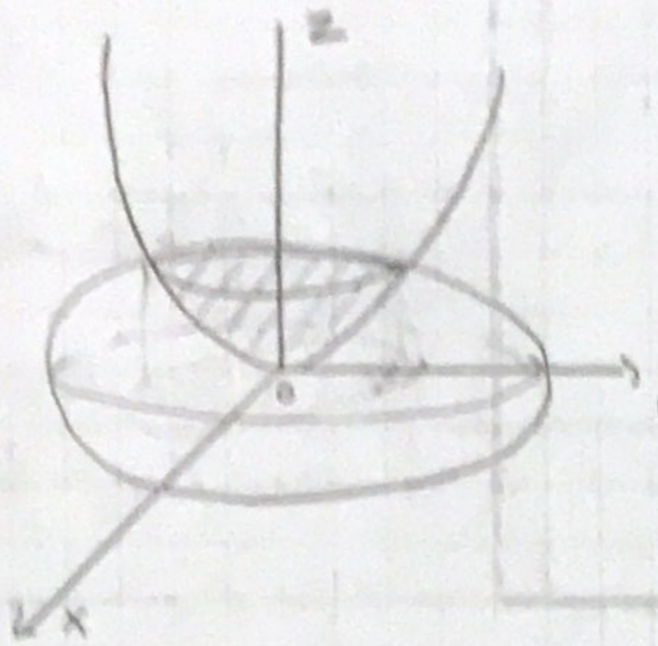
V 08.04.2018. (VIII nedelja) 1

① Izračunati zapreminu tijela ograničenog sa $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ i $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$, $a,b,c\neq 0$



$|z\ (1)\ -\ (2)| \Rightarrow \frac{z}{c}+\frac{z^2}{c^2}=1 \ /\cdot c^2$

$\Rightarrow z^2+c\cdot z-c^2=0$

$z_{1/2}=\frac{-c\pm\sqrt{c^2+4c^2}}{2}$

$z_{1/2}=\frac{-c\pm c\sqrt{5}}{2}$

$z=\frac{-c+c\sqrt{5}}{2}$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{-1+\sqrt{5}}{2}$ - elipsa

$x=ar\cos\varphi$ ← uvodimo eli. koo.

$y=br\sin\varphi$

$z=z$

$r^2\cos^2\varphi+r^2\sin^2\varphi=\frac{\sqrt{5}-1}{2}$

$\Rightarrow r=\sqrt{\frac{\sqrt{5}-1}{2}}$

$\varphi\in(0,2\pi)$

$r\in\left(0,\sqrt{\frac{\sqrt{5}-1}{2}}\right)$

paraboloid: $z=c\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=c\cdot r^2$ (kad uvrstimo koo.)

elipsoid: $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}=c\sqrt{1-r^2}$

$$V=\iiint_V dx\,dy\,dz=\int_0^{2\pi}d\varphi\int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}}dr\int_{c\cdot r^2}^{c\sqrt{1-r^2}}1\cdot abr\cdot dz=\dots$$

$$J=\begin{vmatrix} a\cos\varphi & b\sin\varphi & 0\\ -ar\sin\varphi & br\cos\varphi & 0\\ 0 & 0 & 1\end{vmatrix}=abr$$

