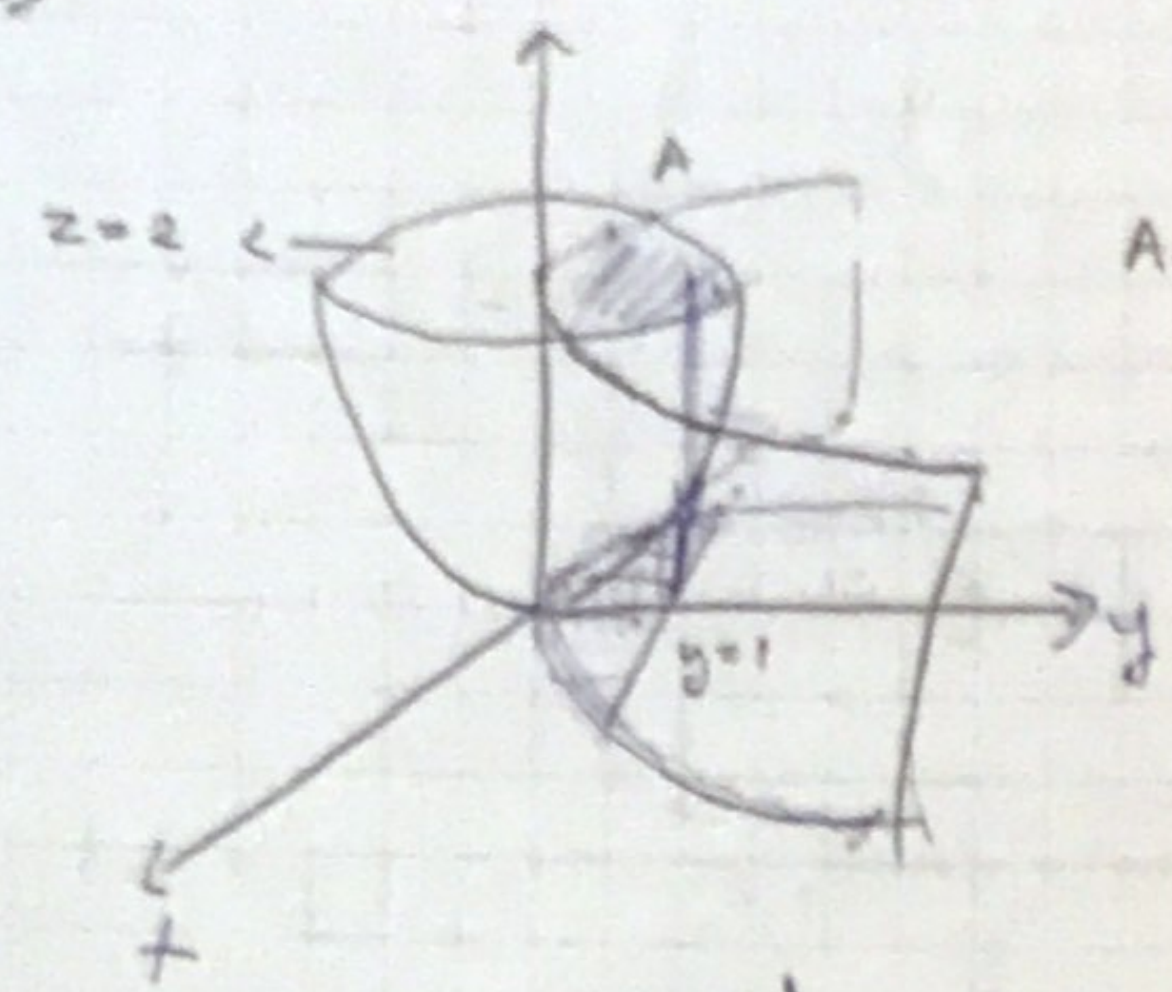
V 26.03.2018. VII nedelja

① Izračunati V tijela ograničenog sa $z=2,\ z=x^2+y^2,\ y=1,\ y=x^2$.

R



$$A: \begin{cases} z=2 \\ x^2+y^2=z \\ y=x^2 \end{cases} \Rightarrow y+y^2=2$$

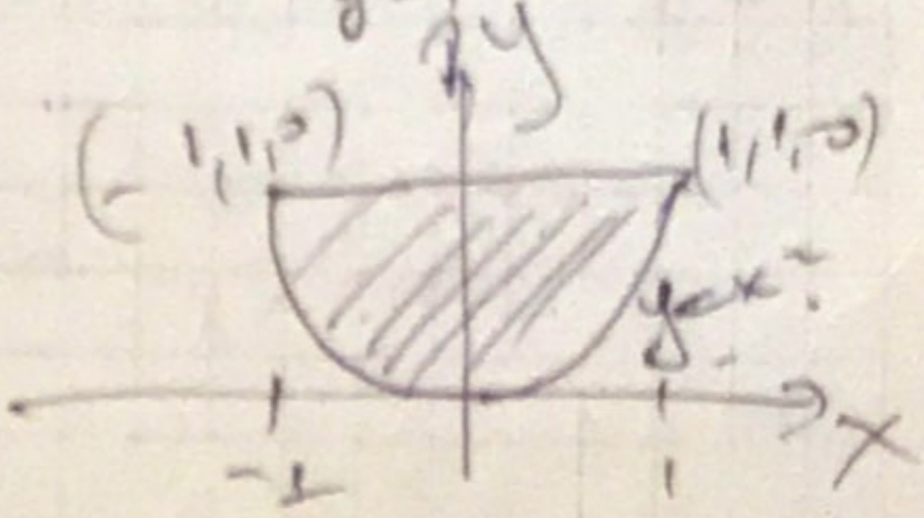
$$y^2+y-2=0$$

$$(y+2)(y-1)=0$$

$$y=1$$

$A(-1,1,2)$

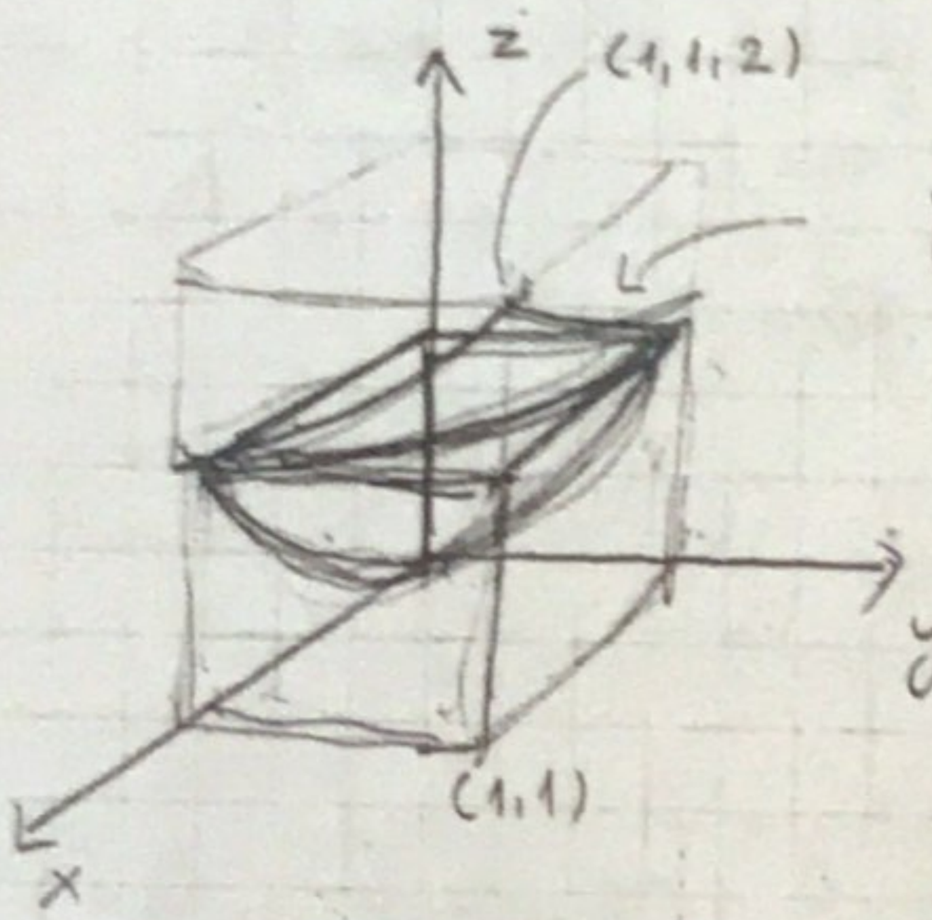
$B(1,1,2)$



$$V=\int_{-1}^{1}dx\int_{x^2}^{1}dy\int_{x^2+y^2}^{2}1\,dz=\int_{-1}^{1}dx\int_{x^2}^{1}2-(x^2+y^2)\,dy=\ldots=\frac{64}{35}$$

② Promjeniti redoslijed integracije $\int_0^1\int_0^1\int_0^{x^2+y^2}f(x,y,z)\,dz\,dy\,dx$ na $z-y-x$

R



?

