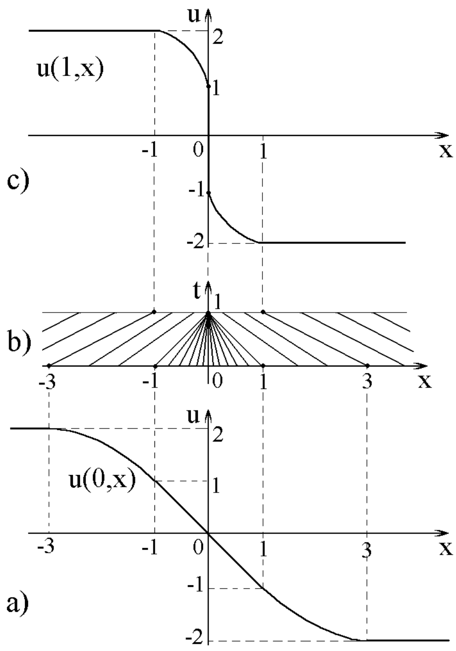

Figure 4. Formation of a strong discontinuity.

doing this, we can choose ψ_1 and ψ_2 in such a way that $-1 < \psi_i'(x) \leqslant 0$, $i = 1, 2$, as $1 < |x| < 3$.

As we have $|u_0'| \leqslant 1$ and $f'' = 1$, the results of the previous section imply the existence of a unique smooth solution $u = u(t, x)$ to problem (3.17) in the strip $0 < t < 1$. As was shown in Section 3.2, in order to construct this solution one has to issue the straight line (see (3.11))

$$x - u_0(y) \cdot t = y, \tag{3.18}$$

starting at every point $(t, x) = (0, y)$ of the line $t = 0$, and one has to assign $u(t, x) = u_0(y)$ at all the points (t, x) of this line.

For $y \leqslant -3$ (for $y \geqslant 3$, respectively) the equation (3.18) determines (see Fig. 4b) the family of parallel straight lines $x = 2t + y$ (or $x = -2t + y$, respectively). Consequently,

$$u(t, x) = 2 \quad \text{for} \quad 0 \leqslant t \leqslant 1, \quad x \leqslant 2t - 3,$$
$$u(t, x) = -2 \quad \text{for} \quad 0 \leqslant t \leqslant 1, \quad x \leqslant 3 - 2t.$$

Further, for $|y| \leqslant 1$ the corresponding straight lines are given by $x + yt = y$, i.e., by $x = y(1 - t)$; on these lines, $u = -y = -x/(1 - t)$. This means that

$$u(t, x) = -x/(1 - t) \quad \text{for} \quad 0 \leqslant t < 1, \quad |x| \leqslant 1 - t.$$

On the set $0 \leqslant t \leqslant 1$, $1 - t < |x| < 3 - 2t$, we cannot write down an explicit formula for $u = u(t,x)$ without defining explicitly the functions ψ_i. Nevertheless, we can guarantee that the straight lines of the form (3.18), corresponding to different values of y from the set $(-3,-1) \cup (1,3)$, do not intersect inside the strip $0 \leqslant t \leqslant 1$ because $|\psi_i'| < 1$ on this set.

For $t = 1$, through each point $(t,x) = (1,x)$ with $x \neq 0$ there passes one and only one straight line (3.18), corresponding to some value y with $|y| > 1$ (see Fig. 4b). Such a line carries the value $u = u_0(y)$ for the solution at the point $(1,x)$. Moreover, if $x \to -0$, then the corresponding value of y tends to -1; and if $x \to +0$, then $y \to 1$. Consequently, at the time instant $t = 1$, we obtain a function $x \mapsto u(1,x)$ which is smooth for $x < 0$ and for $x > 0$, according to the implicit function theorem. As has been pointed out,

$$\lim_{x \to \pm 0} u(1,x) = \lim_{y \to \pm 1} u_0(y) = \mp 1.$$

As to the point $(1,0)$, different characteristics bring different values of u to this point. More precisely, all the lines of the form (3.18) with $|y| \leqslant 1$ (i.e., the lines $x = y(1-t)$) pass through this point; each line carries the corresponding value $u = -y$, so that all the values contained within the segment $[-1,1]$ are brought to the point $(1,0)$.

The graph of the function $u = u(1,x)$ is depicted in Fig. 4c.

To summarize, starting from a smooth function $u(0,x) = u_0(x)$ at the initial instant of time $t = 0$, at time $t = 1$ we obtain the function $x \mapsto u(1,x)$ which turns out to be discontinuous at the point $x = 0$. This kind of discontinuity, where $u(t_0, x_0 + 0) \neq u(t_0, x_0 - 0)$, is called a strong one. Consequently, we can say that the solution of problem (3.17) forms a *strong discontinuity* at the time $t_0 = 1$ at the point $x_0 = 0$.

For the general problem (3.1)–(3.2), whenever $\inf_{y \in \mathbb{R}} [u_0'(y) f''(u_0(y))]$ is negative and it is attained on a non-trivial segment $[y_-, y_+]$, strong discontinuity occurs at the time instant T given by (3.15). In this situation, like in the example just analyzed, all the straight lines (3.11) corresponding to $y \in [y_-, y_+]$ intersect at some point (T, x_0); they bring different values of u to this point.

Problem 3.6. *Show that if*

$$u_0'(y) f''(u_0(y)) = I \quad \forall y \in [y_-, y_+], \quad \textit{where} \quad I = \inf_{y \in \mathbb{R}} [u_0'(y) f''(u_0(y))], \quad I < 0,$$

then the family of straight lines (3.11) *corresponding to* $y \in [y_-, y_+]$ *crosses at one point.*

Instead of a strong discontinuity, a so-called *weak discontinuity* may occur in a solution $u = u(t,x)$ at the time instant T. This term simply means that the function $x \mapsto u(T,x)$ is continuous in x, but fails to be differentiable in x.

Problem 3.7. *Let the infimum* $I = \inf_{y \in \mathbb{R}} [u_0'(y) f''(u_0(y))]$ *be a negative minimum, attained at a single point* y_0. *Let* T *be given by* (3.15). *Show that in this situation, the solution* $u = u(t,x)$, *which is smooth for* $t < T$, *has a weak discontinuity at the point* $(T, y_0 + f'(u_0(y_0))T)$; *in addition, for each* $t > T$ *some of the lines given by* (3.11) *cross.*

4 Generalized solutions of quasilinear equations

As has been shown in the previous section, whatever the smoothness of the initial data is, classical solutions of first-order quasilinear PDEs can develop singularities as time grows. Furthermore, in applications one often encounters problems with discontinuous initial data. The nature of the equations we consider (here, the role of the characteristics is important, because they "carry" the information from the initial datum) is such that we cannot expect that the initial singularities smooth out automatically for $t > 0$. Therefore, it is necessary to extend the notion of a classical solution by considering so-called generalized solutions, i.e., solutions lying in classes of functions which contain functions with discontinuities.

4.1 The notion of generalized solution

There exists a general approach leading to a notion of generalized solution; it has its origin in the theory of distributions. In this approach, the pointwise differential equation is replaced by an appropriate family of integral identities. When restricted to classical (i.e., sufficiently smooth) solutions, these identities are equivalent to the original differential equation. However the integral identities make sense for a much wider class of functions. A function satisfying such integral identities is often called a generalized solution.[3]

The approach we will now develop exploits the Green-Gauss formula.

Theorem 4.1 (The Green–Gauss (Ostrogradskiĭ–Gauss) formula). *Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $w \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega \frac{\partial w}{\partial x_i}\, dx = \int_{\partial\Omega} w \cos(\nu, x_i)\, dS_x.$$

Here $\cos(\nu, x_i)$ is the i-th component of the outward unit normal vector ν (this is the cosine of the angle formed by the direction of the outward normal vector to $\partial\Omega$ and the direction of the i-th coordinate axis Ox_i); and dS_x is the infinitesimal area element on $\partial\Omega$.

Let us apply Theorem 4.1 to the function $w = uv$, $u, v \in C^1(\overline{\Omega})$. Passing one of the terms from the left-hand to the right-hand side, we get the following corollary.

Corollary 4.2 (Integration-by-parts formula). *For any $u, v \in C^1(\overline{\Omega})$,*

$$\int_\Omega v \frac{\partial u}{\partial x_i}\, dx = \int_{\partial\Omega} uv \cos(\nu, x_i)\, dS_x - \int_\Omega u \frac{\partial v}{\partial x_i}\, dx. \tag{4.1}$$

The first term in the right-hand side of (4.1) is analogous to the non-integral term which appears in the well-known one-dimensional integration-by-parts formula.

[3] *NT* — In the literature, these solutions are most usually called "weak" solutions. In the present lectures, the authors have kept the terminology and the approach of S. N. Kruzhkov, designed in order to facilitate the assimilation of the idea of a weak (generalized) solution, and to stress, throughout all the lectures, the distinction and the connections between the classical solutions and the generalized ones.

Assume that a function $u = u(t,x) \in C^1(\Omega)$ is a classical solution of the equation

$$u_t + (f(u))_x = 0, \tag{4.2}$$

$f \in C^1(\mathbb{R})$, in some domain $\Omega \subset \mathbb{R}^2$, e.g., in the strip $\Omega = \Pi_T := \{-\infty < x < +\infty,\ 0 < t < T\}$. This means that substituting $u(t,x)$ into equation (4.2), we obtain a correct identity for all points $(t,x) \in \Omega$. Let us multiply this equation by a compactly supported infinitely differentiable function $\varphi = \varphi(t,x)$. Saying that φ is compactly supported means that $\varphi = 0$ outside of some bounded domain G such that, in addition, $\overline{G} \subset \Omega$. (The space of all compactly supported infinitely differentiable functions on Ω is denoted by $C_0^\infty(\Omega)$.) Since the functions $u = u(t,x)$, $f = f(u(t,x))$, $\varphi = \varphi(t,x)$ are smooth, we can use the integration-by-parts formula (4.1):

$$\begin{aligned} 0 &= \int_\Omega [u_t + (f(u))_x]\,\varphi\, dtdx = \int_G u_t\varphi\, dtdx + \int_G (f(u))_x\varphi\, dtdx \\ &= \int_{\partial G} (u\cos(\nu,t) + f(u)\cos(\nu,x))\,\varphi\, dS - \int_G (u\varphi_t + f(u)\varphi_x)\ dtdx \\ &= -\int_\Omega (u\varphi_t + f(u)\varphi_x)\ dtdx. \end{aligned}$$

Here we took advantage of the fact that $\varphi(t,x) = 0$ for $(t,x) \in \Omega \setminus G$, which is the case, in particular, for $(t,x) \in \partial G$.

Consequently, we have obtained the following assertion: if $u = u(t,x)$ is a smooth solution of equation (4.2) in the domain Ω, then

$$\int_\Omega (u\varphi_t + f(u)\varphi_x)\ dtdx = 0 \quad \forall \varphi \in C_0^\infty(\Omega). \tag{4.3}$$

The relation (4.3) is taken for the definition of a generalized solution (sometimes called a solution in the sense of integral identity or distributional solution) of the equation (4.2). A generalized solution of the equation we consider need not to be smooth. But any classical solution $u = u(t,x)$ of equation (4.2) is also its generalized solution.

The converse fact is also easy to establish: if a function $u = u(t,x)$ is a generalized solution of equation (4.2) which turns out to be smooth (i.e., u belongs to $C^1(\Omega)$ and it satisfies (4.3)), then it is also a classical solution of this equation (i.e, substituting it into equation (4.2) yields a correct equality). Indeed, the calculations above remain true when carried out in the reversed order. Moreover, the fact that the continuous function $[u_t + (f(u))_x]$ satisfies

$$\int_\Omega [u_t + (f(u))_x]\,\varphi\, dtdx = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

implies that $u_t(t,x) + [f(u(t,x))]_x = 0$ for all $(t,x) \in \Omega$.

Problem 4.1. *Justify the latter assertion rigorously.*

4.2 The Rankine–Hugoniot condition

Consider a smooth function $u = u(t,x)$ in a domain $\Omega \subset \mathbb{R}^2_{t,x}$, and associate to this function the vector field $\vec{v} = (u, f(u))$ defined on the same domain. The function u is a classical solution of the equation (4.2) if and only if $\operatorname{div}\vec{v} = 0$; in turn, the latter condition means that the flux of the vector field $\vec{v}$ through the boundary of any domain $G \subset \Omega$ equals zero:

$$\int_{\partial G} (\vec{v}, \nu)\, dS = 0 \qquad \forall G \subset \Omega. \tag{4.4}$$

Here ν is the outward unit normal vector to ∂G, and $(\vec{v}, \nu)$ denotes the scalar product of the vectors $\vec{v}$ and ν. The identity (4.4) is called *a conservation law*.

Now assume we have a piecewise smooth function $u = u(t,x)$ that satisfies equation (4.2) in a neighbourhood of each of its smoothness points. In this case, the conservation law (4.4) need not hold in general (the flux of $\vec{v}$ may be non-zero, if the domain G contains a curve across which $u = u(t,x)$ is discontinuous). We now show that, nevertheless, for any piecewise smooth generalized solution of equation (4.2) (solution in the sense of the integral identity (4.3)), this important physical law does hold. In a sense, the essential feature of the differential equation (4.2) is to express the law (4.4); and this feature is "inherited" by the generalized formulation (4.3).

The proof amounts to the fact that, on every discontinuity curve, a generalized solution satisfies the so-called Rankine–Hugoniot condition. For a piecewise smooth function $u = u(t,x)$ that satisfies equation (4.2) in a neighbourhood of each point of smoothness, this condition is necessary and sufficient for u to be a generalized solution in the sense of the integral identity (4.3). The present section is devoted to the deduction of the aforementioned Rankine–Hugoniot condition.

Let $u = u(t,x)$ be a piecewise smooth generalized solution of equation (4.2) in the domain $\Omega \subset \mathbb{R}^2$, i.e., a solution in the sense of the integral identity (4.3). To be specific, let us assume that Ω is divided into two parts Ω_- and Ω_+, separated by some curve Γ (see Fig. 5); we further assume that in each of these two parts, the function $u = u(t,x)$ is smooth, i.e., $u \in C^1(\Omega_-) \cap C^1(\Omega_+)$, and that there exist one-sided limits u_- and u_+ of the function u as one approaches Γ from the side of Ω_- and from the side of Ω_+, respectively.

Consequently, at each point $(t_0, x_0) \in \Gamma$ of the discontinuity curve Γ, one can define

$$u_-(t_0,x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\Omega_-}} u(t,x) \quad \text{and} \quad u_+(t_0,x_0) = \lim_{\substack{(t,x)\to(t_0,x_0)\\(t,x)\in\Omega_+}} u(t,x).$$

Such discontinuities are called discontinuities of the first kind, or strong discontinuities, or jumps.

Notice that $u = u(t,x)$ is a generalized solution of (4.2) in each of the two subdomains Ω_- and Ω_+, in view of the fact that $C_0^\infty(\Omega_\pm) \subset C_0^\infty(\Omega)$. Moreover, this function is smooth in Ω_- and Ω_+. Therefore, according to what has already been proved, in each of the two subdomains, the function $u = u(t,x)$ is a classical solution of equation (4.2). Let us derive the conditions satisfied by $u = u(t,x)$ along the discontinuity curve Γ.

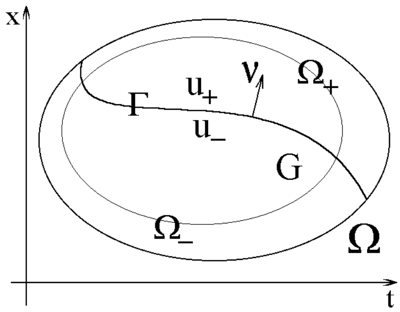

Figure 5. Strong discontinuity (jump).

Proposition 4.3. *Assume that the curve* Γ *contained within the domain* Ω *is represented by the graph of a smooth function* $x = x(t)$*. Then the piecewise smooth generalized solution* $u = u(t, x)$ *of equation* (4.2) *satisfies the following condition on* Γ*, called* the Rankine–Hugoniot condition:

$$\frac{dx}{dt} = \frac{[f(u)]}{[u]} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}\ , \tag{4.5}$$

where $[u] = u_+ - u_-$ *is the jump of the function* u *on the discontinuity curve* Γ*, and* $[f(u)] = f(u_+) - f(u_-)$ *is the jump of* $f = f(u)$*.*

Taking into account the relation $dx/dt = -\cos(\nu, t)/\cos(\nu, x)$*, where* $\cos(\nu, t)$ *and* $\cos(\nu, x)$ *are the components of the unit normal vector* ν *to the curve* $\Gamma = \{(t, x(t))\}$ $($*the vector is oriented to point from* Ω_- *to* Ω_+*; notice that* $\cos(\nu, x) \neq 0)$*, the equality* (4.5) *can be rewritten in the equivalent form*

$$[u]\cos(\nu, t) + [f(u)]\cos(\nu, x) = 0. \tag{4.6}$$

Definition 4.4. *A shock wave* is a discontinuous generalized solution of equation (4.2).

Thus we can say that the Rankine–Hugoniot condition (4.5) relates the speed $\dot{x}$ of propagation of a shock wave with the flux function $f = f(u)$ and the limit states u_+ and u_- of the shock-wave solution $u = u(t, x)$.