Scanned by TapScanner

1) Sferne koordinate

$x = r\cos t \sin\theta$

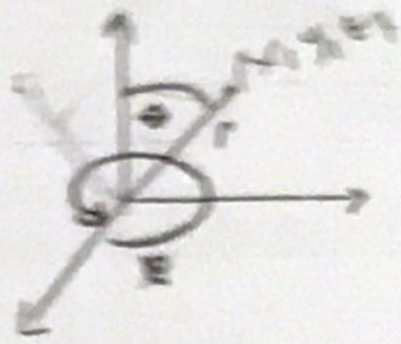
$y = r\sin t \sin\theta$

$z = r\cos\theta$

$0 \le t \le 2\pi$, $r \ge 0$

$0 \le \theta \le \pi$

$|J| = r^2 \sin\theta$



Tod kopam kopam kopam

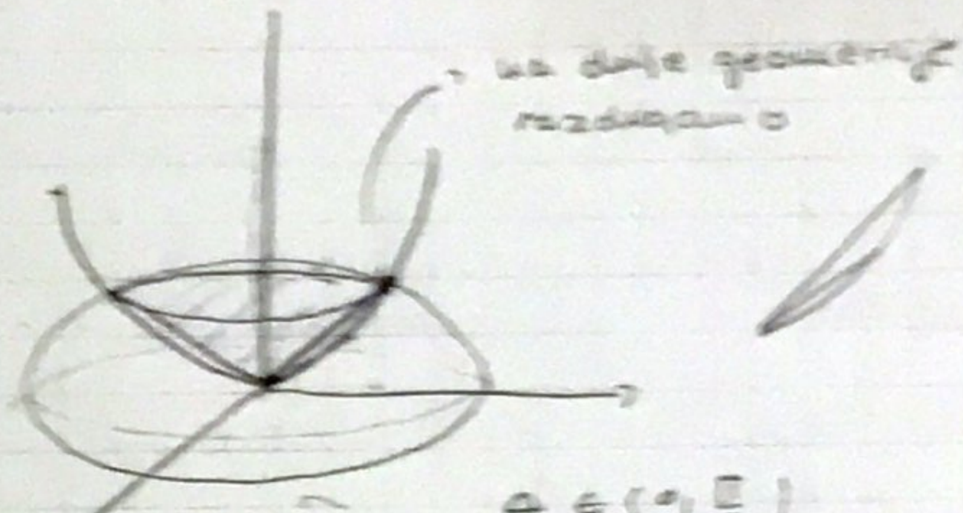
$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2\sin\theta\, dr\, d\theta\, dt = \frac{4\pi}{3}$

II način za 1. del:

$x = a r\cos t \sin\theta$

$y = b r \sin t \sin\theta$

$z = c r\cos\theta$



$\theta \in (0, \frac{\pi}{2})$

$t \in (0, 2\pi)$

r iz zunanje od elipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow r^2 = 1$

$\underbrace{r^2\cos^2 t \sin^2\theta}_{\frac{x^2}{a^2}} + \underbrace{r^2\sin^2 t \sin^2\theta}_{\frac{y^2}{b^2}} + \underbrace{r^2\cos^2\theta}_{\frac{z^2}{c^2}} = 1$

$r^2\sin^2\theta = r\cos\theta$

$\sin^2\theta = \cos\theta$

$1 - \cos^2\theta = \cos\theta$

$z = \cos\theta \Rightarrow z^2 + z - 1 = 0$

$z_{1,2} = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2} \overset{(*)}{=} \frac{\sqrt{5}-1}{2}$

$z > 0$ ker $\cos\theta > 0$ $(\theta \in (0, \frac{\pi}{2}))$

$\cos\theta^* = \frac{\sqrt{5}-1}{2}$ $\Rightarrow$ $\theta^* = \arccos\frac{\sqrt{5}-1}{2}$

$V = \int_0^{2\pi} dt \int_0^{\theta^*} d\theta \int_{\text{do elipse}}^{1} abc\, r^2\sin\theta\, dr + \int_0^{2\pi} dt \int_{\theta^*}^{\frac{\pi}{2}} d\theta \int_0^{\frac{\cos\theta}{\sin^2\theta}} abc\, r^2\sin\theta\, dr$