0

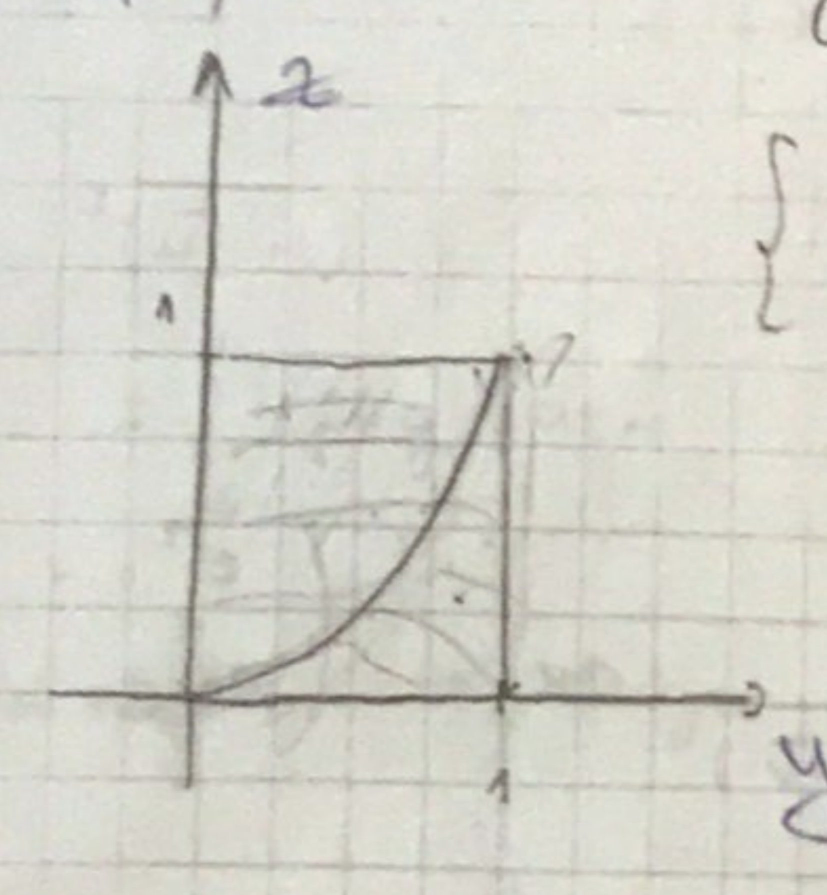
$$\begin{cases} z=1 \\ x^2+y^2=1 \end{cases}$$

$x^2+y^2=2 \rightarrow$ zato uzimamo $z=2$

$\;\;\|\quad\;\|$

$\;\;1\quad\;1$

$z\in(0,1)$



Oyz

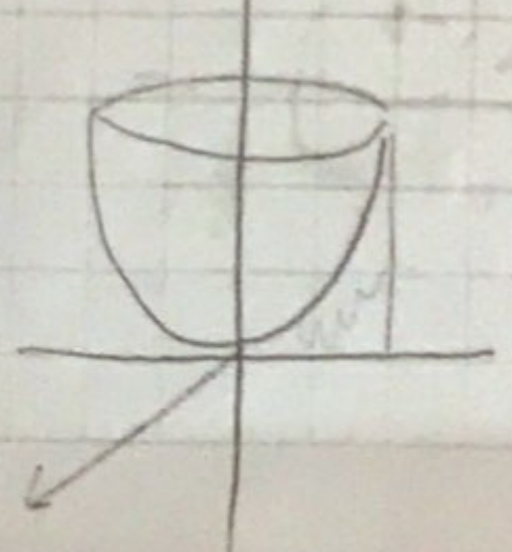
$$\begin{cases} x=0 \\ x^2+y^2=z \end{cases} \Rightarrow z=y^2$$