Proof of Proposition 4.3. Let us prove the formula (4.6). By the definition of a generalized solution, for any "test" function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(t, x) = 0$ for $(t, x) \notin G$, $\overline{G} \subset \Omega$, we have

$$\begin{aligned} 0 &= \int_\Omega (u\varphi_t + f(u)\varphi_x)\ dtdx \\ &= \int_{\Omega_- \cap G} (u\varphi_t + f(u)\varphi_x)\ dtdx + \int_{\Omega_+ \cap G} (u\varphi_t + f(u)\varphi_x)\ dtdx. \end{aligned}$$

The functions $u = u(t,x)$, $f = f(u(t,x))$, and $\varphi = \varphi(t,x)$ are smooth in the domains $\Omega_-\cap G$ and $\Omega_+\cap G$. Since these domains are bounded, while integrating on these domains we can transfer derivatives according to the multi-dimensional integration-by-parts formula (4.1). Notice that the boundaries of these domains consist of Γ and of parts of ∂G. The integrals over ∂G are equal to zero due to the fact that $\varphi(t,x) = 0$ for $(t,x) \in \partial G$. Thus, we have

$$\begin{aligned}
0 = &-\int_{\Omega_-\cap G} (u_t\varphi + (f(u))_x\varphi)\ dtdx + \int_{\Gamma\cap G} (u_- \cos(\nu,t) + f(u_-)\cos(\nu,x))\,\varphi\, dS \\
&- \int_{\Omega_+\cap G} (u_t\varphi + (f(u))_x\varphi)\ dtdx + \int_{\Gamma\cap G} (u_+ \cos(-\nu,t) + f(u_+)\cos(-\nu,x))\,\varphi\, dS \\
= &-\int_{\Omega_-} (u_t + (f(u))_x)\,\varphi\, dtdx - \int_{\Omega_+} (u_t + (f(u))_x)\,\varphi\, dtdx \\
&- \int_{\Gamma} \Big((u_+ - u_-)\cos(\nu,t) + (f(u_+) - f(u_-))\cos(\nu,x)\Big)\varphi\, dS.
\end{aligned}$$

Here we used the fact that ν is the outward unit normal vector to the part Γ of the boundary of the domain $\Omega_- \cap G$; thus $-\nu$ is the outward unit normal vector to the part Γ of the boundary of $\Omega_+ \cap G$. As was already mentioned, $u = u(t,x)$ is a classical solution in both domains Ω_- and Ω_+, i.e., equation (4.2) holds for $(t,x) \in \Omega_- \cup \Omega_+$. Therefore, we have

$$\int_{\Gamma} ([u]\cos(\nu,t) + [f(u)]\cos(\nu,x))\,\varphi\, dS = 0 \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{4.7}$$

Consequently, the equality (4.6) is satisfied at all points $(t,x) \in \Gamma$ where the discontinuity curve Γ is smooth (i.e., at the points $(t,x) \in \Gamma$ where the normal vector $\nu = (\cos(\nu,t), \cos(\nu,x))$ depends continuously on the point of Γ). □

The converse of the statement of the above theorem also holds true. Precisely, let a function $u = u(t,x)$ be a classical solution of equation (4.2) in each of the domains Ω_- and Ω_+. Assume that the function u has a discontinuity of the first kind on the curve Γ separating Ω_- from Ω_+ and that the Rankine–Hugoniot condition holds on the discontinuity curve Γ. Then u is a generalized solution of equation (4.2) in the domain $\Omega = \Omega_- \cup \Gamma \cup \Omega_+$. Indeed, starting from (4.7) and using the fact that

$$u_t + (f(u))_x = 0 \qquad \text{for} \quad (t,x) \in \Omega_- \cup \Omega_+,$$

we can reverse all the calculations of the above proof. This eventually leads to the integral identity (4.3), which is the definition of a generalized solution.

Problem 4.2. *Justify rigorously the above statement.*

Theorem 4.5. *Assume that $u = u(t,x)$ is a piecewise smooth function*[4] *defined in a domain Ω with a finite number of components $\Omega_1, \Omega_2, \ldots, \Omega_m$ where u is smooth, and, ac-*

[4] *NT* — Throughout the lectures, the term “piecewise smooth” refers exactly to the situation described in the assumption formulated in the present paragraph. This framework is sufficient to illustrate the key ideas of generalized solutions. In general, there may exist discontinuous generalized solutions with a much more complicated structure, but they are far beyond our scope.

cordingly, with a finite number of curves of discontinuity of the first kind $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$*, so that we have*

$$\Omega = \Big(\bigcup_{i=1}^{m} \Omega_i\Big) \bigcup \Big(\bigcup_{i=1}^{k} \Gamma_i\Big)$$

(*see Fig.* 6 *which corresponds to the case of a strip domain* $\Omega = \Pi_T$).

The function $u = u(t, x)$ *is a generalized solution of equation* (4.2) *in the domain* Ω *in the sense of the integral identity* (4.3) *if and only if* u *is a classical solution of this equation in a neighbourhood of each smoothness point of* u (*i.e., on each of the sets* Ω_i, $i = 1, \ldots, m$) *and, moreover, the Rankine–Hugoniot condition* (4.6) *is satisfied on each discontinuity curve* Γ_i, $i = 1, \ldots, k$ *except for the finite number of points where some of the curves* Γ_i *intersect one another.*

For the proof, it is sufficient to consider the restriction of the function u to each discontinuity curve Γ_i and the two smoothness components Ω_{i_1}, Ω_{i_2} adjacent to Γ_i; then we can exploit the assertions already shown in Proposition 4.3 and in Problem 4.2.

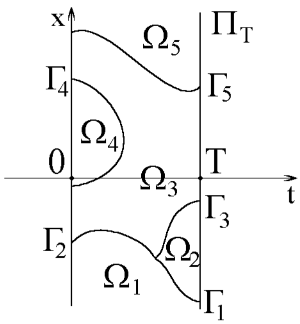

Figure 6. Piecewise smooth solution.

Proposition 4.6. *Let* $u = u(t, x)$ *be a piecewise smooth generalized solution of equation* (4.2) *in the domain* Ω *in the sense of the integral identity* (4.3). *Then the vector field* $\vec{v} = (u, f(u))$ *satisfies the conservation law* (4.4).

Proof. Assume that Ω_i are the components of smoothness of u. Let G be an arbitrary subdomain of the domain Ω. For all i, the flux of the vector field $\vec{v} = (u, f(u))$ through $\partial(\Omega_i \cap G)$ is equal to zero, because u is a classical solution of equation (4.2) in the subdomain Ω_i and thus also in the subdomain $\Omega_i \cap G$. Therefore, we can represent zero as the sum of these fluxes over all boundaries $\partial(\Omega_i \cap G)$. Thanks to the Rankine–Hugoniot condition (4.6), on each discontinuity curve Γ_j the total flux (i.e., the sum of the fluxes from the two sides of Γ_j) of the vector field $\vec{v}$ across the curve $\Gamma_j \cap G$ is equal to zero. Consequently, the sum of the fluxes across all the boundaries $\partial(\Omega_i \cap G)$ is equal to the flux of the vector field $\vec{v}$ through ∂G. This proves (4.4). □

As has been mentioned in Remark 3.1, the area delimited by the graph of a classical solution $u = u(t, x)$ of the problem (3.1)–(3.2) remains constant as a function of time

$t \geqslant 0$, whenever this area is finite. It turns out that also the generalized solutions obey this property. Thus the process of formation of a shock wave (a process that can be visualized as an “overturning” of the graph) occurs in such a way that the part which is “cut off’ has area equal to the area of the “extra” part (see Fig. 7); this equality of the two areas is a direct consequence of the Rankine–Hugoniot condition.

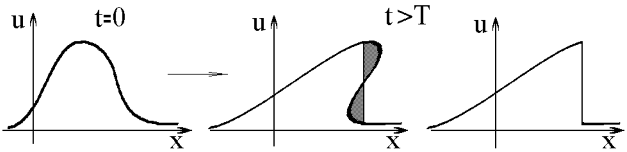

Figure 7. Area-preserving “overturning” of the graph.

Proposition 4.7. *Assume that $u = u(t, x)$ is a piecewise smooth function with compact support in x, such that $x = x(t)$ is the unique discontinuity curve of u and such that u is a generalized solution of equation* (4.2). *Denote*

$$S(t) = \int_{-\infty}^{+\infty} u(t, x)\, dx.$$

Then the function $S = S(t)$ is independent of t, i.e., $S(t) \equiv$ const.

Proof. Indeed, we can write

$$S(t) = \int_{-\infty}^{x(t)} u(t, x)\, dx + \int_{x(t)}^{+\infty} u(t, x)\, dx,$$

where $x = x(t)$ is the curve of discontinuity of the generalized solution $u = u(t, x)$. As previously, we denote by $u_\pm = \lim_{x\to x(t)\pm 0} u(t, x)$ the one-sided limits (limits along the x-axis) of the solution u on the discontinuity curve. Then

$$\begin{aligned}
\frac{dS}{dt} &= u(t, x(t) - 0) \cdot \dot{x}(t) + \int_{-\infty}^{x(t)} u_t(t, x)\, dx \\
&\quad - u(t, x(t) + 0) \cdot \dot{x}(t) + \int_{x(t)}^{+\infty} u_t(t, x)\, dx \\
&= (u_- - u_+) \cdot \dot{x}(t) - \int_{-\infty}^{x(t)} \Big(f(u(t, x))\Big)_x\, dx - \int_{x(t)}^{+\infty} \Big(f(u(t, x))\Big)_x\, dx \\
&= (u_- - u_+) \cdot \dot{x}(t) \\
&\quad - f(u(t, x(t) - 0)) + f(u(t, -\infty)) - f(u(t, +\infty)) + f(u(t, x(t) + 0)) \\
&= (f(u_+) - f(u_-)) - (u_+ - u_-) \cdot \dot{x}(t).
\end{aligned} \tag{4.8}$$

In these calculations, in addition to the equation (4.2) itself, we took advantage of the fact that u has compact support in x, so that $f(u(t, -\infty)) = f(u(t, +\infty)) = f(0)$.

Now if $u_+ = u_-$, then from (4.8) we clearly have

$$\frac{dS}{dt} = 0.$$

In the case $u_+ \neq u_-$, we have the same conclusion thanks to the Rankine–Hugoniot condition (4.5). □

Problem 4.3. *Prove the analogous result for the case where a piecewise smooth generalized* (*in the sense of the integral identity* (4.3)) *solution* $u = u(t,x)$ *of equation* (4.2) *has a finite number of discontinuity curves* $x = x_j(t)$, $j = 1, \ldots, N$.

Remark 4.8. If a function $u = u(t,x)$ has a weak discontinuity on the curve Γ, i.e, u is continuous across Γ and only its derivatives $u_t\,, u_x$ are discontinuous on Γ, then the Rankine–Hugoniot condition (4.6) is trivially satisfied (indeed, $[u] = 0$ and, consequently, also $[f(u)] = 0$). Therefore, a continuous function $u = u(t,x)$, which is piecewise smooth in a domain Ω and is a classical solution of equation (4.2) in a neighbourhood of each smoothness point, is also a generalized solution of (4.2) in the whole domain Ω (it is clear that the function $u = u(t,x)$ is not a classical solution in Ω, since it is not differentiable at the points $(t,x) \in \Gamma \subset \Omega$).

Remark 4.9. Formally, passing to the limit in (4.5) as $u_\pm \to u$, we infer that

$$\frac{dx}{dt} = f'(u(t,x)), \tag{4.9}$$

on a weak discontinuity curve $\Gamma = \{(t,x) \mid x = x(t)\}$ of $u = u(t,x)$; this means that a weak discontinuity propagates along a characteristic.

Let us provide a rigorous justification of this fact.

Let $\Gamma = \{(t,x) \mid x = x(t)\}$ be a weak discontinuity curve separating two classical solutions $u = u(t,x)$ and $v = v(t,x)$ of equation (4.2). Then

$$u(t,x(t)) \equiv v(t,x(t)). \tag{4.10}$$

Differentiating (4.10) with respect to t, we obtain

$$u_t(t,x(t)) + u_x(t,x(t)) \cdot \frac{dx}{dt} = v_t(t,x(t)) + v_x(t,x(t)) \cdot \frac{dx}{dt}$$

Here and in the sequel, u_x, v_x, u_t, v_t denote the corresponding limits of the derivatives as the point (t,x) tends to the weak discontinuity curve Γ. (The existence of these limits follows from the definition of a weak discontinuity.) Expressing the t-derivatives from the equation (4.2), we have

$$u_x(t,x(t)) \cdot \frac{dx}{dt} - f'(u(t,x(t)))u_x = v_x(t,x(t)) \cdot \frac{dx}{dt} - f'(v(t,x(t)))v_x.$$

Hence, taking into account (4.10), we obtain

$$\Big(u_x(t,x(t)) - v_x(t,x(t))\Big)\left(\frac{dx}{dt} - f'(u(t,x(t))\right) = 0.$$

Since the curve $x = x(t)$ is a weak discontinuity curve, the relation $u_x(t,x) \neq v_x(t,x)$ holds on this curve; thus (4.9) follows.

Exercise 4.1. *Is it true that the following functions* $u = u(t,x)$ *are generalized solutions* (in *the sense of the integral identity* (4.3)) *of equation* (4.2) *in the strip* Π_T (*remind that* $\Pi_T = \{-\infty < x < +\infty,\ 0 < t < T\}$), *for*

(i) $$f(u) = u^2/2, \qquad u(t,x) = \begin{cases} 0 & \text{for } x < t, \\ 1 & \text{for } x > t; \end{cases}$$

(ii) $$f(u) = u^2/2, \qquad u(t,x) = \begin{cases} 0 & \text{for } x < t, \\ 2 & \text{for } x > t; \end{cases}$$

(iii) $$f(u) = u^2/2, \qquad u(t,x) = \begin{cases} 2 & \text{for } x < t, \\ 0 & \text{for } x > t; \end{cases}$$

(iv) $$f(u) = -u^2, \qquad u(t,x) = \begin{cases} 1 & \text{for } x < 0, \\ -1 & \text{for } x > 0; \end{cases}$$

(v) $$f(u) = -u^2, \qquad u(t,x) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0; \end{cases}$$

(vi) $$f(u) = u^3, \qquad u(t,x) = \begin{cases} 1 & \text{for } x < 0, \\ -1 & \text{for } x > 0; \end{cases}$$

(vii) $$f(u) = u^3, \qquad u(t,x) = \begin{cases} -1 & \text{for } x < t, \\ 1 & \text{for } x > t; \end{cases}$$

(viii) $$f(u) = u^3, \qquad u(t,x) = \begin{cases} 1 & \text{for } x < t, \\ -1 & \text{for } x > t? \end{cases}$$

Exercise 4.2. *Construct some non-trivial generalized solutions in the strip* Π_T *for the equations*

(i) $u_t - (u^3)_x = 0,$

(ii) $u_t - u^2 \cdot u_x = 0,$

(iii) $u_t + \sin u \cdot u_x = 0,$

(iv) $u_t - (e^u)_x = 0,$

(v) $u_t + (e^u)_x = 0,$

(vi) $u_t + u_x/u = 0$

(by non-trivial, we mean a generalized solution that cannot be identified with a classical solution upon modifying its values on a set of Lebesgue measure zero).

4.3 Example of non-uniqueness of a generalized solution

It turns out that extending the notion of solution of equation (4.2) by replacing this equation with the integral identity (4.3) (let us stress again that this identity expresses in a generalized way the conservation law (4.4) for the vector field $\vec{v} = (u, f(u))$) may result in non-uniqueness of a generalized solution to a Cauchy problem. In order to observe this loss of uniqueness of a solution, let us consider equation (4.2) with the flux function $f(u) = u^2$ and with the zero initial datum:

$$u_t + 2uu_x = 0, \qquad x \in \mathbb{R}, \quad 0 < t < T, \tag{4.11}$$

$$u\big|_{t=0} = 0. \tag{4.12}$$

The function $u(t,x) \equiv 0$ is a classical solution, and thus it is also a generalized solution of the above problem. Nonetheless, we can construct non-zero generalized solutions of the problem considered. Assign (see Fig. 8)

$$u_\delta(t,x) = \begin{cases} 0 & \text{for } x < -\delta t, \\ -\delta & \text{for } -\delta t < x < 0, \\ +\delta & \text{for } 0 < x < +\delta t, \\ 0 & \text{for } x > +\delta t, \end{cases} \qquad \text{where } \delta > 0. \tag{4.13}$$

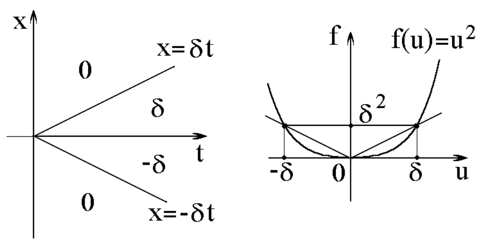

Figure 8. One-parameter family of "wrong" solutions.

Formula (4.13) defines the function $u_\delta = u_\delta(t,x)$ with four components of smoothness; on each of these, u_δ is a classical solution of equation (4.11) (it is clear that, in general, any constant satisfies equation (4.2) whatever be the flux function $f = f(u)$). Let us check the Rankine–Hugoniot condition on each of the three lines of discontinuity of the first kind (which are $x = 0$ and $x = \pm\delta t$):

as $x = 0$, we have $u_- = -\delta$, $u_+ = \delta$, and

$$\frac{dx}{dt} = 0 = \frac{\delta^2 - (-\delta)^2}{\delta - (-\delta)} = \frac{f(u_+) - f(u_-)}{u_+ - u_-};$$

as $x = -\delta t$, we have $u_- = 0$, $u_+ = -\delta$, and

$$\frac{dx}{dt} = -\delta = \frac{(-\delta)^2 - 0^2}{(-\delta) - 0} = \frac{f(u_+) - f(u_-)}{u_+ - u_-};$$

as $x = \delta t$, we have $u_- = \delta$, $u_+ = 0$, and

$$\frac{dx}{dt} = \delta = \frac{0^2 - \delta^2}{0 - \delta} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

Notice that, in the case of piecewise constant solutions, the Rankine–Hugoniot condition has a simple geometrical interpretation. Let us draw the graph of the flux function $f = f(u)$ respective to the axes (u, f), oriented parallel to the axes (t, x). Next, mark the points $(u_-, f(u_-))$ and $(u_+, f(u_+))$ on the graph (see Fig. 9). Then the segment connecting the two points must be parallel to the discontinuity line $x = x(t) = kt$. Indeed, the slope of this segment is equal to $\frac{f(u_+)-f(u_-)}{u_+-u_-}$, while the slope of the discontinuity line equals $\frac{dx}{dt} = k$; the equality between the two slopes is exactly what the Rankine–Hugoniot condition (4.5) expresses.

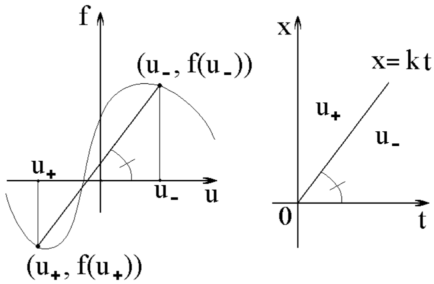

Figure 9. Geometrical interpretation of the Rankine–Hugoniot condition.

This geometrical point of view facilitates the graphical representation of the generalized solutions $u_\delta(t, x)$ of equation (4.11), as constructed above. Marking the points $(0, 0)$, $(\pm\delta, \delta^2)$ and joining them by segments in the way Fig. 8 shows, we obtain the slopes of the discontinuity lines in u_δ.

Exercise 4.3. *Construct a generalized solution of the problem* (4.11)–(4.12) *which is piecewise constant and has three discontinuity lines* $($*as in the solution* $u_\delta = u_\delta(t, x))$, *different from any of the solutions* (4.13). *For the solution constructed, verify analytically the Rankine–Hugoniot relation on all the discontinuity lines.*

Let us point out that it is not possible to construct a piecewise constant generalized solution of problem (4.11)–(4.12) with exactly two discontinuity lines. Indeed, such a solution would have two distinct jumps: a jump from the state 0 (on the left from the discontinuity line) to some constant state δ (on the right), and the jump from δ (now on the left) to 0 (now on the right). According to the Rankine–Hugoniot condition, these jumps can only occur along straight lines of the form $x = \frac{f(\delta)-f(0)}{\delta-0}t+C$, $C \in \mathbb{R}$. Since the solution also obeys the zero initial datum, the constant C should be the same for the two jumps. Thus both jumps cancel each other, because they occur along one and the same line; thus our piecewise constant solution is in fact equal to zero.