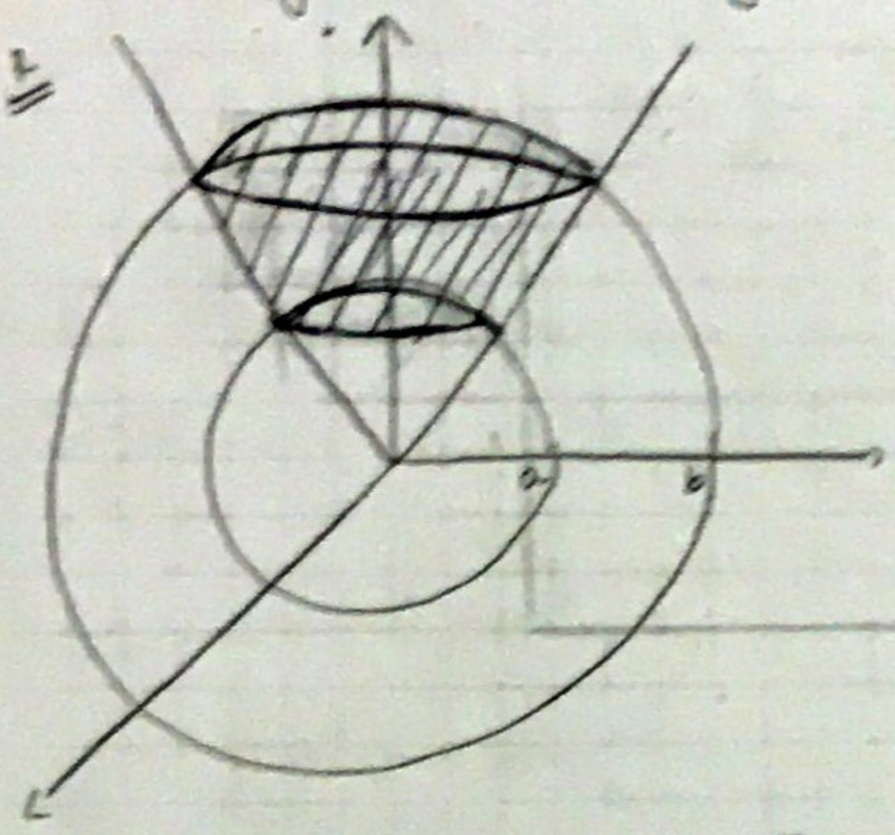
$r^2\sin^2\theta = r\cos\theta$

$r = \frac{\cos\theta}{\sin^2\theta}$

Scanned by TapScanner

② Izračunati zapreminu tijela ograničenog sa $z = \sqrt{x^2+y^2}$, $a^2 = x^2+y^2+z^2$, $b^2 = x^2+y^2+z^2$, $0 < a < b$.

I



$x = r\cos t \sin\theta$

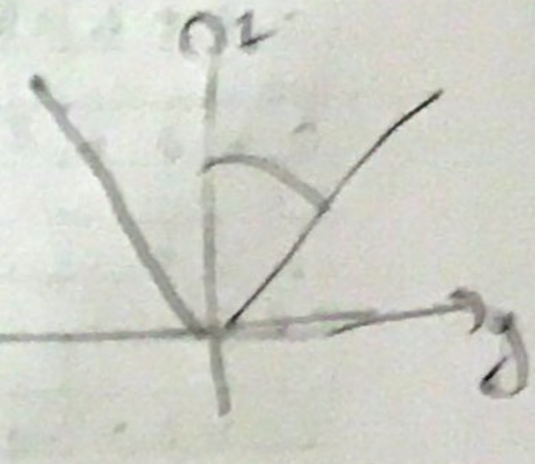
$y = r \sin t \sin\theta$

$z = r\cos\theta$

$t \in (0, 2\pi)$

$\theta \in (0, \frac{\pi}{4})$

$r \in (a, b)$



$$V = \iiint dx\,dy\,dz = \int_0^{2\pi} dt \int_0^{\frac{\pi}{4}} d\theta \int_a^b r^2 \sin\theta\, dr =$$

$$= 2\pi \cdot \frac{b^3-a^3}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$

II način (preko cil. koo.)