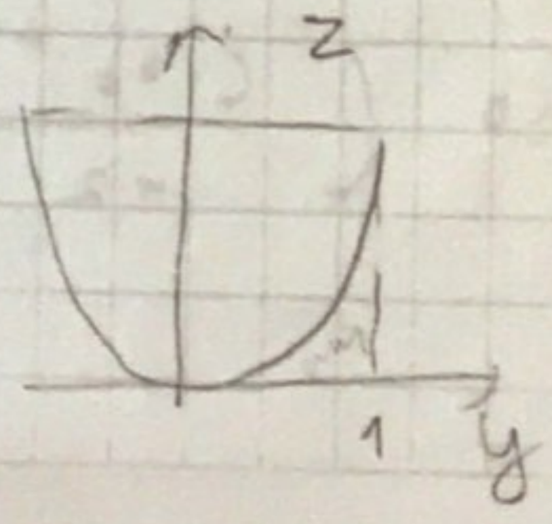
$z\in(0,2)$

$z=x^2+y^2$, $z=1$ $\Big\}\ x^2+y^2=1$

$z=x^2+y^2 \Rightarrow x=\pm\sqrt{z-y^2}$

$$\rightarrow \int_0^1 dz\int_0^{\sqrt{z}}dy\int_{\sqrt{z-y^2}}^{1}dx \;+$$

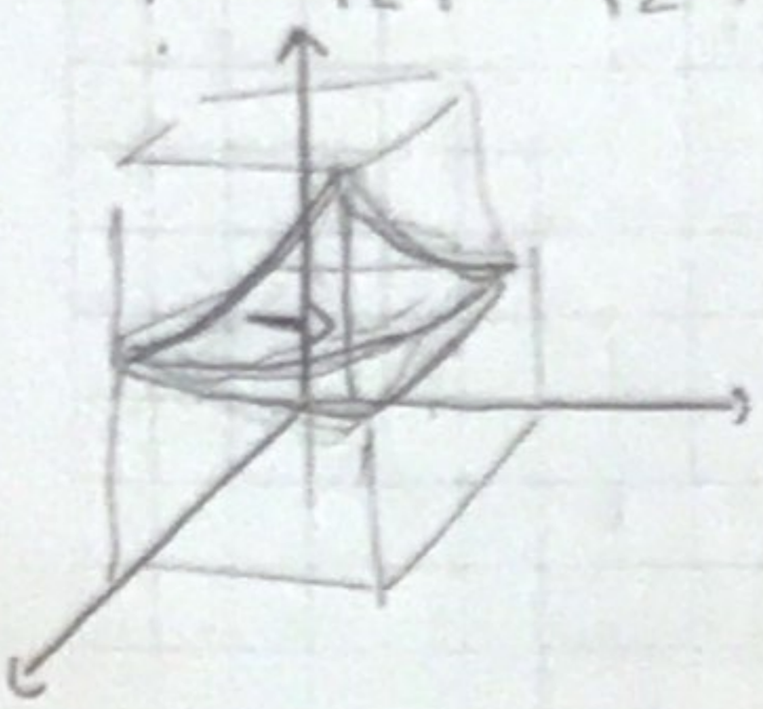


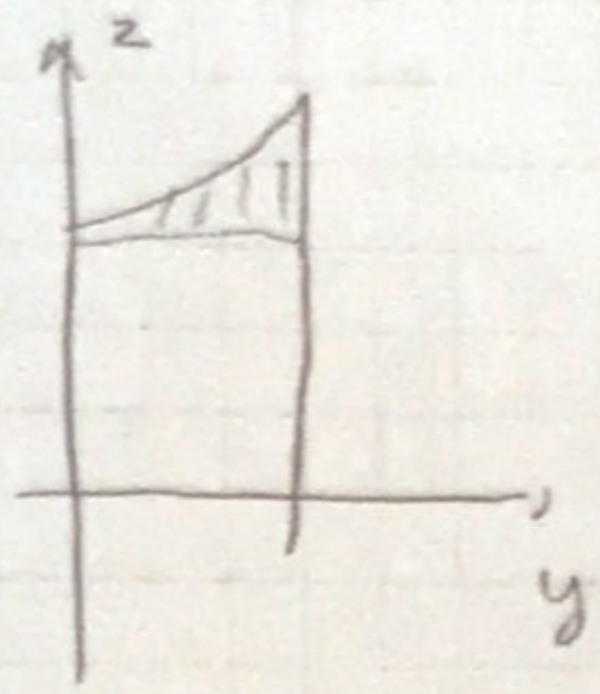


$$\rightarrow \int_0^1 dz\int_{\sqrt{z}}^{1}dy\int_0^{1}f(x,y,z)\,dx \;+$$

vihna

$$\int_1^2 dz \int_{\sqrt{z-1}}^1 dy \int_{\sqrt{z-y^2}}^1 f\,dx$$





$$\begin{cases} x = 1 \\ x^2 + y^2 = z \end{cases}$$

$$\Rightarrow y = \sqrt{z-1}$$

Smjena promjenljive

451
452

Teorema: Neka je $g: U \to V$ difeomorfizam otvorenog skupa $U \subseteq \mathbb{R}^n$ na otvoren skup $V \subseteq \mathbb{R}^n$. $\bar{A} \subseteq U$, $A \in J$, $f: V \to \mathbb{R}$ realna f-ja. Tada je f integrabilna na $g(A)$ akko je $(f\circ g)|\det g'|$ integrabilna na A i važi $\int_{g(A)} f = \int_A (f\circ g)|\det g'|$.

1) Polarne koordinate

$x = r\cdot\cos\varphi$

$y = r\sin\varphi$

smjena: $g(r,\varphi) = (\underbrace{r\cos\varphi}_{x}, \underbrace{r\sin\varphi}_{y})$

gledamo f-ju (x,y) preko (r,φ)

$$\frac{D(x,y)}{D(r,\varphi)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

$f\circ g = f(r\cos\varphi, r\sin\varphi)$

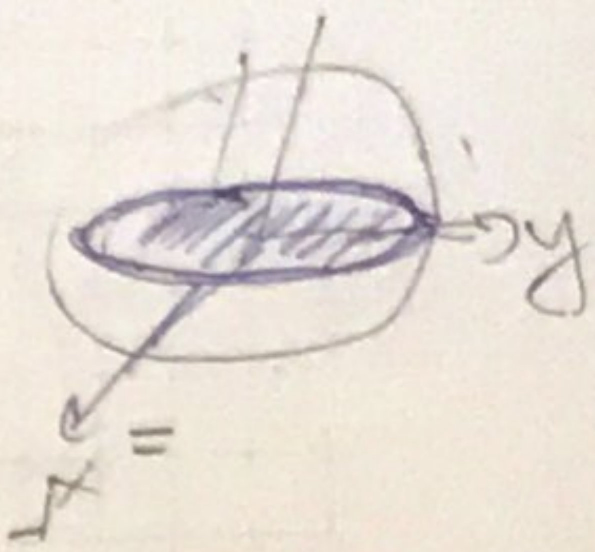
primjer Krug poluprečnika R

$$\iint_D 1\,dx\,dy = \int_0^R \int_0^{2\pi} 1\cdot r\,d\varphi\,dr = 2\pi\,\frac{r^2}{2}\Big|_0^R = 2\pi\,\frac{R^2}{2} = \pi\cdot R^2$$

r se kreće od 0-R

ugao se kreće od 0 do 2π

primjer 2 Zapremina lopte R $\left(x^2+y^2+z^2\le R^2\right)$



$$\iint_{D_R} dx\,dy \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz = 2\iint_{D_R}\sqrt{R^2-x^2-y^2}\,dx\,dy =$$

$x = r\cos\varphi$

$y = r\sin\varphi$

disk

(tu ćemo primjeniti polarne koordinate)

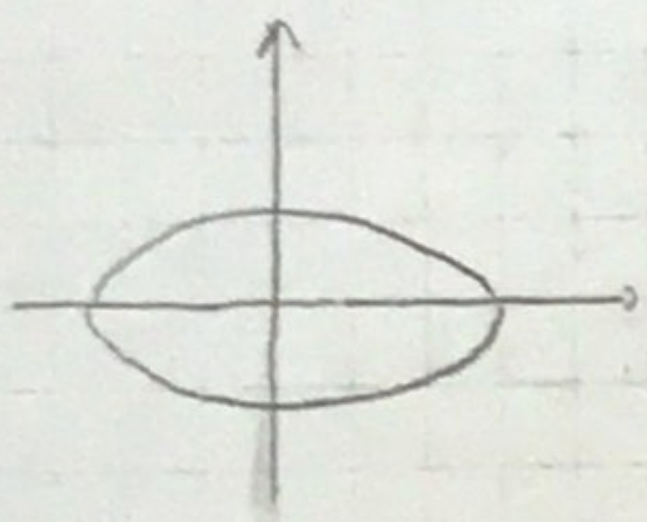
$$= 2\int_0^R\int_0^{2\pi}\sqrt{R^2-r^2}\cdot r\,d\varphi\,dr = \dots = \frac{4\pi R^3}{3}$$

$R^2-r^2=s$

primjer 3 Površina figure ograničene sa elipsom.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=r^2\le 1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$x = ar\cos\varphi$ $\quad \varphi\in(0,2\pi)$

$y = br\sin\varphi$ $\quad r\in(0,1)$

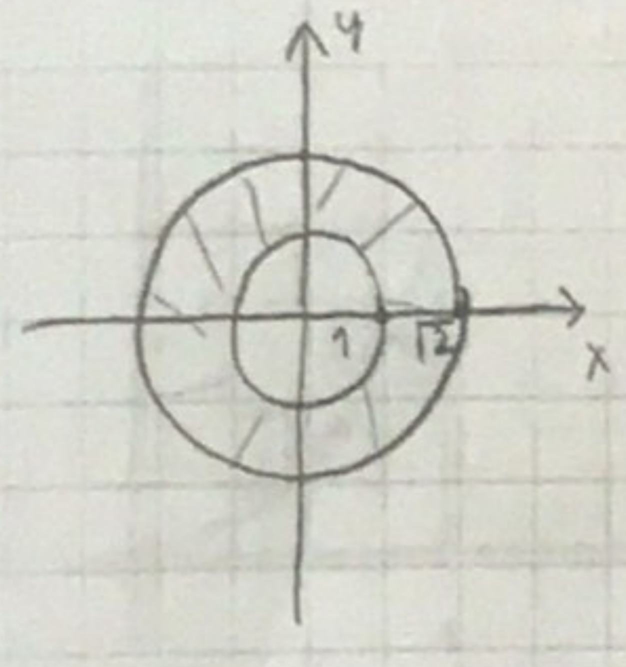
$|J| = abr$

$$\Rightarrow \iint 1\,dx\,dy = \int_0^1\int_0^{2\pi} abr\,d\varphi\,dr = ab\,2\pi\cdot\frac{1}{2} = \pi ab$$

1. Naći integral $\displaystyle\iint_{1\le x^2+y^2\le 2=(\sqrt{2})^2} xy\,dx\,dy$.

$t=xy$

R:



$x = r\cos\varphi$

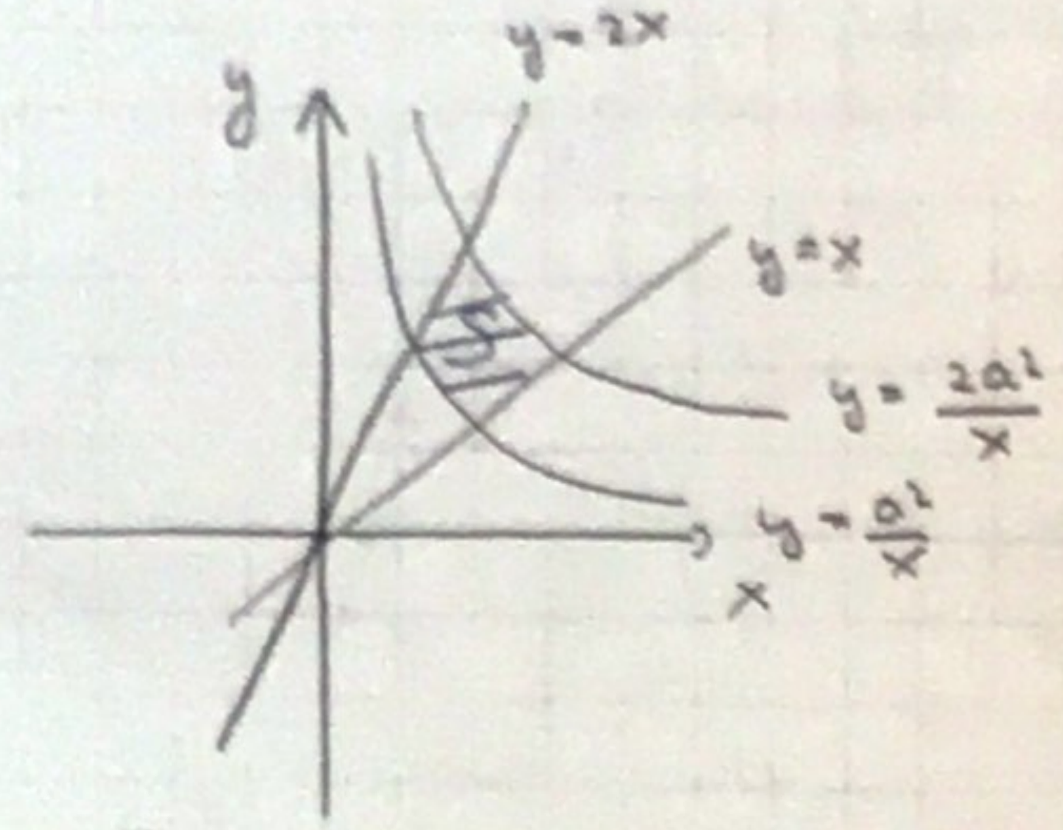
$y = r\sin\varphi$

$|J| = r$

$$\int_1^{\sqrt{2}}\int_0^{2\pi} r^3\cos\varphi\sin\varphi\,d\varphi\,dr = \int 0 = 0$$

(2.) Odrediti površinu figure ograničene sa $xy = a^2$, $y = x$, $xy = 2a^2$, $y = 2x$, $y > 0$, $x > 0$.

R



$u = xy$ $\qquad x = x(u,v)$

$v = \frac{y}{x}$ $\qquad y = y(u,v)$

$a^2 \le u \le 2a^2$

$1 \le v \le 2$

To inv. presl.

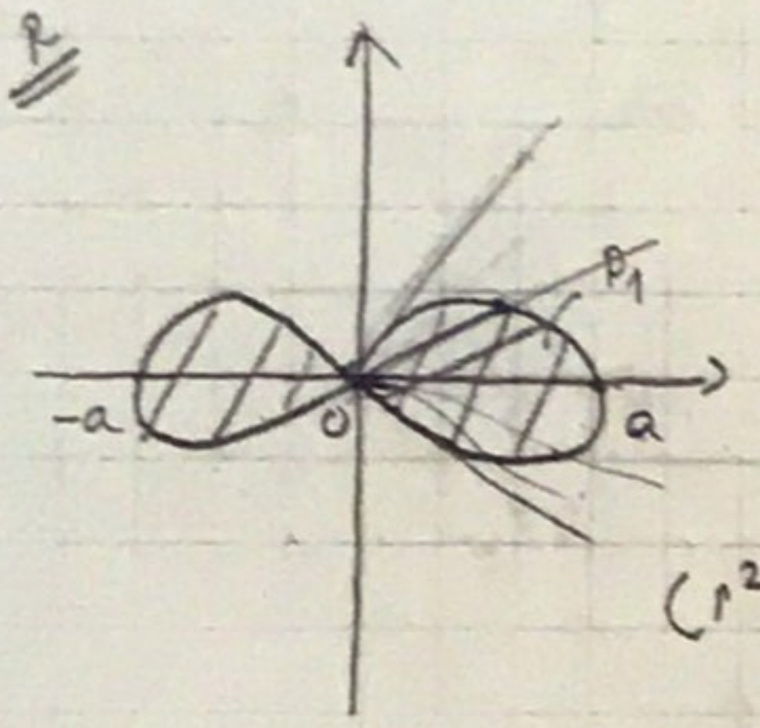
$$\frac{D(x,y)}{D(u,v)} \overset{\downarrow}{=} \frac{1}{\frac{D(u,v)}{D(x,y)}} = \frac{1}{\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}} = \frac{1}{\frac{y}{x} + \frac{y}{x}} = \frac{1}{2\frac{y}{x}} = \frac{1}{2v}$$

$$\iint_S 1\,dx\,dy = \int_{a^2}^{2a^2} du \int_1^2 \frac{1}{2v}\,dv = \frac{1}{2}\int_{a^2}^{2a^2} \ln 2\,du = \frac{\ln 2 \cdot a^2}{2}$$

(3.) $(x^2+y^2)^2 = a^2(x^2-y^2)$, $a > 0$ Lemiskata

Izračunati njenu P.

R



$x = r\cos\varphi$

$y = r\sin\varphi$

[kad znamo φ kako se kreće, znaćemo i r]

$$(r^2\cos^2\varphi + r^2\sin^2\varphi)^2 = a^2 r^2 \cos 2\varphi$$

$$r^4 = a^2 r^2 \cos 2\varphi$$

$$r^2 = a^2 \cos 2\varphi$$

$$r = a\underbrace{\sqrt{\cos 2\varphi}}_{\ge 0} \Rightarrow \boxed{r \in [0, a\sqrt{\cos 2\varphi}]}$$

$\cos 2\varphi \ge 0$) na ovaj način nađemo ugao

$$2\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\boxed{\varphi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)}$$

J

$$P = 2P_1 = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{a\sqrt{\cos 2\varphi}} r\,dr = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{r^2}{2}\Big|_0^{a\sqrt{\cos 2\varphi}} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2\cos(2\varphi) = \dots$$

4) Izračunati površinu: (unutrašnjost zadanog tijela)

a) $\frac{x^2}{k^2} + \frac{y^2}{h^2} = \frac{ax}{k} + \frac{by}{h}$, $a, b, k, h > 0$

b) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{2xy}{c^2}$, $xy > 0$

a) $\frac{x^2}{k^2} - \frac{ax}{k} + \frac{y^2}{h^2} - \frac{by}{h} = 0$

$$\left(\frac{x}{k} - \frac{a}{2}\right)^2 + \left(\frac{y}{h} - \frac{b}{2}\right)^2 = \frac{a^2+b^2}{4}$$

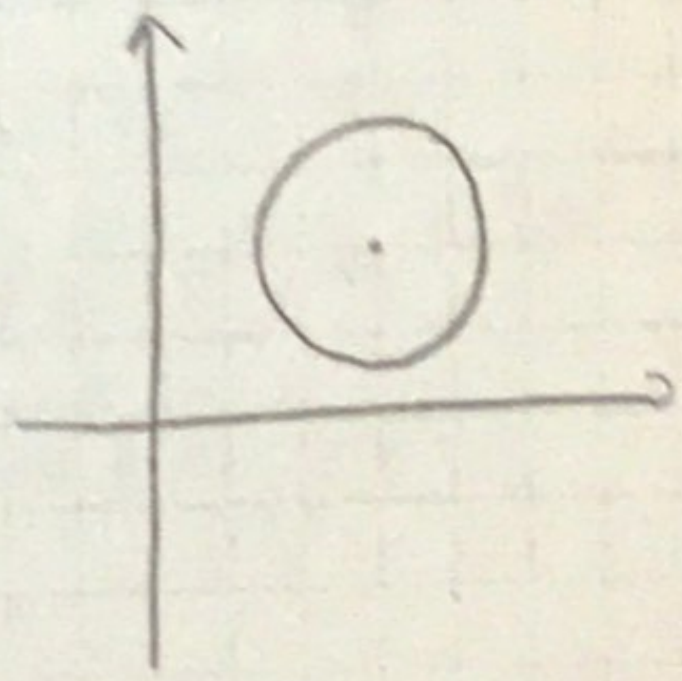
smjena: $\frac{x}{k} - \frac{a}{2} = r\cos\varphi \Rightarrow x = \frac{ka}{2} + kr\cos\varphi$