Exercise 4.4. *Construct piecewise constant generalized solutions of* (4.11)–(4.12) *with more than three discontinuity lines.*

Exercise 4.5. *Is it possible to construct a solution as in the previous exercise but with an even number of discontinuity lines, each of these lines being a ray originating from the point* $(0,0)$ *of the* (t,x)*-plane ?*

In order to construct a non-zero generalized solution of the Cauchy problem

$$u_t + (f(u))_x = 0, \qquad u\big|_{t=0} = 0, \tag{4.14}$$

with an arbitrarily chosen flux function $f = f(u)$, it is sufficient to pick two numbers α and β, $\alpha < 0 < \beta$, in such a way that the points $(0, f(0))$, $(\alpha, f(\alpha))$ and $(\beta, f(\beta))$ are not aligned. Then we join these points pairwise by straight line segments, as it was described above for the case $f(u) = u^2$ (see Fig. 8), and obtain the slopes of the discontinuity rays in the plane (t,x) for the solution to be constructed. Since $\alpha < 0 < \beta$, the slope of the segment joining $(\alpha, f(\alpha))$ with $(\beta, f(\beta))$ is always the intermediate one among the three slopes. Thus the construction produces a piecewise constant solution with the zero initial datum and the two intermediate states α, β.

Exercise 4.6. *Justify carefully that the above construction leads to a piecewise constant generalized solution of problem* (4.14). *Show that if, e.g.,* $0 < \alpha < \beta$*, then the analogous construction yields a non-trivial generalized solution with the initial datum* $u_0(x) \equiv \alpha$.

The above construction breaks down in the case where such non-aligned points on the graph of $f = f(u)$ cannot be found. This corresponds exactly to the case of an affine flux function, i.e., $f(u) = au + b$, $a, b \in \mathbb{R}$. In the latter case, our quasilinear problem is in fact linear:

$$u_t + au_x = 0, \qquad u|_{t=0} = u_0(x). \tag{4.15}$$

In the case where u_0 is smooth (this applies, in particular, to $u_0 \equiv 0$), the unique classical solution of this problem is easily constructed by the method of Section 2; the solution takes the form $u(t,x) = u_0(x - at)$.

Problem 4.4. *Show that for any piecewise smooth solution of equation* $u_t + au_x = 0$, $a =$ const*, the curves of discontinuity are the characteristics of the equation, i.e., the lines* $x = at + C$*. Then, prove the uniqueness of a piecewise smooth solution of the Cauchy problem* (4.15) *with a piecewise smooth initial datum* u_0*. Precisely, show that this solution is given by the equality* $u(t,x) = u_0(x - at)$.

It can be shown that this solution is unique not only within the class of classical solutions, but also within the class of generalized ones; but this is beyond the scope of these notes. In particular, the zero solution is the unique generalized solution of problem (4.14) in the case of a linear flux function $f = f(u)$.

Exercise 4.7. *Construct non-trivial generalized solutions of the problem* (4.14) *with* $f(u) = u^3$, *then with* $f(u) = \sin u$. *Is it possible to construct such solutions with more than three discontinuity lines ?*

It should be understood that, from the physical point of view, all the non-trivial generalized solutions to the problem (4.11)–(4.12) or to the problem (4.14) are "wrong"; notwithstanding the fact that these functions satisfy the PDE in the sense of the integral identity (4.3) and comply with the conservation law (4.4), the only "physically correct" solution of the above problems should be, unquestionably, the solution $u(t,x) \equiv 0$. Consequently, we should also devise a mathematical condition which would select, among all the generalized solutions, the unique "correct" solution. This condition, called the entropy increase condition, will now be formulated.

5 The notion of generalized entropy solution

As exposed in the previous sections, in the study of the Cauchy problem for the equation

$$u_t + \big(f(u)\big)_x = 0 \tag{5.1}$$

with the initial data

$$u\big|_{t=0} = u_0(x), \tag{5.2}$$

we encounter the following situation:

1) There exist some bounded smooth (infinitely differentiable) initial data u_0 such that the unique classical solution $u = u(t,x)$ remains a smooth function up to some critical instant of time T, but the limit

$$u(T,x) = \lim_{t \to T-0} u(t,x)$$

is only a piecewise smooth function with discontinuities of the first kind. The equation (5.1) is one of the so-called "hyperbolic" equations, and the smooth solutions of these equations are determined by the "information" propagated from the initial manifold along the characteristics. Thus it happens that this "information" itself leads to the appearance of discontinuities of the first kind. In this case, it is natural to expect that the solution remains discontinuous as well on some time interval $[T, T+\delta]$. This means that, in order to construct a nonlocal theory of the Cauchy problem (5.1)–(5.2), discontinuous solutions must be introduced into our consideration.

2) One natural approach for introducing such generalized solutions relies on the ideas of the theory of distributions (this approach was discussed in Section 4.1). Even

in a class as wide as the class of all locally bounded measurable functions in Π_T, one could consider generalized solutions $u = u(t, x)$ in the sense of the integral identity

$$\int_{\Pi_T} [u\varphi_t + f(u)\varphi_x]\,dx\,dt = 0, \tag{5.3}$$

which should hold for all “test” functions $\varphi \in C_0^\infty(\Pi_T)$; the initial datum (5.2) should be taken, say, “in the $L_{1,\mathrm{loc}}$ sense” (see (5.31) in Section 5.5 for the exact definition).

Nonetheless, as we have demonstrated in the previous section, so defined generalized solutions of the Cauchy problem may fail to be unique (even for the case $u_0(x) \equiv 0$). It is clear that the non-uniqueness stems from the fact that the “wrong” solutions u_δ, $\delta \neq 0$, have discontinuities. One could guess that not all the discontinuities are admissible; but how can we find the appropriate restrictions on the discontinuities?

5.1 Admissibility condition on discontinuities: the case of a convex flux function

Let us make the additional assumption

$$f'' \geqslant 0, \qquad f \in C^3(\mathbb{R}), \qquad u_0 \in C^2(\mathbb{R}).$$

Problem 5.1. *With the help of* (3.8) *or of* (3.12)*, using Problem* 3.2 *or Problem* 3.3*, show that in this case,* $u \in C^2(\Pi_T)$ *where* $[0, T)$ *is the maximal interval of existence of a classical solution.*

Now let us exploit the following consideration, which is purely mathematical: we try to reveal such properties of the smooth (for $t < T$) solutions that do not weaken (or which are conserved) while time approaches the critical value $t = T$. Such properties will therefore characterize the naturally arising singularities of a solution u. Denote $p = u_x(t, x)$ and differentiate the equation (5.1) in x. We have

$$0 = p_t + f'(u) \cdot p_x + f''(u) \cdot p^2 \geqslant p_t + f'(u)p_x\,.$$

Along any characteristics $x = x(t)$, $\dot{x} = f'\left(u(t, x(t)\right)$ (recall that the characteristics fill the whole domain Π_T of existence of a smooth solution), the latter inequality reads as

$$0 \geqslant p_t + \frac{dx}{dt}p_x = \frac{dp(t, x(t))}{dt}\,,$$

that is, the function p does not increase along the characteristics $x = x(t)$. Thus,

$$p(t, x(t)) \leqslant p(0, x(0)) = u_x(0, x(0)) \leqslant \sup_{x\in\mathbb{R}} u_0'(x) =: K_0\,.$$

Consequently, at any point $(t, x) \in \Pi_T$ there holds

$$p(t, x) = u_x(t, x) \leqslant K_0\,. \tag{5.4}$$

As the derivative $u_x(T,x)$ is not defined for some values of x, we pass to the following equivalent form of the inequality (5.4):

$$\frac{u(t,x_2)-u(t,x_1)}{x_2-x_1} \leqslant K_0 \qquad \forall x_1, x_2\,. \tag{5.5}$$

A similar inequality was introduced in the works of O. A. Oleĭnik (see [37]); the inequality played the role of the admissibility condition in the theory of generalized solutions. From (5.5) it follows that $u(t,x_2)-u(t,x_1) \leqslant K_0(x_2-x_1)$ for $x_1 < x_2$; thus at the limit as $x_2 \to x^* + 0$, $x_1 \to x^* - 0$, where x^* is a discontinuity point of $u(T,x)$, we have

$$u_+ = u(t,x^*+0) < u(t,x^*-0) = u_-\,. \tag{5.6}$$

(Rigorously speaking, passing to the limit implies $u_+ \leqslant u_-$, but $u_+ \neq u_-$ since we assumed that x^* is a discontinuity point.)

Let us require (5.6) to be satisfied at every point of discontinuity of a generalized solution $u = u(t,x)$ (the solution is assumed to be piecewise smooth). It is natural to interpret this condition as an *admissibility condition* on strong discontinuities (jumps) within the class of piecewise smooth solutions.

Remark 5.1. In the example of non-uniqueness exposed above (see Section 4.3) for the Cauchy problem (4.11)–(4.12), where we have $f''(u) = 2 > 0$, the solutions u_δ, $\delta > 0$, of the form (4.13) fail to verify the admissibility condition (5.6) on the discontinuity line $x = 0$. The unique admissible solution of this problem will be the function $u(t,x) \equiv 0$, which is the classical solution of the problem considered.

If $f''(u) \leqslant 0$, then substituting $u = -v$ into equation (5.1) we obtain the equation $v_t + (\tilde{f}(v))_x = 0$, where $\tilde{f}(v) \equiv -f(-v)$; notice that $\tilde{f}''(v) = -f''(-v) \geqslant 0$. For the solution $v = v(t,x)$ of the above equation, we should have $v_+ < v_-$, according to the admissibility condition (5.6). We conclude that in the case $f''(u) \leqslant 0$, the admissibility condition is the inequality $u_+ = -v_+ > -v_- = u_-$, converse to the inequality (5.6).

To summarize, for the case of a convex or a concave flux function $f = f(u)$, we have deduced the following condition for admissibility of discontinuities. Let u_-, respectively u_+, be the one-sided limit of a generalized solution $u = u(t,x)$ as the discontinuity curve is approached from the left, respectively from the right, along the x-axis. Then

- in the case of a convex function $f = f(u)$ (for instance, $f(u) = u^2/2, e^u, \ldots$), generalized solutions of equation (5.1) may have jumps from u_- to u_+ only when $u_- > u_+$;
- in the case of a concave function $f = f(u)$ ($f(u) = -u^2, \ln u, \ldots$), jumps from u_- to u_+ are only possible when $u_- < u_+$.

Let us provide a “physical” explanation of the admissibility condition obtained for the case where the monotonicity of f' is strict. At any point of an admissible discontinuity curve $x = x(t)$, consider the slopes $f'(u_+)$ and $f'(u_-)$ of the characteristics

$x = f'(u_\pm)t + C$ which impinge at this point from the two sides of the discontinuity. Consider also the slope $\omega = \frac{dx}{dt} = \frac{f(u_+)-f(u_-)}{u_+-u_-}$ of the discontinuity curve (more exactly, the slope of its tangent line); notice that ω is equal to the value $f'(\tilde{u})$ at some point $\tilde{u}$ which lies strictly between u_+ and u_-. These three slopes satisfy the so-called *Lax admissibility condition*

$$f'(u_+) < \omega = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = f'(\tilde{u}) < f'(u_-). \tag{5.7}$$

Indeed, if f is strictly convex, then f' is a monotone increasing function, and the admissibility condition for this case of a convex flux function f ensures that $u_+ < \tilde{u} < u_-$. Similarly, if f is strictly concave, then the admissibility condition yields $u_+ > \tilde{u} > u_-$, so that we get (5.7) again, since f' is a monotone decreasing function in this case.

Condition (5.7) is a particular case of the admissibility condition which is fundamental for the theory of systems of conservation laws. It was first formulated by the American mathematician P. D. Lax (see [30]).

Therefore, we observe that, as t grows, the characteristics approach the discontinuity curve from both sides (see Fig. 10a); none of the two characteristics can move away from it (the case where the characteristics move away from the discontinuity curve as t grows is depicted in Fig. 10b). This means that those discontinuities are admissible which are due to the fact that characteristics of a smooth solution (smooth from each side of the discontinuity curve) tend to have intersections as t grows (the intersections eventually occur on the discontinuity curve). On the contrary, the situation when the discontinuity curve is "enforced", with some of the characteristics originating out of the discontinuity curve as time grows, is not admissible.

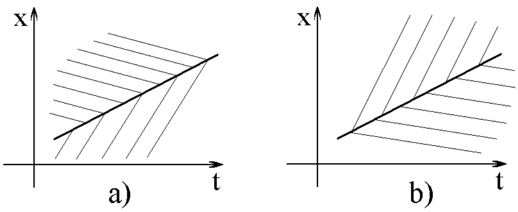

Figure 10. Lax condition: admissible and non-admissible discontinuity curves.

Example 5.2. Let us illustrate the above statement with the example of the Hopf equation (1.1), i.e., the equation (5.1) with $f(u) = u^2/2$. This equation describes the displacement of freely moving particles (see Section 1). Assume that the particles situated, at the initial instant of time, in a neighbourhood of $+\infty$ (i.e., particles with the x-coordinate larger than some sufficiently large value), move with a velocity u_+; assume that the particles initially located in a neighbourhood of $-\infty$ have a velocity u_-; and let $u_+ < u_-$. The latter constraint means that, as time passes, collisions are inevitable, and eventually, a shock wave will form. The velocity of propagation of this shock wave created by particle collisions will be equal to

$$\omega = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = \frac{u_+^2/2 - u_-^2/2}{u_+ - u_-} = \frac{u_+ + u_-}{2}.$$

When the initial velocity profile is a monotone non-increasing function, it can be justified that for sufficiently large t, we obtain a generalized solution of the Hopf equation of the following form:

$$u(t,x) = \begin{cases} u_- & \text{for } x < \omega t + C, \\ u_+ & \text{for } x > \omega t + C. \end{cases} \tag{5.8}$$

This solution can be interpreted as follows. The particles with velocities u_- and u_+ collide when the quicker one (with the velocity u_-) overtakes the slower one (of velocity u_+); this collision is not elastic, and the two particles agglomerate into one single particle. After the collision, the particles continue to move with the velocity $(u_+ + u_-)/2$, creating a shock wave. The velocity of propagation of this wave is calculated with the help of the law of momentum conservation: this velocity is the arithmetic mean of the particles' velocities before the collision. Let us point out that such collisions induce a lost of the kinetic energy of the particles (we will further discuss this question later).

If, on the contrary, the speeds of the particles near $+\infty$ and near $-\infty$ were related by the inequality $u_+ > u_-$ and if the initial velocity distribution were a smooth monotone non-decreasing function, then no collision of particles would ever occur: at any time instant $t > 0$, the velocity distribution $u(t,\cdot)$ would be a smooth non-decreasing function as at the time $t = 0$, and no shock wave might form (see Section 3.1). Therefore, in the case $u_+ > u_-$, the function u given by (5.8), although it does satisfy the integral identity (5.3), is not a physically correct solution of the Hopf equation.

5.2 The vanishing viscosity method

In order to generalize the admissibility condition of the previous section to the case of a flux function $f = f(u)$ which is neither convex nor concave, we make the following observation and reformulate this condition in the terms of the respective location of the graph and the chords of convex or concave functions. We see that the jump between u_- and u_+ is admissible in the sense of the previous section if $u_- > u_+$ (respectively, $u_- < u_+$) and the graph of the flux function f is situated under the chord (respectively, above the chord) joining the points $(u_-, f(u_-))$ and $(u_+, f(u_+))$ (see Fig. 11).

It turns out that the above reformulation of the admissibility rule for convex/concave flux functions remains appropriate for the case of an arbitrary flux function f.

For a rather rigorous justification of this statement, let us use "physical" (more exactly, "fluid dynamics") considerations based on the concepts of an ideal gas and a viscous gas. If $x = x(t)$ is the trajectory of a particle of an ideal gas in a tube aligned with the x-axis, and if the function $u = u(t,x)$ represents the velocity of the particle that occupies the space location x at the time instant t, then (see Section 1)

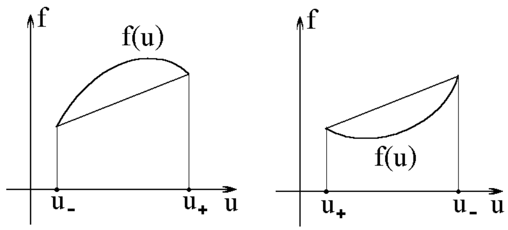

Figure 11. Visualization of admissible jumps, I.

$\dot{x}(t) = u\,(t, x(t))$, $\ddot{x}(t) = \frac{du}{dt} = 0$; this calculation previously led us to the Hopf equation (1.1). But, ideal gases "do not exist"; they only exist theoretically, as limits when the viscosity of a real gas is neglected because of its smallness.

If $\varepsilon > 0$ is the viscosity coefficient of a real gas, then (under certain assumptions) the force of viscous friction which acts on the particle $x(t)$ at time t and relative to the mass unit can be taken to be $\varepsilon u_{xx}(t, x(t))$. Then $\ddot{x} = \frac{du}{dt} = \varepsilon u_{xx}$, and instead of the Hopf equation we obtain the so-called Burgers equation[5]

$$u_t + uu_x = \varepsilon u_{xx}\,. \tag{5.9}$$

It is natural to admit that — this is what actually takes place — all admissible generalized solutions of the Hopf equation can be obtained as the limit of some solutions $u^\varepsilon = u^\varepsilon(t, x)$ of the equation (5.9) as the viscosity coefficient ε tends to 0. The procedure of introducing the term εu_{xx} into a first-order equation and the subsequent study of the limits of the solutions u^ε as $\varepsilon \to +0$ is called the *"vanishing viscosity" method*.

Before we continue with the application of the vanishing viscosity method to a justification of the general admissibility condition formulated above, let us point out an important method of "linearization" (in a sense) of the Burgers equation (5.9). Observe that we have $u_t = (\varepsilon u_x - u^2/2)_x$; thus we can introduce a potential $U = U(t, x)$, determined from the equality

$$dU = u\,dx + \big(\varepsilon u_x - u^2/2\big)\;dt.$$

In this case

$$U_x = u, \qquad U_t = \varepsilon u_x - u^2/2 = \varepsilon U_{xx} - (U_x)^2/2,$$

i.e., the function U satisfies the equation

$$U_t + \frac{1}{2}(U_x)^2 = \varepsilon U_{xx}\,. \tag{5.10}$$

In (5.10), let us make the substitution $U = -2\varepsilon \ln z$. Then

$$U_t = -2\varepsilon\frac{z_t}{z}, \quad U_x = -2\varepsilon\frac{z_x}{z}, \quad U_{xx} = -2\varepsilon\frac{z_{xx}}{z} + 2\varepsilon\frac{(z_x)^2}{z^2}.$$

[5] *NT* — In the western literature, it is customary to call this equation, "the Burgers equation with viscosity"; accordingly, the term "Burgers equation" then designs what is called the Hopf equation in our lectures.