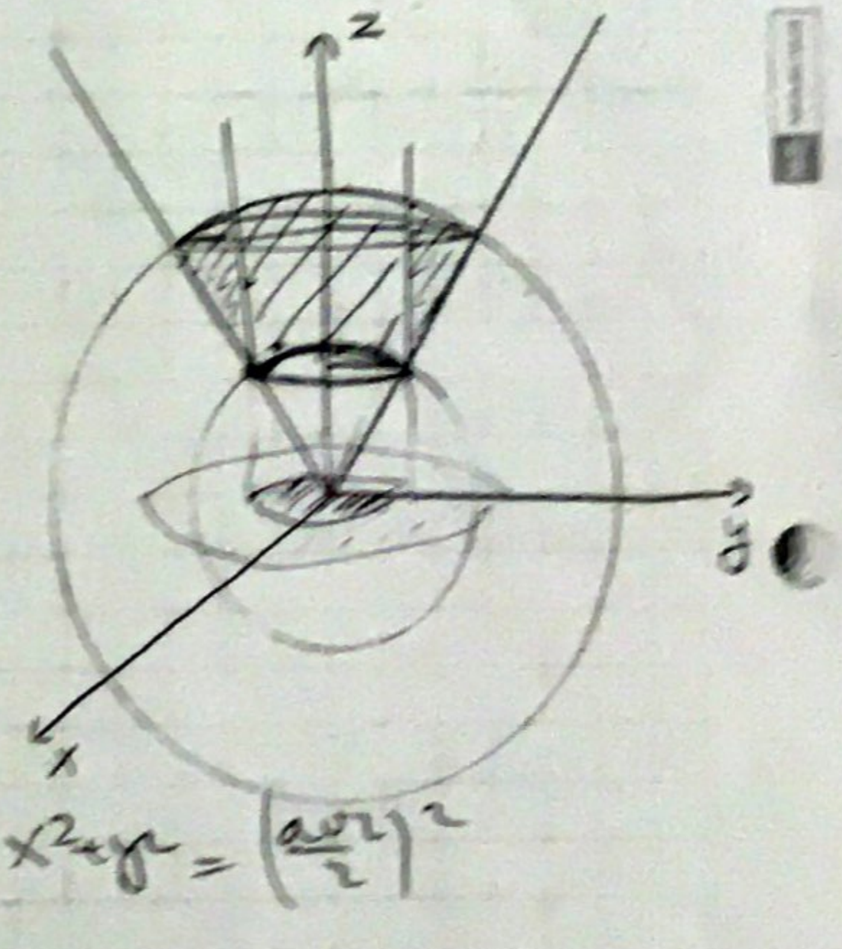
$x = r\cos t$

$y = r \sin t$

$z = z$

$$\begin{cases} z^2 = x^2+y^2 \\ a^2 = x^2+y^2+z^2 \end{cases} \Rightarrow 2z^2 = a^2$$

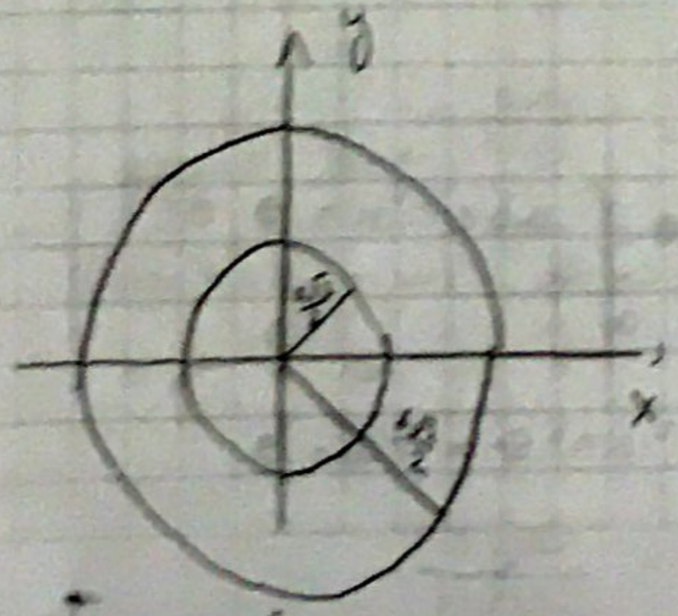
$$z = \frac{a\sqrt{2}}{2}$$



$x^2+y^2 = \left(\frac{a\sqrt{2}}{2}\right)^2$

$$\begin{cases} z^2 = x^2+y^2 \\ b^2 = x^2+y^2+z^2 \end{cases} \Rightarrow z = \frac{b\sqrt{2}}{2}$$