(iz ovoga se nađe Jakobijan)

$\frac{y}{h} - \frac{b}{2} = r\sin\varphi \Rightarrow y = \frac{hb}{2} + hr\sin\varphi$

$r^2 = \frac{a^2+b^2}{4}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$r = \frac{\sqrt{a^2+b^2}}{2}$

$\frac{x}{a} = r\cos\varphi$

$\frac{y}{b} = r\sin\varphi$

$$\iint_D 1\,dx\,dy = \int_0^{\frac{\sqrt{a^2+b^2}}{2}} \int_0^{2\pi} kh\,r\,dr\,d\varphi = \pi(k \cdot h)\frac{a^2+b^2}{4}$$

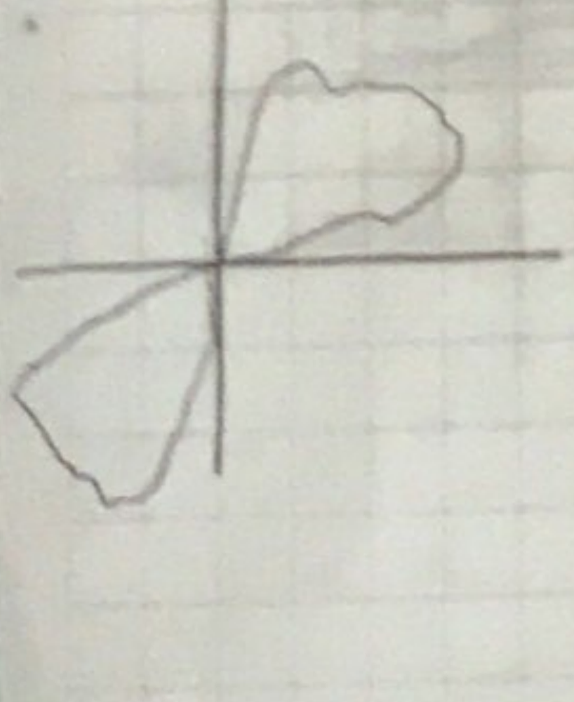
II način

$\frac{x}{k} - \frac{a}{2} = \frac{\sqrt{a^2+b^2}}{2} r\cos\varphi$

$\frac{y}{h} - \frac{b}{2} = \frac{\sqrt{a^2+b^2}}{2} r\sin\varphi$

$r \in (0,1)$

b) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{2xy}{c^2}$



$x = ar\cos\varphi$

$y = br\sin\varphi$

$$(r^2)^2 = \frac{2ab\,r^2\cos\varphi\sin\varphi}{c^2}$$

$$r^2 = \frac{ab\sin 2\varphi}{c^2}$$

$$r = \frac{\sqrt{ab\sin 2\varphi}}{c}$$

$\sin 2\varphi > 0$ $\qquad t \in (0, 4\pi)$

$\sin(t) > 0$

za $t \in (0, \pi)$ $\wedge$ $\qquad t \in (2\pi, 3\pi)$

$\qquad \varphi \in (\pi, \frac{3\pi}{2})$

$0 < 2\varphi < \pi$

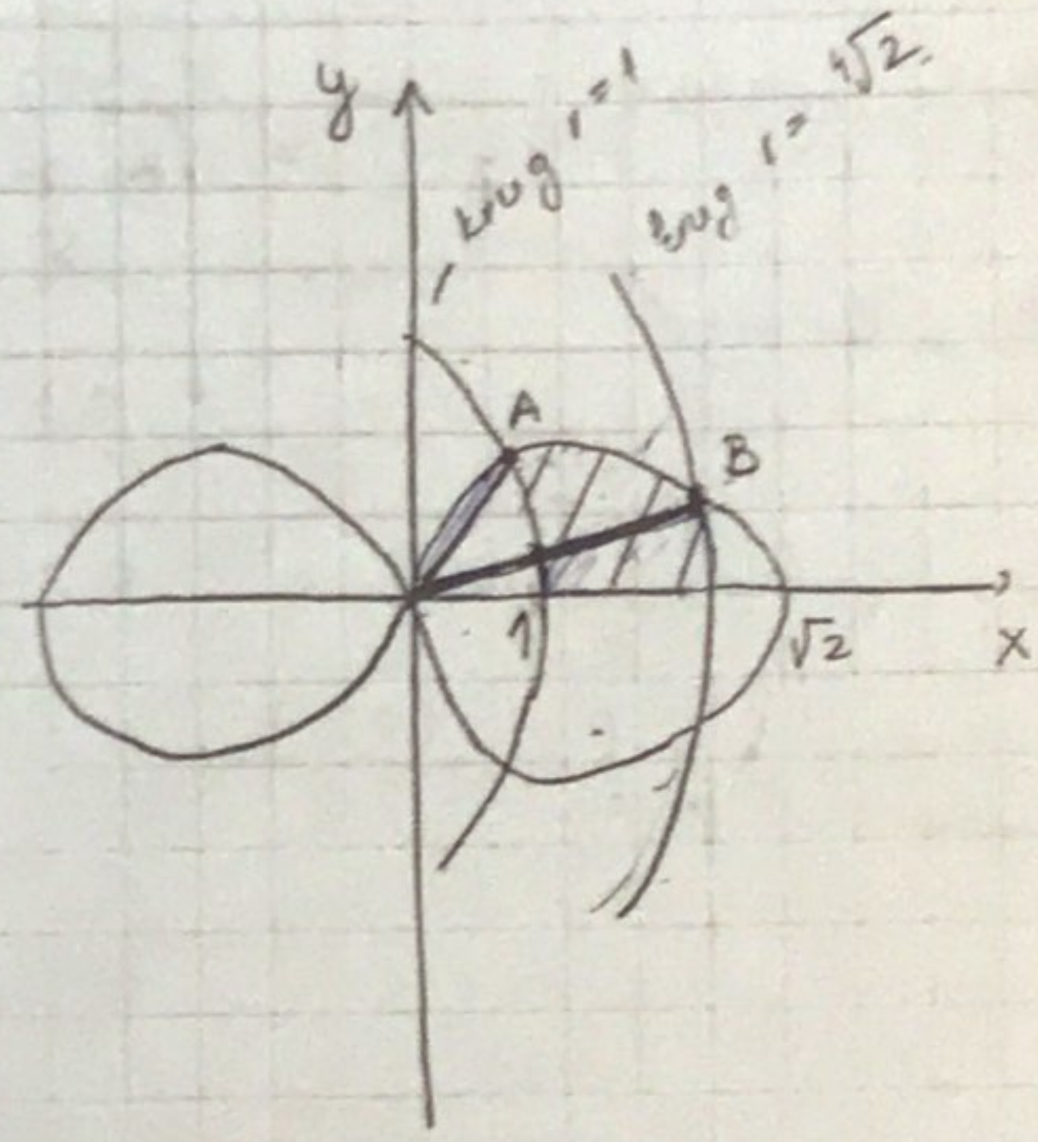
$0 < \varphi < \frac{\pi}{2}$

$$\Rightarrow P = 2 \cdot \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\sqrt{ab}}{c} \cdot \sqrt{\sin 2\varphi}} \underbrace{abr}_{\text{Jakobijan}} \cdot dr = \ldots = \frac{a^2 b^2}{c^2}$$

5. Izračunati površinu oblasti $G = \{(x,y) \mid (x^2+y^2)^2 \le 2(x^2-y^2),\ 1 \le x^2+y^2 \le \sqrt{2}\}$.

$\underbrace{}_{\text{lemniskata}}$, $a = \sqrt{2}$

R:



$(x^2+y^2)^2 = 2(x^2-y^2)$

$x = r\cos\varphi$

$y = r\sin\varphi$

$r^4 = 2r^2\cos 2\varphi$

$r^2 = 2\cos 2\varphi$

$r = \sqrt{2}\sqrt{\cos 2\varphi}$

$$A: \begin{cases} x^2+y^2 = 1 \\ (x^2+y^2)^2 = 2(x^2-y^2) \end{cases} \Leftrightarrow \quad 1 = 2(x^2-y^2)$$