Equation (5.10) then rewrites as

$$-2\varepsilon\frac{z_t}{z} + 2\varepsilon^2\frac{(z_x)^2}{z^2} = -2\varepsilon^2\frac{z_{xx}}{z} + 2\varepsilon^2\frac{(z_x)^2}{z^2},$$

so that we are reduced to a linear equation for the function $z = z(t,x)$, which is the classical heat equation:

$$z_t = \varepsilon z_{xx}\,. \tag{5.11}$$

Remark 5.3. The linearization method pointed out hereabove was first used by the Russian mechanicist V. A. Florin in 1948 in his investigation of a physical application. Later on, in the 1950th, this method was rediscovered by the American scholars E. Hopf and S. Cole; nowadays the transformation is often named after them (it would be more correct to speak about the Florin–Hopf–Cole transformation).

It follows from the above substitution that a solution of equation (5.9) has the form

$$u = U_x = -2\varepsilon\frac{z_x}{z}\,,$$

where $z = z(t,x)$ is a solution of the heat equation (5.11).

As is well-known from the theory of second-order linear PDEs, solutions of the Cauchy problem for the heat equation (5.11), even with initial data that are only piecewise continuous, become infinitely differentiable for $t > 0$. Hence, solutions of the Burgers equation (5.9) are also infinitely differentiable functions, and, consequently, they cannot include shock waves.

Now assume that the so-called "simple wave", given by

$$u(t,x) = u_- + \frac{u_+ - u_-}{2}[1 + \mathrm{sign}(x-\omega t)] = \begin{cases} u_- & \text{for } x < \omega t, \\ u_+ & \text{for } x > \omega t, \end{cases} \tag{5.12}$$

where $\omega =$ const, is a generalized solution of equation (5.1) in the sense of the integral identity (5.3). For this to hold, it is necessary and sufficient that the Rankine–Hugoniot condition

$$\omega \equiv \frac{dx}{dt} = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \tag{5.13}$$

holds on the discontinuity line $x(t) = \omega t$.

For this case, the idea of the vanishing viscosity method can be applied as follows. Let us consider a solution $u = u(t,x)$ of the form (5.12) as admissible, if it can be obtained as a pointwise limit (for $x \neq \omega t$) of solutions $u^\varepsilon = u^\varepsilon(t,x)$ of the equation

$$u^\varepsilon_t + \big(f(u^\varepsilon)\big)_x = \varepsilon u^\varepsilon_{xx} \tag{5.14}$$

as $\varepsilon \to +0$. (The approach developed below has been suggested by I. M. Gel'fand [18]).

Taking into account the special structure of the solution $u = u(t,x)$, let us seek a solution of (5.14) under the form

$$u^\varepsilon(t,x) = v(\xi), \qquad \xi = \frac{x - \omega t}{\varepsilon}. \tag{5.15}$$

Substituting this ansatz into equation (5.14), we infer that the function $v = v(\xi)$ satisfies the equation

$$-\omega v' + (f(v))' = v''. \tag{5.16}$$

On the other hand, it is clear that the function $u^\varepsilon = v\left(\frac{x-\omega t}{\varepsilon}\right)$ converges pointwise (for $x \neq \omega t$) to a function $u = u(t,x)$ of the form (5.12) as $\varepsilon \to +0$ if and only if the function $v = v(\xi)$ satisfies the boundary conditions

$$v(-\infty) = u_- , \qquad v(+\infty) = u_+ . \tag{5.17}$$

Remark 5.4. One cannot hope for uniqueness of such a function $v = v(\xi)$. Indeed, if v is a solution of the problem (5.16)–(5.17), then the functions $\tilde{v} = v(\xi - \xi_0)$ are also solutions of this problem, for all $\xi_0 \in \mathbb{R}$.

Integrating (5.16), we obtain

$$v' = -\omega v + f(v) + C = F(v) + C, \qquad C = \text{const}. \tag{5.18}$$

The ODE (5.18) is autonomous, of first-order, and its right-hand side $F(v) + C$ is smooth; thus (5.18) admits a solution which tends to constant states u_- (as $\xi \to -\infty$) and u_+ (as $\xi \to +\infty$) if and only if the following conditions are satisfied:

(i) u_- and u_+ are stationary points of this equation, i.e., the right-hand side of equation (5.18) is zero at these points:

$$F(u_-) + C = F(u_+) + C = 0,$$

so that $C = -F(u_-) = -F(u_+)$. Upon rewriting the equality $F(u_-) = F(u_+)$ under the form $f(u_-) - \omega u_- = f(u_+) - \omega u_+$, we see that it coincides with the Rankine–Hugoniot condition (5.13).

(ii) There is no stationary point in the open interval between u_- and u_+; moreover, the right-hand side $F(v) - F(u_-) = F(v) - F(u_+)$ of (5.18) restricted to this interval should be

a) positive if $u_- < u_+$ (then the solution increases):

$$F(v) - F(u_-) > 0 \quad \forall v \in (u_-, u_+) \quad \text{if } u_- < u_+; \tag{5.19}$$

b) negative if $u_- > u_+$ ($v = v(\xi)$ decreases):

$$F(v) - F(u_+) < 0 \quad \forall v \in (u_+, u_-) \quad \text{if } u_+ < u_-. \tag{5.20}$$

When the above conditions are satisfied, the solutions of equation (5.16) with the desired boundary behaviour are given by the formula

$$\int_{v_0}^{v} \frac{dw}{F(w) - F(u_-)} = \xi - \xi_0, \qquad v_0 = \frac{u_+ + u_-}{2}.$$

Our point is that the relations (5.19)–(5.20) express analytically the admissibility condition.

Now let us interpret this condition geometrically. Substituting $F(v) = f(v) - \omega v$ into (5.19) and (5.20), we have

$$f(v) - f(u_-) > \omega(v - u_-) \quad \forall v \in (u_-, u_+) \quad \text{if} \quad u_- < u_+,$$
$$f(v) - f(u_+) < \omega(v - u_+) \quad \forall v \in (u_+, u_-) \quad \text{if} \quad u_+ < u_-,$$

which, in view of the Rankine–Hugoniot condition (5.13), amounts to

$$\frac{f(u) - f(u_-)}{u - u_-} > \omega = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \quad \forall u \in (u_-, u_+) \quad \text{if} \quad u_- < u_+, \tag{5.19′}$$

$$\frac{f(u) - f(u_+)}{u - u_+} < \omega = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \quad \forall u \in (u_+, u_-) \quad \text{if} \quad u_+ < u_-. \tag{5.20′}$$

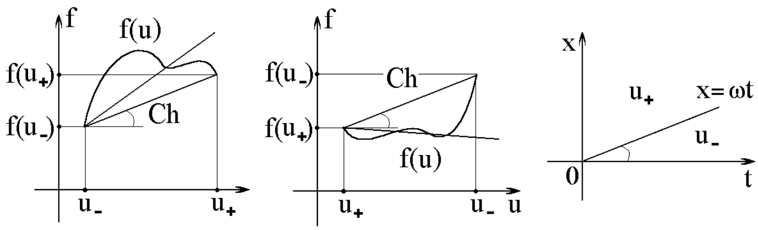

Figure 12. Visualization of admissible jumps, II.

Let us represent the graph of a flux function $f = f(u)$ (see Fig. 12). Condition (5.19′) means that the chord *Ch* with the endpoints $(u_-, f(u_-)), (u_+, f(u_+))$ has a smaller slope (the slope is measured as the inclination of the chord with respect to the positive direction of the u-axis) than the slope of the segment joining the point $(u_-, f(u_-))$ with the point $(u, f(u))$, where u runs over the interval (u_-, u_+)). Consequently, the point $(u, f(u))$ and thus the whole graph of $f = f(u)$ on the interval (u_-, u_+) lies above the chord *Ch*. In the same way, condition (5.20′) signifies that the graph of $f = f(u)$ for $u \in (u_+, u_-)$ is situated below the chord *Ch*.

Remark 5.5. Upon varying the values u_-, u_+ and also the function $f = f(u)$, one can construct different convergent sequences of admissible generalized solutions of the form (5.15). It is natural to consider as admissible also the pointwise limits of the admissible solutions. Therefore, it is clear that any situation where the graph of $f = f(u)$ touches the chord *Ch* should also be considered as admissible.

In conclusion, we obtain that a solution u of the equation (5.1) may have a jump from u_- to u_+ (a jump in the direction of increasing x) when the following **jump admissibility condition** holds:

- in the case $u_- < u_+$, the graph of the function $f = f(u)$ on the segment $[u_-, u_+]$ is situated **above** the chord (in the non-strict sense) with the endpoints $(u_-, f(u_-))$ and $(u_+, f(u_+))$;
- in the case $u_- > u_+$, the graph of the function $f = f(u)$ on the segment $[u_+, u_-]$ is situated **below** the chord (in the non-strict sense) with the endpoints $(u_-, f(u_-))$ and $(u_+, f(u_+))$.

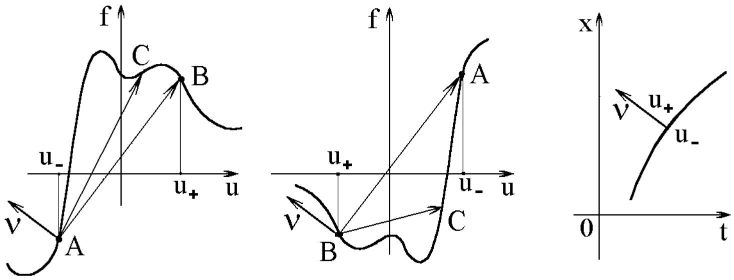

Figure 13. Visualization of admissible jumps, III.

Let us give another analytical expression of the condition obtained. Consider a curve on which the solution has a jump from u_- to u_+. In coordinates (u, f) we draw the graph of the function $f = f(u)$ on the interval between u_- and u_+ and the chord joining the endpoints of this graph. As in Fig. 8 and Fig. 9 (see Section 4.3), we mean that the axes (u, f) are aligned with the axes (t, x). Now on the same graph, let us situate the unit normal vector $\nu = (\cos(\nu, t), \cos(\nu, x))$ to the discontinuity curve (see Fig. 13). Introduce the points $A = (u_-, f(u_-))$, $B = (u_+, f(u_+))$, and let the point $C = (u, f(u))$ run along the graph. The vector ν is orthogonal to the vector $\overrightarrow{AB}$ (this is an expression of the Rankine–Hugoniot condition (5.13)) and is oriented "upwards", i.e., $\cos(\nu, x) > 0$ (this is because we have chosen the normal which forms an acute angle with the positive direction of the x-axis). The condition stating that the graph of the function $f = f(u)$ on the interval between u_- and u_+ is located over the chord ("over", in the non-strict sense) means exactly that the angle between the vectors $\overrightarrow{AC}$ (or, equivalently, $\overrightarrow{BC}$) and ν does not exceed $\pi/2$, that is, the scalar product $(\overrightarrow{AC}, \nu)$ of these vectors is nonnegative. Thus for the case $u_- < u_+$, we have

$$(u - u_-)\cos(\nu, t) + (f(u) - f(u_-))\cos(\nu, x) \geqslant 0 \qquad \forall u \in (u_-, u_+). \tag{5.21}$$

Similarly, the condition stating that the graph is located under the chord ("under", in the non-strict sense) means that the angle between the same vectors as before is greater than or equal to $\pi/2$, that is, the scalar product $(\overrightarrow{BC}, \nu)$ of these vectors is non-positive. Thus for the case $u_- > u_+$, we have

$$(u - u_+)\cos(\nu, t) + (f(u) - f(u_+))\cos(\nu, x) \leqslant 0 \qquad \forall u \in (u_+, u_-). \tag{5.22}$$

Remark 5.6. The admissibility conditions deduced with the vanishing viscosity approach agree perfectly with the conditions obtained in the previous section for the case of a convex/concave flux function $f = f(u)$. Indeed the convexity (respectively, the concavity) of a function means, by definition, that the chord joining two arbitrary points of the graph of the function lies above (respectively, lies below) the graph itself.

In the sequel of these lectures, unless an additional precision is given, by a *solution* of equation (5.1) we will tacitly mean a piecewise smooth function that satisfies the integral identity (5.3) and, in addition, the admissibility condition formulated in the present section.

Exercise 5.1. *Examine the question of admissibility of each of the jumps* $($*jumps satisfying the Rankine–Hugoniot condition* $(5.13))$ *present in the solutions* $u = u(t, x)$ *to the corresponding equations of the form* (5.1):

(i) *for the generalized solutions* $u = u(t, x)$ *given in Exercise* 4.1;

(ii) *for the generalized solutions* $u = u(t, x)$ *constructed in Exercise* 4.2;

(iii) *for the generalized solutions* $u = u(t, x)$ *constructed in Exercise* 4.7.

5.3 The notion of entropy and irreversibility of processes

The jump admissibility conditions obtained in the previous sections are often called entropy-increase type conditions.[6] Where does this name come from? The reason is, the equations we study model nonlinear physical phenomena (called "processes" in the sequel) which are time-irreversible, and the function which characterizes this irreversibility is called "entropy".

The Hopf equation (1.1) is, certainly, the simplest model for the displacement of a gas in a tube; in more correct (more precise) models, also the pressure of the gas is present, moreover, the density of the gas enters the equations when the gas is compressible. The entropy function S is expressed with the help of the two latter quantities characterizing the gas, namely the pressure and the density. In the field of fluid dynamics, already in the 19th century it has been known that the entropy function does not decrease in time across the front of a shock wave Γ:

$$S_+ = S(t+0, x) \geqslant S_- = S(t-0, x), \qquad (t, x) \in \Gamma. \tag{5.23}$$

Therefore, all the inequalities that express irreversibility of processes in nature are called "inequalities of the entropy increase type". For the simplest gas dynamics equation, which is the Hopf equation, the role of entropy is played by the kinetic energy of the particle located at the point x at the time instant t:

$$S(t, x) \equiv \frac{1}{2} u^2(t, x).$$

[6] *NT* — In the literature on conservation laws, one often speaks of "entropy dissipation conditions". This term refers to the inequalities such as (5.28), (5.30) or (5.42) below. Each of these inequalities states the decrease (the *dissipation*) and not the increase of another quantity related to various functions called "entropies".

Let us show that inequality (5.23) for this "entropy" function S does hold across an admissible shock wave.

For the case of the Hopf equation (i.e., for $f(u) = u^2/2$), the Rankine–Hugoniot condition (5.13) has the form

$$\frac{u_- + u_+}{2} = \frac{dx}{dt}. \tag{5.24}$$

Since the flux function $f(u) = u^2/2$ is convex, the jump admissibility condition reduces to the inequality

$$u_- - u_+ > 0. \tag{5.25}$$

If $dx/dt > 0$, then (according to Fig. 14) we have $S_- = u_+^2/2$ and $S_+ = u_-^2/2$. Multiplying inequality (5.25) by the expression $(u_- + u_+)/2$ (this expression is positive thanks to (5.24)), we have $(u_-^2 - u_+^2)/2 > 0$, thus $S_- < S_+$.

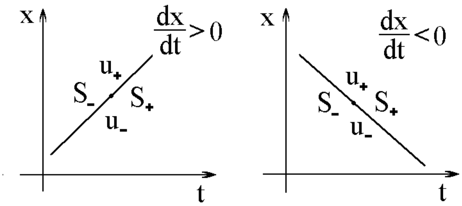

Figure 14. Increase of S for the Hopf equation.

Similarly, if $dx/dt < 0$, then (see Fig. 14)

$$S_- = \frac{1}{2}(u_-)^2 < \frac{1}{2}(u_+)^2 = S_+.$$

5.4 Energy estimates

Let us provide another characterization of irreversibility for equation (5.1), a characterization which has a clear physical meaning. Consider the full kinetic energy of the particle system under consideration:

$$E(t) = \int_{-\infty}^{+\infty} \frac{1}{2} u^2(t,x)\, dx. \tag{5.26}$$

For smooth (and, say, compactly supported) initial data, there exists a classical solution u of problem (5.1)–(5.2) on some time interval $[0, T)$, $T > 0$; moreover, for all fixed t, this solution has compact support in x. In the present section, we will only consider those solutions u of equation (5.1) for which the kinetic energy (5.26) is finite (this holds, e.g., in the above situation where $u = u(t,x)$ is of compact support in the variable x).

Proposition 5.7. *For classical solutions of equation* (5.1) *there holds*

$$E(t) \equiv \text{const},$$

i.e., the kinetic energy (5.26) *is a* first integral *of the equation* (5.1).

Proof. Since we have assumed that $u(t, \pm\infty) = 0$, we have

$$\begin{aligned}\frac{dE}{dt} &= \int_{-\infty}^{+\infty} uu_t\,dx = -\int_{-\infty}^{+\infty} u(f(u))_x\,dx \\ &= -uf(u)\Big|_{x=-\infty}^{x=+\infty} + \int_{-\infty}^{+\infty} f(u)u_x\,dx = \int_{u(t,-\infty)}^{u(t,+\infty)} f(u)\,du = 0.\end{aligned}$$ □

Now consider the corresponding equation with viscosity:

$$u_t^\varepsilon + (f(u^\varepsilon))_x = \varepsilon u_{xx}^\varepsilon \tag{5.27}$$

Proposition 5.8. *Let* $u^\varepsilon \not\equiv 0$ *be a solution of equation* (5.27) *such that, in addition,* u^ε, u_x^ε, *and* u_{xx}^ε *decay to zero as* $x \to \pm\infty$ *at a sufficiently high rate, and uniformly in* t. *Then the full kinetic energy* $E = E(t)$ *of this solution is a decreasing function of time.*

Proof. As in the proof of the previous proposition, we find

$$\begin{aligned}\frac{dE}{dt} &= \int_{-\infty}^{+\infty} u^\varepsilon u_t^\varepsilon\,dx \\ &= \int_{-\infty}^{+\infty} u^\varepsilon\left(\varepsilon u_{xx}^\varepsilon - (f(u^\varepsilon))_x\right)\,dx = -\varepsilon\int_{-\infty}^{+\infty} (u_x^\varepsilon)^2\,dx \leqslant 0.\end{aligned} \tag{5.28}$$

Notice that we have the equality sign in (5.28) only in the case of a function u^ε that is constant in x. Since we assume that this function decays to zero as $x \to \infty$, we have $dE/dt < 0$ unless $u^\varepsilon \equiv 0$. □

Recall (see Section 5.2) that admissible generalized entropy solutions u of equation (5.1) were obtained as limits of solutions u^ε of equations (5.27); on the latter solutions, the kinetic energy is dissipated. Therefore, it can be expected that also on the limiting solutions u, the kinetic energy does not increase with time.

Proposition 5.9. *Assume that* $u = u(t, x)$ *is a piecewise smooth admissible generalized entropy solution of equation* (5.1) *with one curve of jump discontinuity* $x = x(t)$. *Then the speed of decrease of the kinetic energy* $E = E(t)$ *of this solution is equal, at any instant of time* $t = t_0$, *to the area* $S(t_0)$ *delimited by the graph of the flux function* $f = f(u)$ *on the segment* $[u_-, u_+]$ (*or on the segment* $[u_+, u_-]$) *and by the chord joining the endpoints* $(u_-, f(u_-))$ *and* $(u_+, f(u_+))$ *of this graph* (*see Fig.* 15):

$$\frac{dE}{dt}(t_0) = -S(t_0). \tag{5.29}$$

As previously, by $u_\pm = u_\pm(t_0)$ *we denote the one-sided limits (as* $x \to x(t_0)$*) of the function* $x \mapsto u(t_0, x)$ *as the point approaches the discontinuity position* $x(t_0)$.

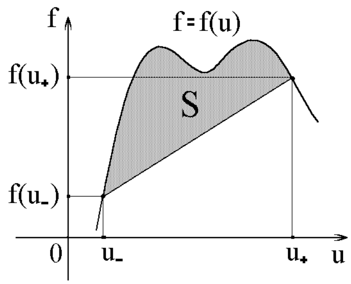

Figure 15. Area that determines the energy decrease rate.

Proof. To be specific, consider the case where $u_- < u_+$ and, consequently, the graph of the function $f = f(u)$ on the segment $[u_-, u_+]$ lies above the corresponding chord. Then

$$S = \int_{u_-}^{u_+} f(u)\,du - \frac{f(u_+) + f(u_-)}{2}(u_+ - u_-).$$

On the other hand,

$$\frac{dE}{dt} = \frac{d}{dt}\int_{-\infty}^{+\infty} \frac{1}{2}u^2(t,x)\,dx = \frac{d}{dt}\left(\int_{-\infty}^{x(t)} \frac{1}{2}u^2(t,x)\,dx + \int_{x(t)}^{+\infty} \frac{1}{2}u^2(t,x)\,dx\right)$$

$$= \frac{1}{2}u_-^2 \cdot \dot{x}(t) + \int_{-\infty}^{x(t)} uu_t(t,x)\,dx - \frac{1}{2}u_+^2 \cdot \dot{x}(t) + \int_{x(t)}^{+\infty} uu_t(t,x)\,dx$$

$$= \frac{u_-^2 - u_+^2}{2} \cdot \dot{x}(t) - \int_{-\infty}^{x(t)} u\,(f(u))_x\;dx - \int_{x(t)}^{+\infty} u\,(f(u))_x\;dx$$

$$= \frac{u_-^2 - u_+^2}{2} \cdot \dot{x}(t) - uf(u)\Big|_{x=-\infty}^{x=x(t)} + \int_{-\infty}^{x(t)} f(u)u_x\,dx$$

$$- uf(u)\Big|_{x=x(t)}^{x=+\infty} + \int_{x(t)}^{+\infty} f(u)u_x\,dx.$$

Thanks to the Rankine–Hugoniot condition (5.13) and taking into account the fact that $u(t,\pm\infty) = 0$, we have

$$\frac{dE}{dt} = \frac{u_-^2 - u_+^2}{2} \cdot \frac{f(u_+) - f(u_-)}{u_+ - u_-} - u_- f(u_-) + \int_0^{u_-} f(u)\,du + u_+ f(u_+) + \int_{u_+}^{0} f(u)\,du$$

$$= u_+ f(u_+) - u_- f(u_-) - \frac{(u_+ + u_-)(f(u_+) - f(u_-))}{2} - \int_{u_-}^{u_+} f(u)\,du$$

$$= \frac{(u_+ - u_-)(f(u_+) + f(u_-))}{2} - \int_{u_-}^{u_+} f(u)\,du = -S.$$

□

Remark 5.10. If the solution contains several shock waves (i.e., several jump discontinuities), then *on each of the discontinuity curves* the energy is lost (dissipated) according to the inequality (5.29). (The proof of this fact is left to the reader.)

Conclusion. We see that, according to Proposition 5.7, we have $E(t) = \text{const} = E(0)$ on smooth solutions $u = u(t,x)$ of the equation (5.1), up to the critical instant of time T (the instant when singularities arise in the solutions), i.e., up to the time T there is no dissipation of the kinetic energy; the kinetic energy stays constant on $[0,T)$.

However, when shock waves appear, according to (5.29), we have

$$\frac{dE}{dt} < 0,$$

so that the kinetic energy dissipates (on a shock wave, a part of it is transformed into heat). Consequently, the evolution of admissible generalized solutions with shock waves is related to the decrease of the kinetic energy; this is what makes the physical processes modelled by equation (5.1) irreversible.

The readers who sometimes spend vacations at the sea are probably acquainted with this phenomenon. Near the shore, if the sea is calm and the waves are temperate, the sea temperature near the surface is almost the same as the air temperature above. When the wind becomes stronger, waves become foamy, turbulent structures occur; these "broken waves" can be seen as shock waves on the sea surface. In this case, after some time, one can observe that the temperature of the surface layer of the sea has become higher than the air temperature. This heating phenomenon is conditioned by the heat production that occurs on the shock waves.

From the purely mathematical point of view, this situation stems from the fact that equation (5.1) does not change under the simultaneous change of t into $-t$ and of x into $-x$ (similarly, any of the shift transformations along the axes, namely $x \to x - x_0$ or $t \to t - T$, does not change the equation); in this case, it is said that the equation remains invariant under the corresponding transformation. Consequently, along with any *smooth*, as $t < T$, solution $u = u(t,x)$ of equation (5.1), the transformed function $\tilde{u}(t,x) \equiv u(T-t,-x)$ will also be a *smooth* solution of the same equation.

The same property holds for generalized solutions (in the sense of integral equality (5.3); the admissibility condition is not required), because the identity (5.3) is invariant under the same transformations.

If, on the contrary, $u = u(t,x)$ is an *admissible discontinuous generalized* solution of equation (5.1), then the corresponding function $\tilde{u}$ **will not be** an *admissible generalized ("entropy")* solution of the equation considered. This is because the entropy increase condition is not invariant under the transformation which includes the time reversal (the entropy increase condition is then replaced by the converse entropy decrease condition). Therefore, the simultaneous change of t into $T-t$ and of x into $-x$ is not allowed in the presence of discontinuous solutions. Hence, an admissible discontinuous generalized solution $u = u(t,x)$ is transformed into the non-admissible ("wrong") discontinuous generalized solution $\tilde{u}(t,x) \equiv u(T-t,-x)$.

5.5 Kruzhkov's definition of a generalized solution

In the preceding sections we discussed the requirements which one should impose on jumps (i.e., discontinuities of the first kind occurring along smooth curves) of generalized solutions (in the sense of the integral identity (5.3)) of equation (5.1). However, this kind of restrictions is only meaningful for piecewise smooth functions; in this case the notion of a jump, i.e., a discontinuity curve with one-sided limits of a solution on this curve, is meaningful. In contrast, while defining a generalized solution $u = u(t, x)$ of this equation in the sense of the integral identity (5.3), we only need that the integrals in (5.3) make sense. Clearly, the latter assumption is by far less restrictive compared with the assumption of piecewise smoothness of the function $u = u(t, x)$. Therefore, a natural question arises, namely, how could one define an admissible generalized solution to the Cauchy problem (5.1)–(5.2), so that the new notion includes both the integral identity and a condition of the entropy increase type (we need some generalization of the entropy increase conditions stated above as we want to extend them to solutions which may not be piecewise smooth). The answer to this question was given by S. N. Kruzhkov (see [25, 26]), and the answer applies not only to the problem we consider in these lectures but also to a wider class of equations and systems. In the same works of S. N. Kruzhkov, the existence and uniqueness of an admissible generalized solution, in the sense of the new definition, was proved.

Let us now give the aforementioned definition. One of the widest spaces of functions in which generalized solutions of our problem can be searched is the space of bounded measurable functions $u = u(t, x)$ defined in the strip $\Pi_T = [0, T) \times \mathbb{R}_x$.

Definition 5.11. A bounded measurable function $u = u(t, x) : \Pi_T \to \mathbb{R}$ is called a *generalized entropy solution*[7] (in the sense of Kruzhkov) of the problem (5.1)–(5.2) if

(i) for any constant $k \in \mathbb{R}$ and any nonnegative test function $\varphi = \varphi(t, x) \in C_0^\infty(\Pi_T)$ there holds the inequality

$$\int_{\Pi_T} \left[|u - k|\varphi_t + \text{sign}(u - k)\big(f(u) - f(k)\big)\varphi_x \right] \, dx\, dt \geqslant 0; \tag{5.30}$$

(ii) there holds $u(t, \cdot) \to u_0$ as $t \to +0$ in the topology of $L_{1,\text{loc}}(\mathbb{R})$, i.e.,

$$\forall\, [a, b] \subset \mathbb{R}, \qquad \lim_{t \to +0} \int_a^b |u(t, x) - u_0(x)| \, dx = 0. \tag{5.31}$$

Proposition 5.12. *If a function $u = u(t, x)$ is a generalized entropy solution in the sense of Definition* 5.11 *of problem* (5.1)–(5.2)*, then it is also a generalized solution of equation* (5.1) *in the sense of the integral identity* (5.3).

Proof. Note that the function taking everywhere a constant value k is a classical solution and, therefore, it is also a generalized solution of equation (5.1). It follows that for

[7] *NT* — The western literature refers to "Kruzhkov entropy solutions" or merely to "entropy solutions".

any test function $\varphi \in C_0^\infty(\Pi_T)$, there holds

$$\int_{\Pi_T} [k\varphi_t + f(k)\varphi_x] \, dx\, dt = 0. \tag{5.32}$$

This identity can also be checked by a direct calculation.

Choose a value $k >$ ess-sup $_{(t,x)\in\Pi_T} u(t,x)$ in (5.30) . We have

$$\int_{\Pi_T} \big[(k-u)\varphi_t + (f(k) - f(u))\varphi_x\big] \, dx\, dt \geqslant 0$$

for any function $\varphi \in C_0^\infty(\Pi_T)$, $\varphi(t,x) \geqslant 0$. Taking into account (5.32), we conclude that

$$-\int_{\Pi_T} \big[u\varphi_t + f(u)\varphi_x\big] \, dx\, dt \geqslant 0. \tag{5.33}$$

Then taking $k <$ ess-inf $_{(t,x)\in\Pi_T} u(t,x)$, we obtain in the same way

$$\int_{\Pi_T} [u\varphi_t + f(u)\varphi_x] \, dx\, dt \geqslant 0. \tag{5.34}$$

Comparing the inequalities (5.33) and (5.34), we arrive at the equality

$$\int_{\Pi_T} [u\varphi_t + f(u)\varphi_x] \, dx\, dt = 0 \qquad \forall \varphi(t,x) \in C_0^\infty(\Pi_T), \ \varphi(t,x) \geqslant 0.$$

This is the integral identity we were aiming at, except that we need it for an arbitrary (not necessarily nonnegative) function $\phi \in C_0^\infty(\Pi_T)$. Therefore, in order to conclude the proof, it remains to notice that any function $\varphi \in C_0^\infty(\Pi_T)$ can be represented as the difference $\varphi = \varphi_1 - \varphi_2$ of two *nonnegative* test functions φ_1 and φ_2. It is sufficient to take a nonnegative function $\varphi_1 \in C_0^\infty(\Pi_T)$ with $\varphi_1 \equiv \sup_{\Pi_T} \varphi$ on the support of φ. Since the relation (5.3) holds for both φ_1 and φ_2, it also holds true for φ. □

Proposition 5.13. *Let $u = u(t,x)$ be a piecewise smooth function that is a generalized entropy solution of equation* (5.1) *in the sense of Definition* 5.11. *Then on each discontinuity curve Γ (given by the equation $x = x(t)$) the adequate admissibility condition,* (5.21) *or* (5.22)*, holds.*

Proof. Fix a point $(t_0, x_0) \in \Gamma$, $x_0 = x(t_0)$, on the discontinuity curve Γ. As usual, denote by $u_\pm(t_0, x_0)$ the one-sided limits of $u(t_0, x)$ on Γ as x approaches x_0. To be specific, assume that $u_-(t_0, x_0) < u_+(t_0, x_0)$. Let us fix an arbitrary number $k \in (u_-, u_+)$ and choose a small neighbourhood $O \subset \Pi_T$ of the point (t_0, x_0) such that

$$u(t,x) < k \quad \text{for} \quad (t,x) \in O_- \equiv \{(t,x) \in O \mid x < x(t)\}, \tag{5.35}$$

$$u(t,x) > k \quad \text{for} \quad (t,x) \in O_+ \equiv \{(t,x) \in O \mid x > x(t)\}. \tag{5.36}$$

This is always possible since we consider a piecewise smooth solution. Moreover, without loss of generality, we can assume that u is smooth in each of the subdomains O_+ and O_-.

From (5.30) it follows that for any test function $\varphi \in C_0^\infty(O)$, $\varphi(t,x) \geqslant 0$, there holds

$$\int_O \big[|u-k|\varphi_t + \operatorname{sign}(u-k)\,(f(u)-f(k))\,\varphi_x\big]\,dx\,dt \geqslant 0. \tag{5.37}$$

Let us split the latter integral over the domain O into the sum of integrals over O_- and O_+. Taking into account (5.35)–(5.36), we obtain

$$-\int_{O_-}\big[(u-k)\varphi_t + (f(u)-f(k))\,\varphi_x\big]\,dx\,dt + \int_{O_+}\big[(u-k)\varphi_t + (f(u)-f(k))\,\varphi_x\big]\,dx\,dt \geqslant 0.$$

Now let us transfer the t and x derivatives according to the integration-by-parts formula (4.1). In addition to the integrals over the domains O_- and O_+, also integrals over their boundaries will arise, that is, we will get integrals over ∂O and over $\Gamma \cap O$. As φ is compactly supported in O, the integral over ∂O is zero. Consequently, we obtain

$$\begin{aligned}
&\int_{O_-}\big[u_t + (f(u))_x\big]\varphi\,dx\,dt \\
&\qquad - \int_{\Gamma\cap O}\big((u_- - k)\cos(\nu,t) + (f(u_-)-f(k))\cos(\nu,x)\big)\varphi\,dS \\
&- \int_{O_+}\big[u_t + (f(u))_x\big]\varphi\,dx\,dt \\
&\qquad - \int_{\Gamma\cap O}\big((u_+ - k)\cos(\nu,t) + (f(u_+)-f(k))\cos(\nu,x)\big)\varphi\,dS \geqslant 0.
\end{aligned}$$

Here ν is the normal vector to the curve Γ pointing from O_- to O_+ (i.e., the outward normal vector to the boundary of O_- and, at the same time, the interior normal vector for O_+). According to Proposition 5.12, the function $u = u(t,x)$ is a generalized (in the sense of the integral identity (5.3)) solution of equation (5.1). Since u is smooth in $O_\pm$, it is also a classical solution of the equation in each of the subdomains O_- and O_+. Consequently, we have in both O_- and O_+ the pointwise identity $u_t + (f(u))_x = 0$. Thus for any nonnegative test function $\varphi \in C_0^\infty(O)$, there holds

$$\int_{\Gamma\cap O}\big((2k - u_- - u_+)\cos(\nu,t) + (2f(k) - f(u_-) - f(u_+))\cos(\nu,x)\big)\varphi\,dS \geqslant 0.$$

This means that for all $k \in (u_-, u_+)$, we have

$$(2k - u_- - u_+)\cos(\nu,t) + (2f(k) - f(u_-) - f(u_+))\cos(\nu,x) \geqslant 0. \tag{5.38}$$

As already mentioned, $u = u(t,x)$ is a generalized solution of equation (5.1). This means, in particular, that the Rankine–Hugoniot condition (5.13) is satisfied along the discontinuity curve Γ (here we take this condition in the equivalent form (4.6)):

$$(u_+ - u_-)\cos(\nu,t) + (f(u_+) - f(u_-))\cos(\nu,x) = 0. \tag{5.39}$$

Taking into account (5.39), we can rewrite inequality (5.38) under the form

$$\begin{aligned} 2\big[(k-u_-)\cos(\nu,t) + (f(k)-f(u_-))\cos(\nu,x)\big] \\ -\big[(u_+-u_-)\cos(\nu,t) + (f(u_+)-f(u_-))\cos(\nu,x)\big] \\ = 2\big[(k-u_-)\cos(\nu,t) + (f(k)-f(u_-))\cos(\nu,x)\big] \geqslant 0 \end{aligned}$$

for all $k \in (u_-, u_+)$. This is exactly the jump admissibility condition (5.21).

As to the case $u_+ < u_-$, transforming the term $\operatorname{sign}(u-k)$ and the term with the absolute value in equality (5.37) in the same vein as before, we obtain the minus signs in front of the same expressions. Accordingly, in place of the relation (5.38), we get

$$(2k - u_- - u_+)\cos(\nu,t) + (2f(k) - f(u_-) - f(u_+))\cos(\nu,x) \leqslant 0$$

for all $k \in (u_+, u_-)$. With the help of (5.39), we obtain the inequality

$$\begin{aligned} 2\big[(k-u_+)\cos(\nu,t) + (f(k)-f(u_+))\cos(\nu,x)\big] \\ +\big[(u_+-u_-)\cos(\nu,t) + (f(u_+)-f(u_-))\cos(\nu,x)\big] \\ = 2\big[(k-u_+)\cos(\nu,t) + (f(k)-f(u_+))\cos(\nu,x)\big] \leqslant 0, \end{aligned}$$

which holds for all $k \in (u_+, u_-)$. This statement coincides with (5.22). □

Finally, let us show that inequality (5.30) can be derived from the vanishing viscosity approach. Indeed, let $u = u(t,x)$ be a limit in the topology of $L_{1,\mathrm{loc}}(\Pi_T)$, as $\varepsilon \to +0$, of classical solutions $u^\varepsilon = u^\varepsilon(t,x)$ to the Cauchy problem consisting of the equation

$$u_t + f'(u)u_x = \varepsilon u_{xx} \tag{5.40}$$

and the initial datum $u(0,x) = u_0(x)$.