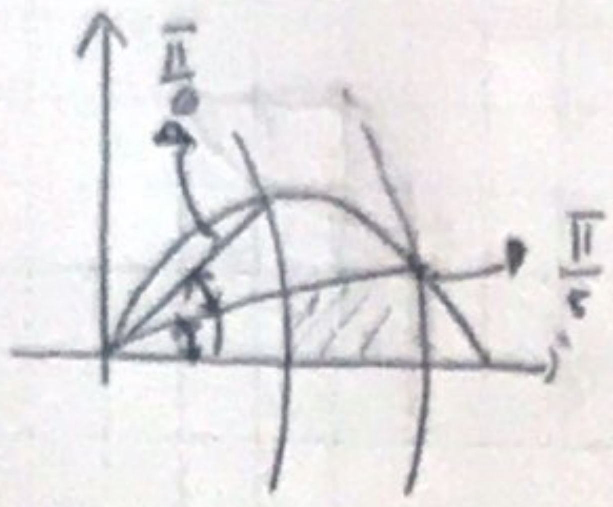
$$\begin{cases} x^2 - y^2 = \frac{1}{2} \\ x^2+y^2 = 1 \end{cases} \Rightarrow \quad \begin{matrix} y^2 = \frac{1}{4} \Rightarrow y = \frac{1}{2} \\ x^2 = \frac{3}{4} \Rightarrow x = \frac{\sqrt{3}}{2} \end{matrix}$$

u toj tački A: $y = r\sin\varphi$ $\wedge$ $r = 1$ (jer se nalazimo u krug $r = 1$)

$$\frac{1}{2} = \sin\varphi \Rightarrow \varphi = \frac{\pi}{6}$$

$$B: \begin{cases} x^2+y^2 = \sqrt{2} \\ (x^2+y^2)^2 = 2(x^2-y^2) \end{cases} \quad \ldots \quad \varphi_2 = \frac{\pi}{8}$$

$$P = 4\left[\int_0^{\frac{\pi}{8}} d\varphi \int_1^{\sqrt[4]{2}} r\,dr + \int_{\frac{\pi}{8}}^{\frac{\pi}{6}} d\varphi \int_1^{\sqrt{2\cos 2\varphi}} r\,dr\right] = 4\left[\frac{\pi}{8}\cdot\frac{1}{2}\,(\sqrt{2}-1) + \int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \frac{(2\cos 2\varphi - 1)}{2}\,d\varphi\right] = \dots$$



⑨ Uopštene polarne koordinate

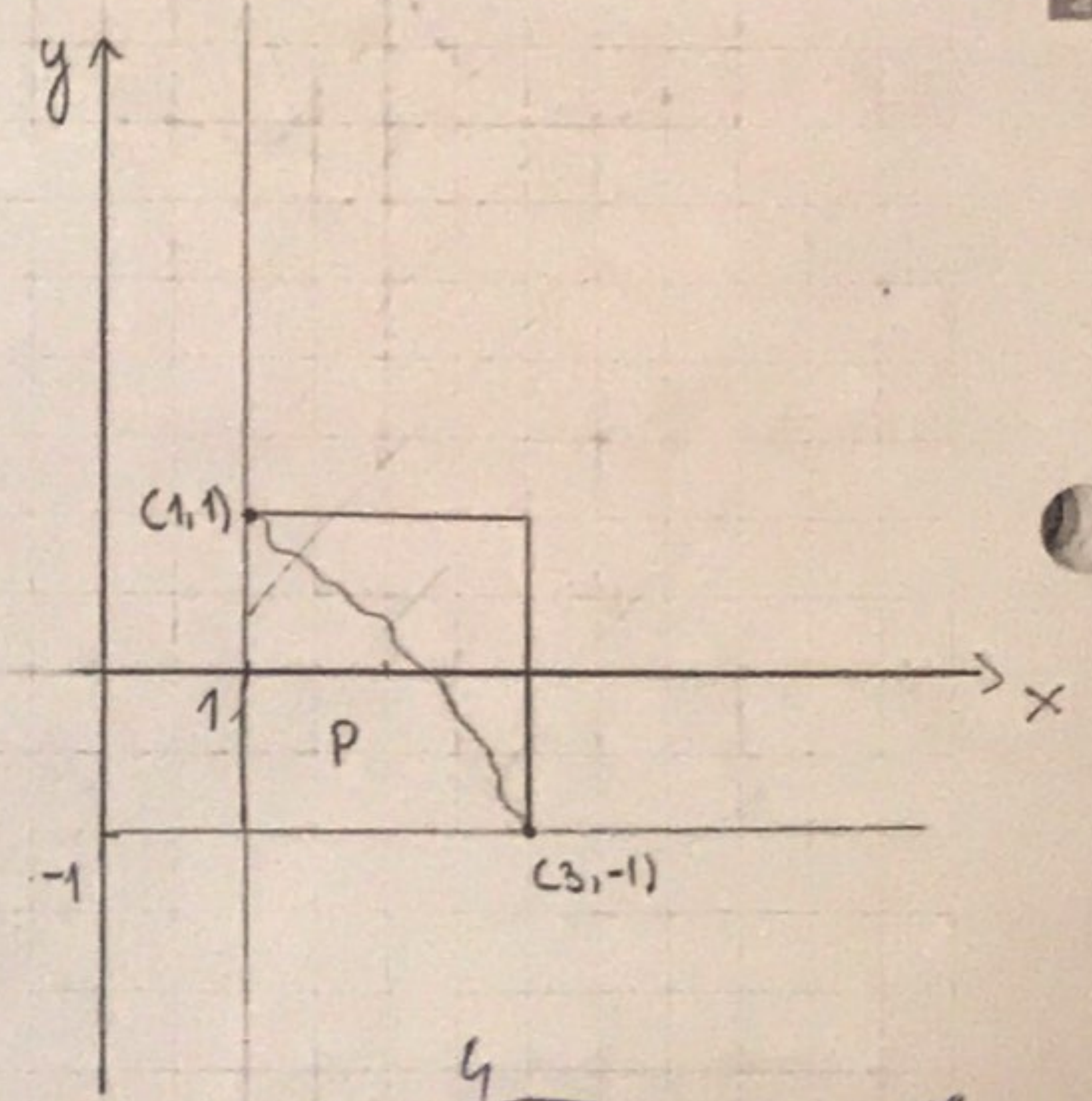
$$x - x_0 = a r^{\beta} \cos^{\alpha}\varphi$$
$$y - y_0 = b r^{\beta} \sin^{\alpha}\varphi \quad , \; a, b, \alpha, \beta \in \mathbb{R}$$
$$J = ab\alpha\beta \cdot \cos^{(\alpha-1)}(\varphi)\sin^{(\alpha-1)}(\varphi)\, r^{2\beta - 1}$$

⑥ Naći površinu oblasti ograničene sa $\sqrt{x-1} + \sqrt{y+1} = \sqrt{2}$, $x = 1$, $y = -1$.

R:
$$x - 1 = r^2 \cos^4\varphi$$
$$y + 1 = r^2 \sin^4\varphi$$
$$\Rightarrow (r^2\cos^4\varphi)^{\frac{1}{2}} + (r^2 \sin^4\varphi)^{\frac{1}{2}} = \sqrt{2}$$
$$r(\cos^2\varphi + \sin^2\varphi) = \sqrt{2}$$
$$\Rightarrow r = \sqrt{2}$$

$x - 1 \geq 0 \Rightarrow x \geq 1$

$y + 1 \geq 0 \Rightarrow y \geq -1$



$1 \leq x \leq 3$

$-1 \leq y \leq 1$

$\varphi \in (0, \frac{\pi}{2})$

$r \in (0, \sqrt{2})$

$|J| = 8r^3 \cos^3\varphi \sin^3\varphi$

$$P = \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} 8r^3 \cos^3\varphi \sin^3\varphi \, dr\, d\varphi = \dots = \frac{1}{12}$$

II
$$\sqrt[4]{x-1} = r\cos\varphi$$
$$\sqrt[4]{y+1} = r\sin\varphi$$
$$x = 1 + r^4\cos^4\varphi$$
$$y = -1 + r^4\sin^4\varphi$$

polarne - u ravni
cilindrične - u prostoru

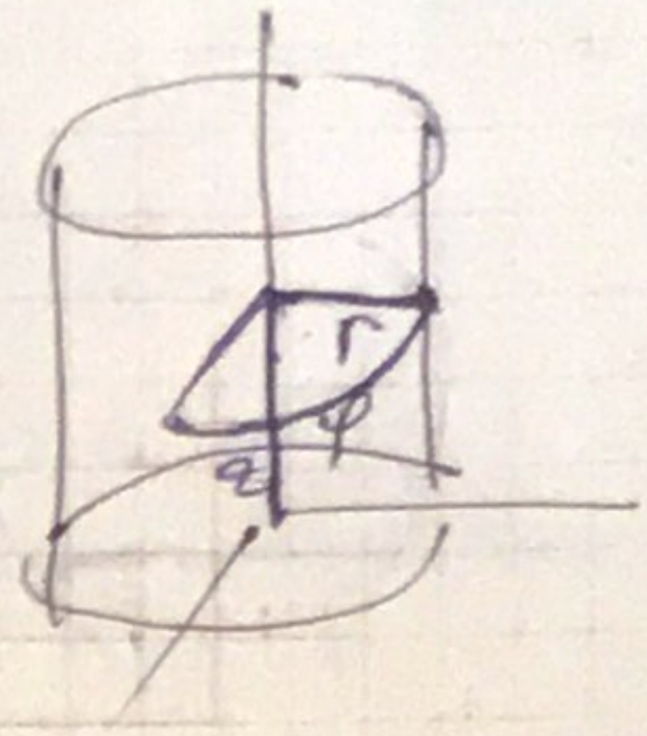
9 Cilindrične koordinate (u prostoru)

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$$|J| = \begin{vmatrix} \cos\varphi & \sin\varphi & 0 \\ -r\sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$