Take any convex function $E = E(u) \in C^2(\mathbb{R})$ and multiply equation (5.40) by $E'(u)$. The equalities

$$E'(u)u_t = \frac{\partial E(u(t,x))}{\partial t}, \qquad f'(u)E'(u)u_x = \frac{\partial}{\partial x}\Big(\int_k^{u(t,x)} f'(\xi)\,E'(\xi)\,d\xi\Big),$$
$$E'(u)u_{xx} = (E(u))_{xx} - E''(u)u_x^2,$$

imply

$$E_t + \Big(\int_k^u f'(\xi)E'(\xi)\,d\xi\Big)_x = \varepsilon\,(E(u))_{xx} - \varepsilon E''(u)u_x^2 \leqslant \varepsilon\,(E(u))_{xx} \tag{5.41}$$

since $E''(u) \geqslant 0$ and $\varepsilon > 0$. Now let us multiply inequality (5.41) by a test function $\varphi = \varphi(t,x) \geqslant 0$ from Definition 5.11 and integrate it over Π_T. Using the integration-by-parts formula, we transfer all the derivatives to the test function φ:

$$-\int_{\Pi_T}\Big[\varphi_t E(u) + \varphi_x \int_k^u f'(\xi)E'(\xi)\,d\xi\Big]\,dx\,dt \leqslant \varepsilon\int_{\Pi_T}\varphi_{xx}E(u)\,dx\,dt$$

Passing to the limit as $\varepsilon \to +0$, we get

$$\int_{\Pi_T} \left[\varphi_t E(u) + \varphi_x \int_k^u f'(\xi) E'(\xi)\, d\xi \right] dx\, dt \geqslant 0. \tag{5.42}$$

Let $\{E_m\}$ be a sequence of C^2-functions approximating the function $u \mapsto |u-k|$ uniformly on $\mathbb{R}$. Substitute $E = E_m(u)$ in the inequality (5.42) and pass to the limit as $m \to \infty$. We can choose E_m in such a way that E'_m is bounded and $E'_m(\xi) \to \operatorname{sign}(\xi - k)$ for all $\xi \in \mathbb{R}$, $\xi \neq k$. Thus, we have

$$\begin{aligned} \int_k^u f'(\xi) E'_m(\xi)\, d\xi \longrightarrow & \int_k^u f'(\xi) \operatorname{sign}(\xi - k)\, d\xi \\ & = \operatorname{sign}(u-k) \int_k^u f'(\xi)\, d\xi = \operatorname{sign}(u-k) \left(f(u) - f(k) \right). \end{aligned}$$

In this way, we deduce (5.30) from (5.42).

Problem 5.2. *Justify in detail the last passage to the limit in the above proof.*

Remark 5.14. In the case of a convex flux function $f = f(u)$, we can replace the integral inequality (5.30) in the definition of a generalized entropy solution by, first, the integral identity (5.3), and, second, the additional admissibility requirement that the inequality (5.42) holds for one fixed strictly convex function $E = E(u)$. Uniqueness of the so defined solution is shown in [39].

In the context of the inequality (5.42), a convex function $E = E(u)$ is called an "entropy" of the equation (5.1); indeed, inequality (5.42) is another variant of the "entropy increase-type conditions" in the sense of Section 5.3.

Remark 5.15. The definition of a generalized entropy solution on the basis of the inequality (5.30) extends to the multi-dimensional analogue of the problem (5.1)–(5.2). In this case, we have $x \in \mathbb{R}^n$,

$$f : \mathbb{R} \to \mathbb{R}^n, \qquad (f(u))_x \equiv \nabla_x f(u(t,x)), \qquad \varphi_x = \nabla_x \varphi,$$

and $(f(u) - f(k))\, \varphi_x$ is the scalar product of the vector $(f(u) - f(k))$ with the gradient of φ with respect to the space variable x. This way to define the notion of a solution $u = u(t,x)$, and also the family of entropies $|u-k|$, $k \in \mathbb{R}$, is often named after S. N. Kruzhkov (Kruzhkov's solutions, the Kruzhkov entropies). These notions were introduced in the works [25, 26]. Also the techniques of existence and uniqueness proofs, techniques deeply rooted in the physical context of the problem, were set up in these papers.

6 The Riemann problem (evolution of a primitive jump)

In this section, we consider the so-called Riemann problem for equation (4.2), which is the problem of evolution from a simplest piecewise constant initial datum. That is, we will construct admissible generalized solutions $u = u(t, x)$ of the following problem in a strip $\Pi_T = \{-\infty < x < +\infty, 0 < t < T\}$:

$$u_t + (f(u))_x = 0, \qquad u\big|_{t=0} = u_0(x) = \begin{cases} u_- & \text{for } x < 0, \\ u_+ & \text{for } x > 0, \end{cases} \tag{6.1}$$

where u_- and u_+ are two arbitrary constant states. The solutions we want to construct will be piecewise smooth in Π_T. This means that, first, they will satisfy the equation in the classical pointwise sense on all smoothness components of the solution; and second, they will satisfy both the Rankine–Hugoniot condition (4.5) and the entropy increase condition on each curve of jump discontinuity. These solutions will converge to the function u_0 as $t \to +0$ at all points, except for the point $x = 0$.

The proof of the uniqueness of an admissible generalized solution (in the sense of the integral identity and entropy increase condition) of the problem (6.1) can be found in [27, Lectures 4-6]; its existence is demonstrated below with an explicit construction.

First of all, let us notice that the equation we consider is invariant under the change $x \to kx$, $t \to kt$; moreover, the initial datum also remains unchanged under the action of homotheties $x \to kx$, $k > 0$. Furthermore, the entropy increase condition is also invariant under the above transformations. Admitting the uniqueness of an admissible generalized solution of the above problem, we conclude that any change of variables $x \to kx$, $t \to kt$ with $k > 0$ transforms the unique solution $u = u(t, x)$ of the problem into itself, i.e.,

$$u(kt, kx) \equiv u(t, x) \quad \forall k > 0.$$

This exactly means that the function $u = u(t, x)$ remains constant on each ray $x = \xi t$, $t > 0$, issued from the origin $(0, 0)$, so that $u(t, x)$ depends only on the variable $\xi = x/t$:

$$u(t, x) = u(x/t), \quad t > 0. \tag{6.2}$$

Solutions that only depend on x/t are called *self-similar*. In particular, jump discontinuity curves of self-similar solutions can only be straight rays emanating from the origin $(0, 0)$.

Exercise 6.1. *Find all the self-similar solutions of the equations from Exercise* 4.2 *such that the solutions are smooth in the whole half-plane* $t > 0$.

6.1 The Hopf equation

To start with, consider the Riemann problem (6.1) in the case $f(u) = u^2/2$:

$$u_t + uu_x = 0, \qquad u\big|_{t=0} = u_0(x) = \begin{cases} u_- & \text{for } x < 0, \\ u_+ & \text{for } x > 0. \end{cases} \tag{6.3}$$

First of all, we describe all the smooth self-similar solutions of the Hopf equation. Substituting (6.2) into the equation (6.3), we find

$$-\frac{x}{t^2}\,u'\left(\frac{x}{t}\right)+\frac{1}{t}\,u\left(\frac{x}{t}\right)u'\left(\frac{x}{t}\right)=\frac{1}{t}\,u'\left(\frac{x}{t}\right)\left(u\left(\frac{x}{t}\right)-\frac{x}{t}\right)=0,$$

i.e., either $u'=0$, so that we have $u\equiv C$ where C is a constant, or $u=x/t$. Consequently, the set of all smooth self-similar solutions of the Hopf equation reduces to the constant solutions and to the function x/t.

Now our task is to juxtapose pieces of the above smooth self-similar solutions in a correct way (i.e., respecting the Rankine–Hugoniot and the entropy increase condition on the discontinuity rays), with the goal to comply with the initial datum $u_0=u_0(x)$.

First, let us see which rays can separate two smoothness components of such a solution: two adjacent components may correspond either to two different constant states, or to a constant state and to the restriction of the function x/t on some cone with the vertex $(0,0)$.

It follows from the Rankine–Hugoniot condition (4.5) that two constant functions $u(t,x)\equiv u_1$ and $u(t,x)\equiv u_2,\ u_i=$ const, can only be juxtaposed along the ray

$$x=\frac{f(u_2)-f(u_1)}{u_2-u_1}\,t=\frac{1}{2}\,\frac{u_2^2-u_1^2}{u_2-u_1}\,t=\frac{u_2+u_1}{2}\,t,$$

and because of the entropy increase condition, the jump is admissible only when u jumps from a greater to a smaller value (we mean that the direction of the jump is such that x grows). Consequently, if we specify, e.g., that $u_2>u_1$, then we should have

$$u(t,x)=u_2 \quad \text{for } x<\frac{u_2+u_1}{2}t\,, \quad \text{and} \quad u(t,x)=u_1 \quad \text{for } x>\frac{u_2+u_1}{2}t\,.$$

As to the juxtaposition of a constant $u(t,x)\equiv u_3=$ const and the function $u(t,x)=x/t$, we have the following. If the two functions juxtapose along a ray $x=\xi t$, then the limit of the function x/t on this ray equals ξ, and (4.5) yields

$$\xi=\frac{dx}{dt}=\frac{f(u_3)-f(\xi)}{u_3-\xi}=\frac{1}{2}\,\frac{u_3^2-\xi^2}{u_3-\xi}=\frac{u_3+\xi}{2}\,,$$

so that $\xi=u_3$. The latter means that the function obtained by the juxtaposition turns out to be continuous on the border ray $x=\xi t=u_3t,\ t>0$. Consequently, here the discontinuity is a weak, not a strong one.

Now we can solve completely the Riemann problem for the Hopf equation. Here, two substantially different situations should be considered:

(i) When $u_->u_+$, we can construct a *shock wave* solution, where the two constants u_- and u_+ are joined across the ray $x=\frac{u_2+u_1}{2}t$, according to the Rankine–Hugoniot condition (see Fig. 16):

$$u(t,x)=\begin{cases} u_- & \text{for } x<\frac{u_-+u_+}{2}t,\\ u_+ & \text{for } x>\frac{u_-+u_+}{2}t.\end{cases} \tag{6.4}$$

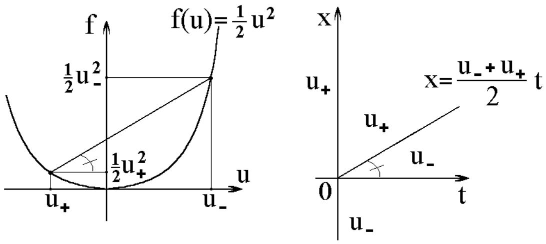

Figure 16. Shock-wave solution to the Riemann problem.

As has already been mentioned, the jump discontinuity in the desired solution is compatible with the admissibility condition of increase of entropy.

(ii) If $u_- < u_+$, we cannot take the shock wave solution analogous to the previous case, because the jump discontinuity would not satisfy the entropy increase condition. Here the function x/t is helpful; it can be combined continuously with the constant states u_- and u_+ (see Fig. 17):

$$u(t,x) = \begin{cases} u_- & \text{for } x \leqslant u_- t, \\ x/t & \text{for } u_- t < x < u_+ t, \\ u_+ & \text{for } x \geqslant u_+ t. \end{cases} \tag{6.5}$$

The so defined solution is indeed continuous in the whole half-plane $t > 0$. The cone determined by the inequalities $u_- t < x < u_+ t$, $t > 0$, in which the smoothing of the initially discontinuous function takes place, is called the *region of rarefaction* of the solution, and the solution (6.5) itself is called a *centered rarefaction wave*.

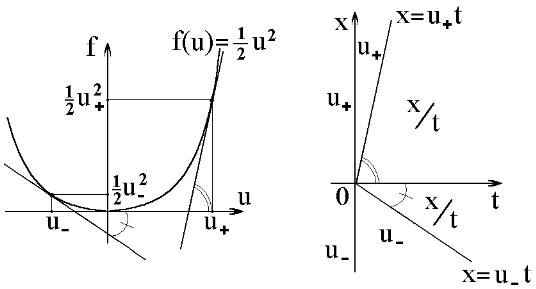

Figure 17. Rarefaction-wave solution to the Riemann problem.

Let us give a comment of geometrical nature to the solutions obtained. Drawing the graph of the function $f(u) = u^2/2$ relative to the axes (u, f), parallel to the axes

(t, x), let us mark the points $(u_-, u_-^2/2)$ and $(u_+, u_+^2/2)$ on the graph. Then, as it has already been mentioned, the discontinuity ray in solution (6.4) is parallel to the segment joining these two points (see Fig. 16). Also observe the following fact (in the sequel, we will see that this is by no means incidental): the lines of weak discontinuity of the solution $u = u(t, x)$ given by (6.5), namely the two rays $x = u_- t$ and $x = u_+ t$, are parallel to the tangent directions to the graph of the function $f(u) = u^2/2$ at the points $(u_-, f(u_-))$ and $(u_+, f(u_+))$, respectively.

Remark 6.1. When $u_- > u_+$, formula (6.5) is meaningless: no function in the upper half-plane $t > 0$ is determined by this formula.

Problem 6.1. *Show that the solution constructed above, given by* (6.4) *or by* (6.5)*, according to the sign of* $(u_- - u_+)$*, is the unique admissible generalized solution of the Riemann problem* (6.3) *within the class of all self–similar piecewise–smooth functions.*

6.2 The case of a convex flux function

In the case where $f = f(u)$ is a smooth strictly convex function, the solution of the Riemann problem (6.1) is almost the same as for the case of the Hopf equation (i.e., as for the case $f(u) = u^2/2$). The only difference is that the non-constant smooth self-similar solution $u(t, x) = x/t$ of the Hopf equation is replaced by the appropriate smooth function $\psi = \psi(x/t)$. Let us find this function ψ. As above, we substitute (6.2) into (6.1) and obtain

$$-\frac{x}{t^2}u' + \frac{1}{t}f'(u)u' = \frac{1}{t}u'(x/t)\left(f'\left(u\left(x/t\right)\right) - x/t\right) = 0.$$

Therefore, besides the constants obtained from the equation $u' = 0$, there exists one more function $u(\xi) = \psi(\xi)$ (here $\xi = x/t$) defined as the solution of the equation

$$f'(\psi) = \xi.$$

That is, ψ is the function inverse to f': we have $\psi = (f')^{-1}$. The inverse function does exist since f is strictly convex, so that f' is a strictly monotone function. The solution $u(t, x) = \psi(x/t)$, which is discontinuous at $(0, 0)$ and continuous for $t > 0$, is a *centered rarefaction wave*.

Remark 6.2. In the previous section, for the particular case of the Hopf equation, we had $f'(u) = u$, so that $\psi(\xi) = (f')^{-1}(\xi) = \xi$.

In the case of a general strictly convex flux function $f = f(u)$, we construct the solution of the Riemann problem (6.1) similar to the case of the Hopf equation, namely:

(i) When $u_- > u_+$, then we can use the shock wave again, juxtaposing the two constant states u_- and u_+ separated by the ray $\frac{x}{t} = \frac{f(u_+)-f(u_-)}{u_+-u_-}$, $t > 0$, the slope of the ray being found from the Rankine–Hugoniot condition:

$$u(t, x) = \begin{cases} u_- & \text{for } x < \frac{f(u_+)-f(u_-)}{u_+-u_-}t, \\ u_+ & \text{for } x > \frac{f(u_+)-f(u_-)}{u_+-u_-}t. \end{cases} \tag{6.6}$$

(Compare with (6.4) and Fig. 16.) The jump in the solution obtained is admissible according to the entropy increase condition.

(ii) When $u_- < u_+$, then the function given by (6.6) is a generalized solution but it does not satisfy the entropy increase condition. Then, similar to the construction of (6.5), we combine the constant states u_- and u_+ with the non-trivial smooth solution $\psi = \psi\,(x/t)$. The rays $x = \xi_- t$ and $x = \xi_+ t$, where the transition occurs, are determined by the requirement of continuity of the solution: $u_\pm = \psi\,(\xi_\pm)$, i.e., $\xi_\pm = f'(u_\pm)$, so that

$$u(t,x) = \begin{cases} u_- & \text{for } x \leqslant f'(u_-)t, \\ \psi\,(x/t) & \text{for } f'(u_-)t < x < f'(u_+)t, \\ u_+ & \text{for } x \geqslant f'(u_+)t. \end{cases} \tag{6.7}$$

The function given by (6.7) is well-defined in the upper half-plane $t > 0$; indeed, the flux function $f = f(u)$ is strictly convex, thus f' is an increasing function, so that $f'(u_-) < f'(u_+)$ whenever $u_- < u_+$.

The rarefaction wave $\psi = \psi(x/t)$, being continuous for $t > 0$, takes all the intermediate values between u_- and u_+. As ψ is defined as the inverse function of f', the condition $\psi\,(x/t) = \hat{u}$ is equivalent to the equality $x = f'(\hat{u})t$ valid for all $\hat{u} \in [u_-, u_+]$. This means that the rarefaction wave $\psi = \psi\,(x/t)$ takes a given intermediate value $\hat{u}$ on the ray $x = f'(\hat{u})t$, $t > 0$. We can see that this ray is parallel to the direction tangent to the graph $f = f(u)$ at the point $(\hat{u}, f(\hat{u}))$ of the graph. Thus in particular, we have justified the statement already noted in the previous section: the rays of weak discontinuity of the solution $u = u(t,x)$ given by formula (6.7) (i.e., the rays $x = f'(u_\pm)t$) are aligned with the directions tangent to the graph $f = f(u)$ at the endpoints $(u_\pm, f(u_\pm))$ (see Fig. 17). (As always, we assume that the axes (u, f) are aligned with the axes (t, x).)

Remark 6.3. Note that the convexity of $f = f(u)$ is only needed on the segment $[u_-, u_+]$ (or $[u_+, u_-]$, if $u_+ < u_-$).