primjer Zapremina cilindra

$x^2+y^2=R^2,\ z\in(0,H)$

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

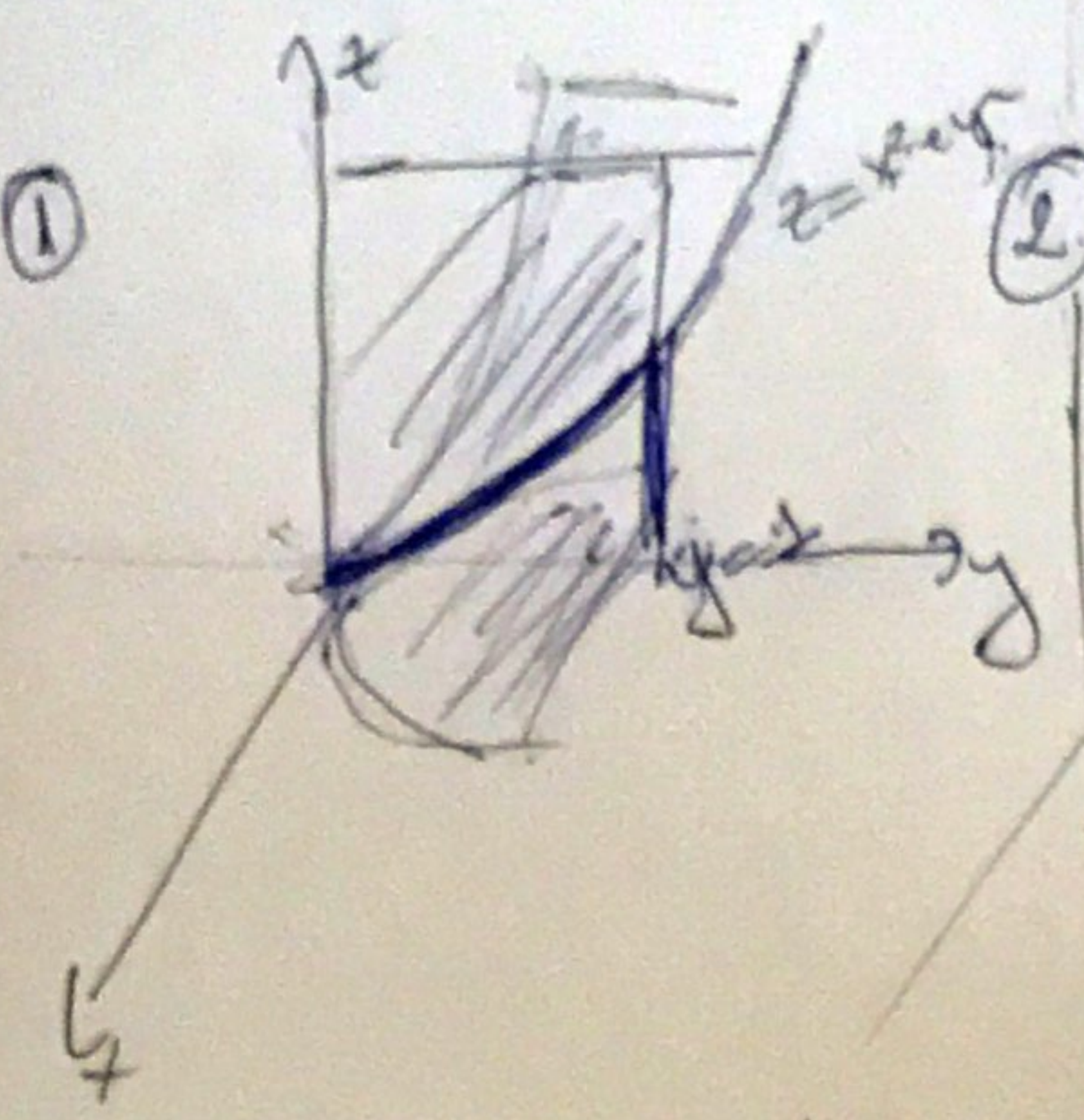
$r \in (0, R)$

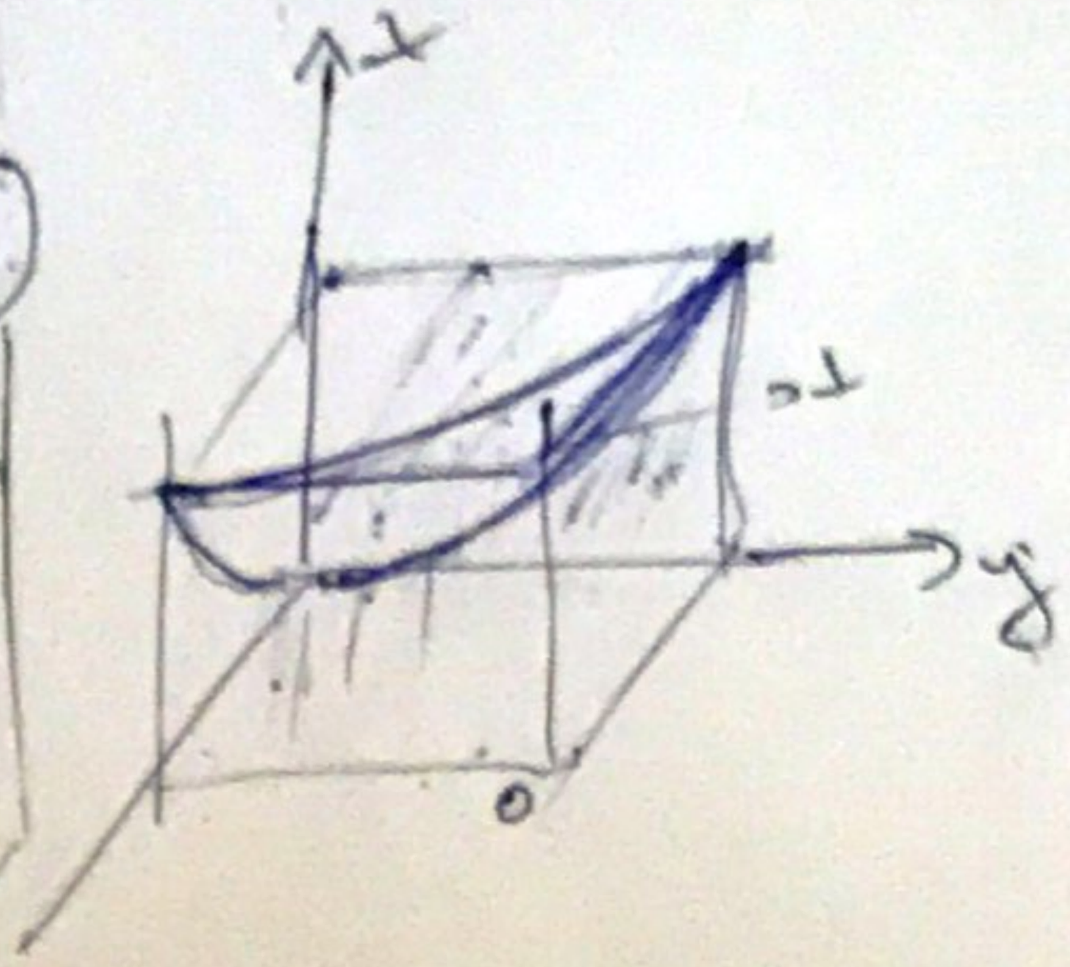
$\varphi \in (0, 2\pi)$

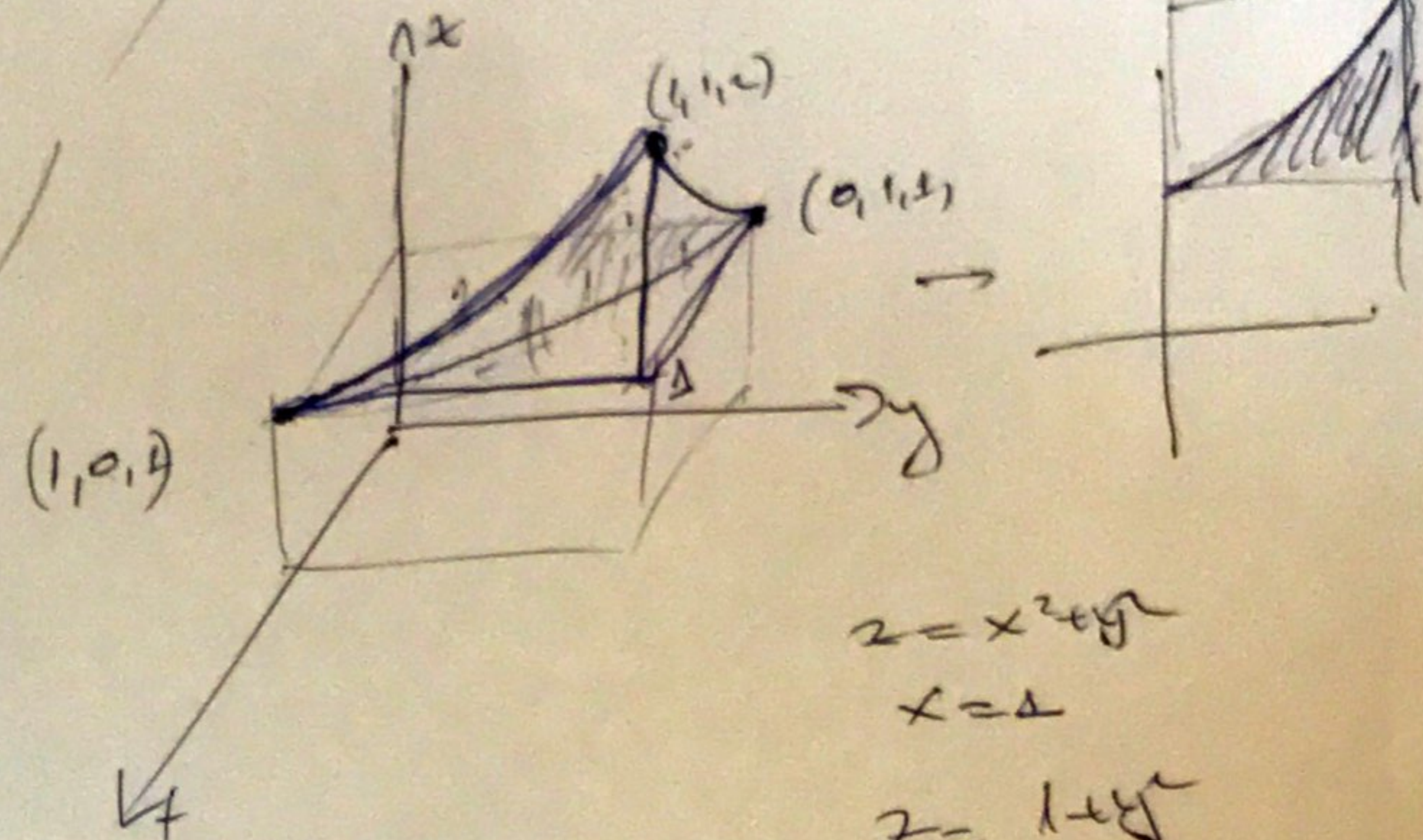
$z \in (0, H)$

$$\Rightarrow V = \int_0^R \int_0^{2\pi} \int_0^H r\,dz\,d\varphi\,dr = 2\pi H \frac{R^2}{2} = \pi H R^2$$

① ②







$z = x^2+y^2$

$x = 1$

$z = 1+y^2$

$y = \sqrt{z-1}$

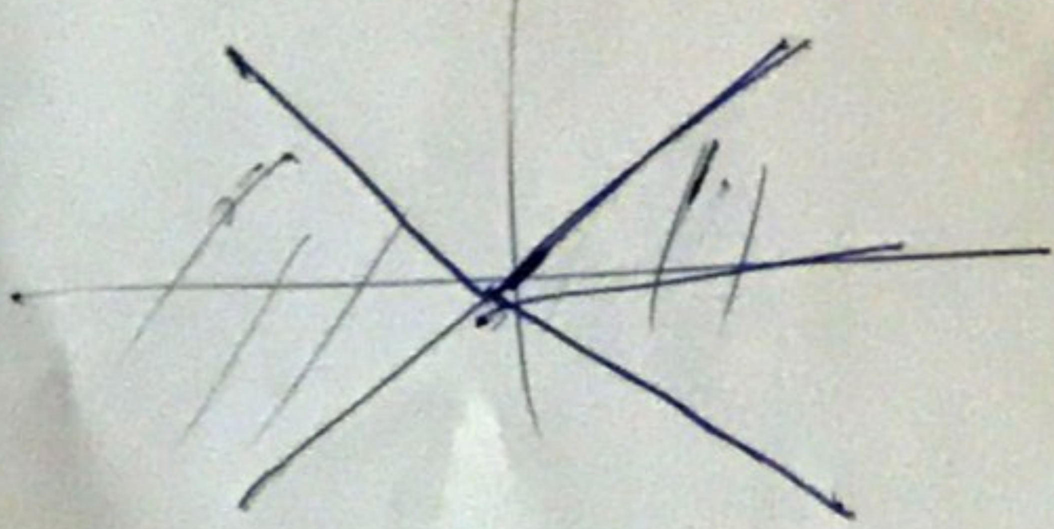
③

$2\varphi \in (0, 4\pi)$

$\underset{t}{\cos 2\varphi} \geq 0$

$2\varphi = t \in (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi) \cup (2\pi, \frac{5\pi}{2}) \cup (\frac{7\pi}{2}, 4\pi)$

$\varphi \in (0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi) \cup (\pi, \frac{5\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi)$



⑤ $x^2+y^2=\sqrt{2}$

$(x^2+y^2)^2 = 2(x^2-y^2) \Rightarrow x^2-y^2=1$

$2x^2 = 1+\sqrt{2}$

$2 \cdot r^2\cos^2\varphi = 1+\sqrt{2}$

$\cancel{2} \cdot \sqrt{2}\, \frac{1+\cos 2\varphi}{\cancel{2}} = 1+\sqrt{2}$

$\cancel{1}+\cos 2\varphi = \cancel{1}+\frac{1}{\sqrt{2}}$

$\cos 2\varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad 2\varphi = \frac{\pi}{4}, \quad \varphi = \frac{\pi}{8}$