Concerning the case of a strictly concave and smooth (on the segment between u_- and u_+) flux function $f = f(u)$, the unique self-similar admissible generalized solution to the Riemann problem is constructed by exchanging, in a sense, the two situations described above. Namely: for the case $u_- < u_+$, we obtain the shock wave (6.6); if $u_- > u_+$, then the solution is given by (6.7) (in this case f' is a decreasing function, consequently, here we have $f'(u_-) < f'(u_+)$). The careful derivation of the formulas is left to the reader:

Problem 6.2. *Solve the Riemann problem* (6.1) *in the case of a general smooth strictly concave flux function $f = f(u)$; represent the piecewise smooth solution graphically* (*as in Fig.* 16 *and* 17)*; check the validity of the Rankine–Hugoniot condition, and of the entropy increase inequality on the jumps.*

Exercise 6.2. *Solve the following Riemann problems:*

(i) $u_t - (u^2)_x = 0$,
$$u\big|_{t=0} = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x > 0 \end{cases} \quad \text{and} \quad u\big|_{t=0} = \begin{cases} 1 & \text{for } x < 0, \\ -1 & \text{for } x > 0; \end{cases}$$

(ii) $u_t + u^2 \cdot u_x = 0$,
$$u\big|_{t=0} = \begin{cases} 0 & \text{for } x < 0, \\ 2 & \text{for } x > 0 \end{cases} \quad \text{and} \quad u\big|_{t=0} = \begin{cases} 2 & \text{for } x < 0, \\ 0 & \text{for } x > 0; \end{cases}$$

(iii) $u_t + \cos u \cdot u_x = 0,\ u\big|_{t=0} = \begin{cases} 0 & \text{for } x < 0, \\ \pi & \text{for } x > 0, \end{cases}$
$$u\big|_{t=0} = \begin{cases} \pi & \text{for } x < 0, \\ 0 & \text{for } x > 0 \end{cases} \quad \text{and} \quad u\big|_{t=0} = \begin{cases} \pi & \text{for } x < 0, \\ 2\pi & \text{for } x > 0; \end{cases}$$

(iv) $u_t + e^u \cdot u_x = 0$,
$$u\big|_{t=0} = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0 \end{cases} \quad \text{and} \quad u\big|_{t=0} = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x > 0; \end{cases}$$

(v) $u_t + (\ln u)_x = 0$,
$$u\big|_{t=0} = \begin{cases} e & \text{for } x < 0, \\ 1 & \text{for } x > 0 \end{cases} \quad \text{and} \quad u\big|_{t=0} = \begin{cases} 1 & \text{for } x < 0, \\ e & \text{for } x > 0. \end{cases}$$

6.3 The case of a flux function with inflexion point

In order to treat the Riemann problem in the case where $f = f(u)$ is neither convex nor concave, let us first give two definitions.

Definition 6.4. The *concave hull* of a function $f = f(u)$ on a segment $[\alpha, \beta]$ is the function
$$\hat{f}(u) = \inf_{\tilde{f} \in \hat{F}} \tilde{f}(u), \quad u \in [\alpha, \beta],$$
where $\hat{F}$ is the family of all concave functions $\tilde{f} = \tilde{f}(u)$ defined on $[\alpha, \beta]$ such that $\tilde{f}(u) \geqslant f(u)$ for all $u \in [\alpha, \beta]$.

Definition 6.5. The *convex hull* of a function $f(u)$ on a segment $[\alpha, \beta]$ is the function
$$\check{f}(u) = \sup_{\tilde{f} \in \check{F}} \tilde{f}(u), \quad u \in [\alpha, \beta],$$
where $\check{F}$ is the family of all convex functions $\tilde{f} = \tilde{f}(u)$ defined on $[\alpha, \beta]$ such that $\tilde{f}(u) \leqslant f(u)$ for all $u \in [\alpha, \beta]$.

Remark 6.6. If f is a concave (respectively, convex) function on $[\alpha, \beta]$, then the function itself is its concave (respectively, convex) hull: $\hat{f} = f$ (respectively, $\check{f} = f$); furthermore, the graph of its convex (respectively, concave) hull is the straight line segment joining the endpoints $(\alpha, f(\alpha))$ and $(\beta, f(\beta))$ of the graph.

Exercise 6.3. *Construct the concave and the convex hulls for the function* $f(u) = u^3$ *on the segment* $[-1, 1]$ *as well as for the function* $f(u) = \sin u$ *on the segment* $[0, 3\pi]$.

To solve the Riemann problem (6.1) for a given smooth flux function $f = f(u)$ in the case $u_- < u_+$, we first construct the convex hull of f on the segment $[u_-, u_+]$. In the case $u_- > u_+$, we construct the concave hull of f on the segment $[u_+, u_-]$.

The graph of any of the hulls consists of some parts of the graph of f, where the graph has the right convexity/concavity direction, and of straight line segments connecting these pieces of the graph of f (see the above exercise). Each of the straight line segments will correspond to a jump ray (thus, to a shock wave) in the solution of the Riemann problem; each of such rays will separate two components of smoothness of the solution. Each of these components can either be a constant state (u_- or u_+), or a smooth self-similar solution of the form $u(t, x) = \psi(x/t)$ (i.e., a centered rarefaction wave). Here $\psi = \psi(\xi)$ is the function (locally) inverse to f', so that $\xi = f'(u)$ (see Section 6.2). Notice that on each segment of strict convexity/concavity of $f = f(u)$ the function f' is indeed invertible.

Example 6.7. Let us construct the solution (i.e., the self-similar admissible generalized solution) of the following Riemann problem:

$$u_t + (u^3)_x = 0, \quad u\big|_{t=0} = \begin{cases} 1 & \text{for } x < 0, \\ -1 & \text{for } x > 0. \end{cases} \tag{6.8}$$

First, because of $u_- = 1 > -1 = u_+$, we construct the concave hull of the flux function $f(u) = u^3$ on the segment $[-1, 1]$. To perform the construction, we draw the tangent line to the graph at the right endpoint $(1, 1)$ of this graph. The tangency point, denoted by $(\hat{u}, \hat{u}^3)$ can be determined from the condition

$$\frac{1 - \hat{u}^3}{1 - \hat{u}} = f'(\hat{u}) = 3\hat{u}^2, \quad \hat{u} \neq 1,$$

i.e., $1 + \hat{u} + \hat{u}^2 = 3\hat{u}^2$, whence $\hat{u} = -1/2$. Notice that the piece of the graph of $f(u) = u^3$ between the left endpoint $(-1, -1)$ of the graph and the tangency point $(-1/2, (-1/2)^3)$ is concave. Thus we see that[8] the graph of the concave hull $\hat{f}$ of the function $f(u) = u^3$ on the segment $[-1, 1]$ consists of: first, the piece of the "cubic parabola" $f = f(u) = u^3$ on the segment $[-1, -1/2]$; and second, the straight line segment that joins the points $(-1/2, -1/8)$ and $(1, 1)$ (see Fig. 18). Therefore, the solution of the Riemann problem under consideration has one and only one ray $x = \xi t$, $t > 0$, on which the solution has a jump. This ray is parallel to the straight line segment in the graph of $\hat{f} = \hat{f}(u)$ (as usual, for the sake of convenient graphical representation, the axes (t, x) are aligned with the axes (u, f)); expressing analytically the slope of the strong discontinuity ray, we have

$$\xi = \frac{1 + 1/8}{1 + 1/2} = \frac{3}{4}\,.$$

[8] *NT* — This conclusion requires some thinking; it is based on several easy-to-justify properties of the concave hull. In particular, one always has $\hat{f}(\alpha) = f(\alpha) = \check{f}(\alpha)$, $\hat{f}(\beta) = f(\beta) = \check{f}(\beta)$, with the notation of the definitions. The reader who analyzed the examples of Exercise 6.3 has already performed this construction.

This ray separates the constant state $u_- = 1$ (taken from the side $x < \frac{3}{4}t$) and a piece of rarefaction $\psi(x/t)$. Here $\psi = \psi(\xi)$ is the function inverse to $\xi = f'(u) = 3u^2$ on the segment $[-1, -1/2]$, so that we have

$$u = \psi(\xi) = -\sqrt{\xi/3}, \qquad 3/4 \leqslant \xi \leqslant 3.$$

The limit of the solution $u = u(t, x)$ from the side $x > \frac{3}{4}t$ on the jump ray $x = \frac{3}{4}t$ equals $\psi(\frac{3}{4}) = -\frac{1}{2}$ (this stems from the fact that $f'(-\frac{1}{2}) = 3(-\frac{1}{2})^2 = \frac{3}{4}$).

As for the case of a convex flux function (see Section 6.2), the juxtaposition of the rarefaction wave $\psi = \psi(x/t)$ and the constant state $u_+ = -1$ occurs continuously, that is, these two smoothness components are separated by the weak discontinuity ray $x = 3t$, $t > 0$. Once more, this ray is aligned with the tangent direction at the point $(u_+, f(u_+)) = \left(u_+, u_+^3\right) = (-1, -1)$ of the graph of the flux function $f(u) = u^3$.

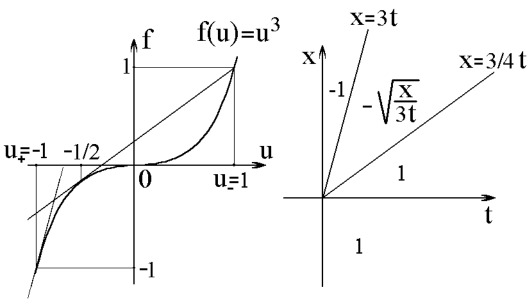

Figure 18. Solution for Example 6.7.

Thus we obtain the following solution of problem (6.8):

$$u(t,x) = \begin{cases} 1 & \text{for } x < \frac{3}{4}t, \\ -\sqrt{\frac{x}{3t}} & \text{for } \frac{3}{4}t < x < 3t, \\ -1 & \text{for } x \geqslant 3t. \end{cases}$$

Exercise 6.4. *Construct the solution of the Riemann problem*

$$u_t + u^2 \cdot u_x = 0, \quad u\big|_{t=0} = \begin{cases} -2 & \text{for } x < 0, \\ 2 & \text{for } x > 0. \end{cases}$$

Example 6.8. Let us solve the Riemann problem

$$u_t + (\sin u)_x = 0, \quad u\big|_{t=0} = \begin{cases} 3\pi & \text{for } x < 0, \\ 0 & \text{for } x > 0. \end{cases}$$

As we have $u_- = 3\pi > 0 = u_+$, we have to construct the concave hull $\hat{f} = \hat{f}(u)$ of the graph of $f(u) = \sin u$ on the segment $[0, 3\pi]$. The graph of $\hat{f}$ (see Fig. 19) consists of two pieces of concavity of the graph of $f(u) = \sin u$, those on the segments $[0, \pi/2]$ and $[5\pi/2, 3\pi]$, and of the horizontal segment joining the points $(\pi/2, 1)$ and $(5\pi/2, 1)$ of the sine curve. We conclude that the solution $u = u(t, x)$ should have one strong discontinuity (jump) along the ray $x = 0$, separating the one-sided limit states

$$\frac{5\pi}{2} = \lim_{x\to-0} u(t,x) \quad \text{and} \quad \frac{\pi}{2} = \lim_{x\to+0} u(t,x).$$

We also see that

$$u(t,x) = \begin{cases} 3\pi & \text{for } x < f'(3\pi)\cdot t = \cos 3\pi \cdot t = -t, \\ 0 & \text{for } x > f'(0)\cdot t = t. \end{cases}$$

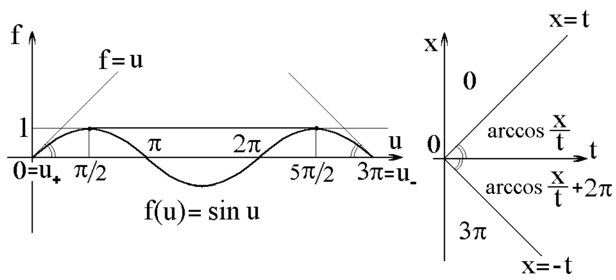

Figure 19. Solution for Example 6.8.

It remains to express u from the equation

$$f'(u) = \cos u = \xi = x/t$$

on the segments $[0, \pi/2]$ and $[5\pi/2, 3\pi]$. By construction, it is not surprising that the function $f'(u) = \cos u$ is monotone on these segments. Solutions of the equation $\cos u = \xi$, $-1 \leqslant \xi \leqslant 1$, are well-known: we have $u = \pm \arccos \xi + 2\pi n, n \in \mathbb{Z}$. On the segment $[0, \pi/2]$, the solution specifies to $u = \arccos \xi$, while on the segment $[5\pi/2, 3\pi]$ we get $u = \arccos \xi + 2\pi$. Recapitulating, the solution we have constructed looks as follows (see Fig. 19):

$$u(t,x) = \begin{cases} 3\pi & \text{for } x \leqslant -t, \\ \arccos x/t + 2\pi & \text{for } -t < x < 0, \\ \arccos x/t & \text{for } 0 < x < t, \\ 0 & \text{for } x \geqslant t. \end{cases}$$

The solution of the Riemann problem will change drastically if we exchange the values u_+ and u_-.

Example 6.9. Construct the solution of the Riemann problem

$$u_t + (\sin u)_x = 0, \quad u\big|_{t=0} = \begin{cases} 0 & \text{for } x < 0, \\ 3\pi & \text{for } x > 0. \end{cases}$$

Now we have to start by constructing the convex hull $\breve{f} = \breve{f}(u)$ of the function $f(u) = \sin u$ on the segment $[0, 3\pi]$ (see Fig. 20). It consists of two segments of the lines issued from the graph's endpoints $(0, 0)$ and $(3\pi, 0)$, the lines being tangent to the sine graph at some points contained within $[\pi, 2\pi]$, each of the segments being taken between the endpoint and the tangency point, and of the convex piece of the sine curve between the two tangency points $(u_1, \sin u_1)$ and $(u_2, \sin u_2)$. Symmetry considerations readily yield the equalities $u_1 + u_2 = 3\pi,\ \sin u_1 = \sin u_2$; also the slopes of the two tangent segments constructed above only differ by their sign. Denote by

$$-k = \frac{f(u_1) - f(0)}{u_1 - 0} = \frac{\sin u_1}{u_1} = f'(u_1) = \cos u_1$$

the slope of the tangent segment passing through the endpoint $(0, 0)$. Then $+k$ is the slope of the other tangent segment. We cannot find explicitly the exact values of u_1, u_2 and k, but we can say that u_1 is the smallest strictly positive solution of the equation $\tan u_1 = u_1$, that $u_2 = 3\pi - u_1$, and that $k = -\cos u_1 = \cos u_2$.

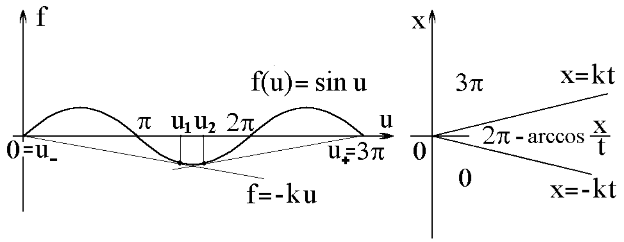

Figure 20. Solution for Example 6.9.

On the segment $[u_1, u_2] \subset [\pi, 2\pi]$, we can invert the function $f'(u) = \cos u$. In this case, $u = (f')^{-1}(\xi) = 2\pi - \arccos \xi,\ -k \leqslant \xi \leqslant k$. Now we can write down the "almost explicit" solution (depicted in Fig. 20):

$$u(t, x) = \begin{cases} 0 & \text{for } x \leqslant -kt, \\ 2\pi - \arccos x/t & \text{for } -kt < x < kt, \\ 3\pi & \text{for } x \geqslant kt. \end{cases}$$

The solution above has two strong discontinuities: the one across the line $x = -kt$ with the jump from 0 to u_1, and the one across the line $x = kt$ with the jump from u_2 to 3π.

Exercise 6.5. *Construct the solution of the Riemann problem*

$$u_t + \sin(2u) \cdot u_x = 0, \quad u\big|_{t=0} = \begin{cases} -5\pi/4 & \text{for } x < 0, \\ 5\pi/4 & \text{for } x > 0. \end{cases}$$

Afterword

In the present lecture course, we have introduced the reader to the notions and tools which underly the nonlocal theory of the Cauchy problem for the one-dimensional (in the space variable) quasilinear conservation law of the form

$$u_t + (f(u))_x = 0. \tag{6.9}$$

As to the nonlocal theory for the multidimensional scalar equation

$$u_t + \operatorname{div}_x f(u) = 0, \qquad x \in \mathbb{R}^n, \tag{6.10}$$

where f is an n-dimensional vector-function, it appeared in a rather complete form at the end of the 1960s (see [25, 26]), for the case where the components $f_i = f_i(u)$ of the flux function vector $f = f(u)$ satisfy a Lipschitz continuity condition. This assumption of Lipschitz continuity results in the effects of finite speed of propagation of perturbations and of finite domain of dependence (at a fixed point (t, x)) on the initial data for the solutions of equation (6.10).

A further challenge in the nonlocal theory of equations (6.9) and (6.10) lies in its generalization to the case where the flux function $f = f(u)$ is merely continuous, i.e., it is not necessarily differentiable. In this case, one expects that purely "parabolic", "diffusive" effects should appear: namely, the effects of infinite speed of propagation of perturbations and of infinite domain of dependence of entropy solutions on the initial data.

Indeed, let us look at the construction of the admissible generalized entropy solution of the Cauchy problem

$$u_t + \left(\frac{|u|^\alpha}{\alpha}\right)_x = 0, \qquad x \in \mathbb{R}, \qquad \alpha \in (0,1), \tag{6.11}$$

$$u\big|_{t=0} = u_0(x) \equiv \frac{[\operatorname{sign}(x+1) - \operatorname{sign} x]}{2} = \begin{cases} 1, & x \in (-1,0), \\ 0, & x \notin (-1,0). \end{cases} \tag{6.12}$$

As we have initially $u_0(x) \geqslant 0$, it can be deduced from Definition 5.11 that the generalized entropy solution $u = u(t,x)$ of the problem (6.11)—(6.12) is also nonnegative. Consequently, in (6.9) the flux function $f(u) \equiv u^\alpha/\alpha$ is concave on the interval of all values that could be possibly taken by the solution $u = u(t,x)$. On the other hand, because of the special ("single-step") structure of the initial function, it can be expected that, for a sufficiently small time interval $0 \leqslant t \leqslant \delta$, the admissible generalized solution of our problem will be determined by the solutions of the two Riemann problems with the initial functions $\operatorname{sign}(x+1)$ and $\operatorname{sign} x$, respectively.

Problem 6.3. *Check that the function*

$$u(t,x) = \begin{cases} 0 & \text{for } x < \frac{t}{\alpha} - 1, \\ 1 & \text{for } \frac{t}{\alpha} - 1 < x \leqslant t, \\ \left(\frac{t}{x}\right)^{\frac{1}{1-\alpha}} & \text{for } x > t \end{cases}$$

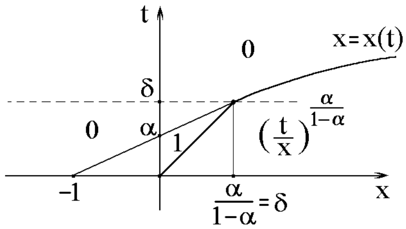

Figure 21. Solution of problem (6.11)–(6.12).

(*see Fig.* 21 *for a graphic representation of this function) defines a piecewise smooth admissible generalized solution of the problem* (6.11)–(6.12) *in the time interval* $0 < t < \frac{\alpha}{1-\alpha} = \delta$.

Problem 6.4. *Extend the above solution* $u = u(t,x)$ *of the problem* (6.11)–(6.12) *to the half-space* $t > \delta = \frac{\alpha}{1-\alpha}$. *More exactly, find the equation of the discontinuity curve* $x = x(t)$, *using for* $t > \delta$ *the ansatz* (*see Fig.* 21)

$$u(t,x) = \begin{cases} 0 & \text{for } x < x(t), \\ \left(\frac{t}{x}\right)^{\frac{1}{1-\alpha}} & \text{for } x > x(t). \end{cases}$$

Consequently, for the compactly supported initial function (6.12), the generalized entropy solution $u = u(t,x)$ of the Cauchy problem for equation (6.11) has in x a non-compact (unbounded) support, for all time $t > 0$ (thus, for an instant of time as small as desired!). It is known that, in the theory of parabolic PDEs (modelling diffusive processes in nature), such effect of infinite speed of propagation leads to non-uniqueness of a solution of the Cauchy problem. What would be the influence of this effect on the theory of nonlocal solvability of the Cauchy problem for equation (6.10), within the class of all essentially bounded measurable functions in the upper half-plane? It turns out that, without any further restriction on the continuous components $f_i = f_i(u)$ of the flux function, there exists at least one generalized entropy solution of the Cauchy problem. Contrarily (as it has been observed for the first time in the work [28]), the property of uniqueness of a generalized entropy solution of this problem can be connected with the product of the moduli of continuity ω_i of the functions f_i. If for all $u, v \in \mathbb{R}$

$$|f_i(u) - f_i(v)| \leqslant \omega_i(|u-v|), \tag{6.13}$$

where ω_i is a concave, strictly increasing and continuous function on $[0,+\infty)$ with $\omega_i(0) = 0$, then it is sufficient that for small ρ

$$\Omega(\rho) \equiv \prod_{i=1}^{n} \omega_i(\rho) \leqslant \text{const}\ \rho^{n-1}; \tag{6.14}$$

i.e., the restriction (6.13)–(6.14) ensures the uniqueness of a generalized entropy solution to a Cauchy problem for equation (6.11).

Further, let us stress that for the equation

$$u_t + \Big(\frac{|u|^\alpha}{\alpha}\Big)_x + \Big(\frac{|u|^\beta}{\beta}\Big)_y = 0, \qquad 0 < \alpha < \beta < 1,$$

the restriction (6.14) (which, for this concrete case, takes the form $\alpha + \beta \geqslant 1$), is both necessary and sufficient for the uniqueness of a generalized entropy solution to the Cauchy problem with general initial datum. The corresponding counterexample was constructed by E. Yu. Panov (see, e.g., [28]).

Notice that in the case $n = 1$ the condition (6.14) imposes no restriction at all on the merely continuous flux function $f = f(u)$: in the one-dimensional situation, a generalized entropy solution to the Cauchy problem is always unique.

Also notice that in the work [28] a rather simple proof of the uniqueness of a generalized entropy solutions is given under the assumption $\Omega(\rho)/\rho^{n-1} \to 0$ as $\rho \to 0$ that is slightly stronger than (6.14).

In conclusion, let us say that the nonlocal theory of first-order quasilinear conservation laws, whose rigorous mathematical treatment started in the 1950th, is yet actively developing. Many interesting problems remain unsolved, even for the one-dimensional equation (6.10). But most topical and interesting are the problems of conservation laws in the vector case, even for the simplest situations. Indeed, let us consider the well-known "wave equation" system

$$\begin{cases} u_t - v_x &= 0, \\ v_t - u_x &= 0. \end{cases}$$

This system was the very first object of research in PDEs (then called "mathematical physics"), in the works of D'Alembert and Euler. In order to take into account certain nonlinear dependencies in the process of wave propagation considered, one replaces the linear expression v_x in the first equation by the nonlinear expression $(p(v))_x$, where p is a function with $p'(v) > 0$. In this case, there arises the so-called "p-system", which is well-known in the theory of hyperbolic systems of conservation laws:

$$\begin{cases} u_t - (p(v))_x &= 0, \\ v_t - u_x &= 0. \end{cases}$$

This system is another simple (although more complex than the Hopf equation (1.1)) but important model in the field of gas dynamics. Alas, nowadays, whatever be the non-linearity $p = p(v)$, nobody in the entire world knows how to define the "correct" entropy solution of this problem.

Thus a slightest nonlinear perturbation of a simple linear system results in an extremely difficult unsolved problem[9] in the field of nonlinear analysis.

[9] *NT* — These are words of S. N. Kruzhkov, spoken out in 1997 shortly before his passing away. Since then,

Hopefully, the topical, simple-to-formulate, both “natural” and difficult field of nonlocal theory of quasilinear conservation laws will yet attract the attention of young, deep-thinking researchers, able to invent new approaches away from the traditional guidelines.

Acknowledgments

The authors would like to express their gratitude to the editors Etienne Emmrich and Petra Wittbold for having provided the possibility to publish the S. N. Kruzhkov lectures and thus make the lectures available to a wide audience. The manuscript benefited much from their careful reading and many helpful remarks.

The present publication would not have been possible without the active involvement by Boris Andreianov. He modestly positioned himself as the translator of the Russian manuscript; in fact, he initiated the present publication and moreover could be considered as its rightful co-author. Along with the translation he revised our original manuscript to the current western mathematical presentation standards. On top of that he has provided comments on the present state of the subject of conservation laws, that

further advance was made in the research on first-order quasilinear equations. New important approaches became standard, which shed more light on the fundamental theory and on advanced qualitative properties of admissible generalized solutions. We refer the interested reader to the monographs and textbooks [11, 14, 22, 32, 33, 35, 47, 48] which appeared since 1996, and to the references therein.

In particular, the problem mentioned here has been, at least partially, solved. A definition of a generalized solution which leads to a complete well-posedness theory, and which applies in particular to the p-system in one space dimension, has been given in the works of A. Bressan and collaborators (see [11]). These results, and the methods developed to achieve the results, represent a breakthrough in the theory of systems of conservation laws, a breakthrough that occurred more than thirty-five years after the pioneering works of S. N. Kruzhkov establishing the notion of entropy solution for the case of one scalar equation.

Yet the most important case for the applications, the one of multi-dimensional systems of conservation laws, remains very far from being solved. We can simply repeat S. N. Kruzhkov’s words, saying that nowadays, in 2008, nobody in the entire world knows how to define the “correct” notion of solution for this problem!

For the case (also discussed in the above Afterword) of a general merely continuous (but not necessarily Lipschitz, nor Hölder continuous) vector flux function $f = f(u)$, in spite of some further progress (see [2, 5, 42]), a difficult open question persists: whether or not there is uniqueness of a generalized entropy solution in $L^\infty(0,T; L^1(\mathbb{R}^n)) \cap L^\infty(\Pi_T)$ without any additional restriction (such as (6.13)–(6.14)) on the flux function $f = f(u)$.

Let us mention, without any tentative of exhaustivity, that in the last fifteen years progress has been achieved: on the study of boundary-value problems for conservation laws (see, e.g., [35, 38]), on the numerical approximation of entropy solutions (see, e.g., [8, 22]), on the study of fine properties of general (not necessarily piecewise smooth, see, e.g., [24])) entropy solutions using methods of geometric measure theory (see, e.g., [15]) and the new tools of kinetic solutions (see, e.g., [9, 10, 19, 34, 43, 45, 47, 51]) and parameterized families of H-measures (see [40, 41, 46]), on the study of linear problems with irregular coefficients (see, e.g., [1]), on the convergence of the vanishing viscosity method (see [7]), on the study of stability of shock waves, on various generalizations of conservation law (6.10) including nonlocal problems, problems with oscillating or discontinuous in (t, x) coefficients, stochastic problems, problems on manifolds, on the related degenerated diffusion problems (see, e.g., [12, 13, 4]), on the study of singular solutions (see, e.g., [44]), of unbounded solutions (see, e.g., [21, 42]), and on the related new notion of renormalized solution (see [6]). Even a theory of “non-Kruzhkov” solutions to conservation laws was constructed (see [33]), stimulated by physical models with a specific notion of admissibility. Much of the above progress was inspired by “physical” considerations and by the investigation of applied problems.

Thus, although the above Afterword does not reflect the most recent challenges in the theory of first-order quasilinear PDEs, S. N. Kruzhkov’s words sound as topical as ever. And it is certain that, after ten more years, the present footnote will look somewhat obsolete with respect to the new front of research.

the authors would simply be unable to survey. The authors do not belong to the scientific school of S. N. Kruzhkov, but are rather his colleagues who have collaborated with S. N. Kruzhkov on the task of creating and promoting the present course of lectures as a new element of the mathematical education at the Moscow Lomonosov State University. Their scientific interests lie in connected, but yet different branches of PDEs with respect to the subject of the lectures. On the contrary, Boris Andreianov learned the subject directly from S. N. Kruzhkov as a student and as a Ph.D. student. Now he continues to work in the field of the first-order quasilinear PDEs. His contribution to the preparation of the present edition is extremely valuable.

Translated from the Russian manuscript
by Boris P. Andreianov (Besançon)

References

[1] L. Ambrosio. The flow associated to weakly differentiable vector fields: recent results and open problems, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.*, **(8) 10**:1 (2007), 25–41.

[2] B. Andreianov, Ph. Bénilan and S. N. Kruzhkov. L^1-theory of scalar conservation law with continuous flux function, J. Funct. Anal., **171**:1 (2000), 15–33.

[3] V. I. Arnol'd. *Supplementary chapters to the theory of ordinary differential equations.* (Russian), Nauka eds, Moscow, 1982. French transl. in *Chapitres supplémentaires de la théorie des équations différentielles ordinaires.* Éditions Mir, Moscow, 1984.

[4] M. Bendahmane and K. H. Karlsen. Renormalized solutions for quasilinear anisotropic degenerate parabolic equations, *SIAM J. Math. Anal.*, **36**:2 (2004), 405–422.

[5] Ph. Bénilan and S. N. Kruzhkov. Conservation laws with continuous flux functions, *NoDEA Nonlin. Diff. Eq. Appl.*, **3**:4 (1996), 395–419.

[6] Ph. Bénilan, J. Carrillo and P. Wittbold. Renormalized entropy solutions of scalar conservation laws, *Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4)* **29**:2 (2000), 313–327.

[7] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems, *Ann. Math.*, (2) **161**:1 (2005), 223–342.

[8] F. Bouchut and B. Perthame. Kruzhkov's estimates for scalar conservation laws revisited, *Trans. Amer. Math. Soc.*, **350**:7 (1998), 2847–2870.

[9] Y. Brenier. Résolution d'équations d'évolution quasilinéaires en dimension N d'espace à l'aide d'équations linéaires en dimension $N+1$. *J. Diff. Eq.*, **50**:3 (1983), 375–390.

[10] Y. Brenier. L^2 formulation of multidimensional scalar conservation laws, to appear in *Arch. Ration. Mech. Anal.*

[11] A. Bressan. *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*, Oxford Univ. Press, Oxford, 2000.

[12] J. Carrillo. Entropy solutions for nonlinear degenerate problems, *Arch. Ration. Mech. Anal.*, **147**:4 (1999), 269–361.

[13] G.-Q. Chen and B. Perthame. Well-posedness for anisotropic degenerate parabolic-hyperbolic equations, *Ann. Inst. H. Poincaré Anal. Non Lin.*, **20**:4 (2003), 645–668.

[14] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, 2nd ed. Springer, Berlin, 2005.

[15] C. De Lellis, F. Otto and M. Westdickenberg. Structure of entropy solutions for multidimensional scalar conservation laws, Arch. Ration. Mech. Anal., **170**:2 (2003), 137–184.

[16] L. C. Evans. *Partial differential equations*, American Math. Soc., Providence, RI, 1998.

[17] A. F. Filippov. *Problems book on differential equations*, Nauka eds., Moscow, 1985. French transl. in A. Philippov. *Recueil de problèmes d'équations différentielles.* Éditions Mir, Moscow, 1976.

[18] I. M. Gel'fand. Some problems in the theory of quasi-linear equations. (Russian), *Uspehi Mat. Nauk*, **14**:2 (1959), 87–158. English transl. in *Russian Mathematical Surveys*, **2** (1959).

[19] Y. Giga and T. Miyakawa. A kinetic construction of global solutions of first order quasilinear equations, *Duke Math. J.*, **50**:2 (1983), 505–515.

[20] E. Godlewski and P.-A. Raviart. *Hyperbolic systems of conservation laws*, Ellipses, Paris, 1991.

[21] A. Yu. Goritskiĭ and E. Yu. Panov. On locally bounded generalized entropy solutions of the Cauchy problem for a first-order quasilinear equation. (Russian), *Tr. Mat. Inst. Steklova* **236**:1 (2002), *Differ. Uravn. i Din. Sist.*, 120–133. English transl. in *Proc. Steklov Inst. Math.*, **236**:1 (2002), 110–123.

[22] H. Holden and N. H. Risebro. *Front tracking for hyperbolic conservation laws*, Springer, New York, 2002.

[23] E. Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.*, **3** (1950), 201–230.

[24] M. V. Korobkov and E. Yu. Panov. Isentropic solutions of quasilinear equations of the first order, *Mat. Sb.*, **197**:5 (2006), 99-124. English transl. in *Sb. Math.*, **197**:5 (2006), 727-752.

[25] S. N. Kružkov. Generalized solutions of the Cauchy problem in the large for first order nonlinear equations. (Russian) , *Dokl. Akad. Nauk. SSSR*, **187**:1 (1969), 29–32. English transl. in Soviet Math. Dokl. **10** (1969), 785–788.

[26] S. N. Kružkov. First order quasilinear equations with several independent variables. (Russian), *Mat. Sb.*, **81**:123 (1970), 228–255. English transl. in *Math. USSR Sb.*, **10** (1970), 217–243.

[27] S. N. Kružkov. *Nonlinear Partial Differential Equations (Lectures). Part II. First-order equations*, Moscow State Lomonosov Univ. eds, Moscow, 1970.

[28] S. N. Kruzhkov and E. Yu. Panov. First-order conservative quasilinear laws with an infinite domain of dependence on the initial data. (Russian), Dokl. Akad. Nauk SSSR, **314**:1 (1990), 79–84. English transl. in *Soviet Math. Dokl.*, **42**:2 (1991), 316–321.

[29] O. A. Ladyženskaya. On the construction of discontinuous solutions of quasi-linear hyperbolic equations as limits of solutions of the corresponding parabolic equations when the "coefficient of viscosity" tends towards zero. (Russian), *Dokl. Akad. Nauk SSSR*, **111**:2 (1956), 291–294.

[30] P. D. Lax. Hyperbolic systems of conservation laws. II, *Comm. Pure Appl. Math.*, **10**:1 (1957), 537–566.

[31] P. D. Lax. Weak solutions of nonlinear hyperbolic equations and their numerical computation. *Comm. Pure Appl. Math.*, **7**:1 (1954), 159–193.

[32] P. D. Lax. *Hyperbolic partial differential equations.* Courant Lect. Notes in Math., 14. American Math. Soc., Providence, RI, 2006.

[33] P .G. LeFloch. *Hyperbolic systems of conservation laws. The theory of classical and nonclassical shock waves*, Lect. Math. ETH Zürich. Birkhäuser, Basel, 2002.

[34] P.-L. Lions, B. Perthame and E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related equations, *J. Amer. Math. Soc.*, **7**:1 (1994), 169–191.

[35] J. Málek, J. Nečas, M. Rokyta and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, Chapman & Hall, London, 1996.

[36] O. A. Oleĭnik. On Cauchy's problem for nonlinear equations in a class of discontinuous functions. (Russian), *Doklady Akad. Nauk SSSR*, **95**:3 (1954), 451–454.

[37] O. A. Oleĭnik. Discontinuous solutions of non-linear differential equations. (Russian), *Uspehi Mat. Nauk*, **12**:3 (1957), 3–73. English transl. in *Russian Mathematical Surveys*, **3** (1957).

[38] F. Otto. Initial-boundary value problem for a scalar conservation law, C. R. Acad. Sci. Paris Sér. I Math., **322**:8 (1996), 729–734.

[39] E. Yu. Panov. Uniqueness of the solution of the Cauchy problem for a first-order quasilinear equation with an admissible strictly convex entropy. (Russian), *Mat. Zametki*, **55**:55 (1994), 116–129. English transl. in *Math. Notes*, **55**:5-6 (1994), 517–525.

[40] E. Yu. Panov. On sequences of measure valued solutions for a first order quasilinear equation. (Russian), *Mat. Sb.*, **185**:2 (1994), 87–106; English transl. in *Russian Acad. Sci. Sb. Math.*, **81**:1 (1995), 211–227.

[41] E. Yu. Panov. Property of strong precompactness for bounded sets of measure valued solutions of a first-order quasilinear equation. (Russian), *Mat. Sb.*, **190**:3 (1999), 109–128; English transl. in Sb. Math., **190**:3 (1999), 427–446.

[42] E. Yu. Panov. On the theory of generalized entropy solutions of the Cauchy problem for a first-order quasilinear equation in the class of locally integrable functions. (Russian), *Izvest. RAN*, **66**:6 (2002), 91–136; English transl. in *Izvestiya: Mathematics*, **66**:6 (2002), 1171–1218.

[43] E. Yu. Panov. On kinetic formulation of first-order hyperbolic quasilinear systems, Ukr. Math. Bull., **1**:4 (2004), 553–568.

[44] E. Yu. Panov and V. M. Shelkovich. δ'-Shock waves as a new type of solutions to systems of conservation laws, *J. Diff. Eq.*, **228**:1 (2006), 49–86.

[45] E. Yu. Panov. On kinetic formulation of measure valued solutions for hyperbolic first-order quasilinear equations with several space variables, in: *Analytical Approaches to Multidimensional Balance Laws*, O. Rozanova ed., Nova Press, 2007.

[46] E. Yu. Panov. Existence of strong traces for quasi-solutions of multidimensional conservation laws, J. Hyperbolic Differ. Equ., **4**:4 (2007), 729–770.

[47] B. Perthame. *Kinetic formulation of conservation laws*, Oxford Univ. Press, Oxford, 2002.

[48] D. Serre. *Systems of conservation laws. 1. Hyperbolicity, entropies, shock waves*. and *Systems of conservation laws. 2. Geometric structures, oscillations, and initial-boundary value problems*. Transl. from the 1996 French original. Cambridge Univ. Press, Cambridge, 1999, and 2000.

[49] J. Smoller. *Shock waves and reaction-diffusion equations*, 2nd ed., Springer, New York, 1994.

[50] A. N. Tihonov and A. A. Samarskiĭ. On discontinuous solutions of a quasilinear equation of first order. (Russian), *Dokl. Akad. Nauk SSSR*, **99**:1 (1954), 27–30.

[51] A. Vasseur. Strong traces for solutions of multidimensional scalar conservation laws, *Arch. Ration. Mech. Anal.*, **160**:3 (2001), 181–193.

[52] A. I. Vol'pert. Spaces BV and quasilinear equations, *Mat. Sb.*, **73** (115) (1967), 255–302. English transl. in *Math. USSR Sb.*, **2** (1967), 225–267.

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