$x^2+y^2 = \left(\frac{b\sqrt{2}}{2}\right)^2$



$|J| = r$

Scanned by TapScanner

$$\int_0^{2\pi} d\varphi \int_0^{\frac{a\sqrt{2}}{2}} dr \int_{\sqrt{a^2-r^2}}^{\sqrt{b^2-r^2}} r\,dz + \int_0^{2\pi} d\varphi \int_{\frac{a\sqrt{2}}{2}}^{\frac{b\sqrt{2}}{2}} dr \int_r^{\sqrt{b^2-r^2}} r\,dz = \cdots$$

primjer Zapremina elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$x = a r \cos\varphi \sin\theta$

$y = b r \sin\varphi \sin\theta$

$z = c \cdot r \cdot \cos\theta$

$r \in (0,1)$

$\varphi \in (0, 2\pi)$

$\theta \in (0, \pi)$

$$\mu(E) = \int_0^{2\pi} \int_0^{\pi} \int_0^1 abc\, r^2 \sin\theta \, dr\, d\theta\, d\varphi = \cdots = \frac{4abc\pi}{3}$$

Scanned by TapScanner

V (IX nedelja)

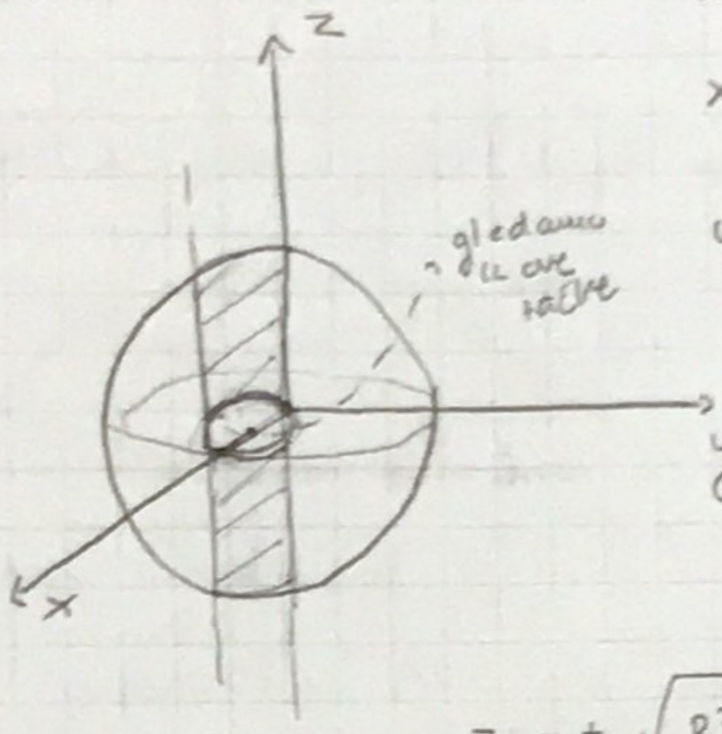
① $x^2+y^2 \le Rx \quad \Rightarrow \quad (x-\frac{R}{2})^2+y^2 \le (\frac{R}{2})^2$

$x^2+y^2+z^2 \le R^2$

$V=?$

→ uvodimo cil. koo.

$x-\frac{R}{2} = r\cos\varphi \qquad x = \frac{R}{2}+r\cos\varphi$



$y = r\sin\varphi$

$z = z$

$r \in (0, \frac{R}{2})$

$\varphi \in (0, 2\pi)$

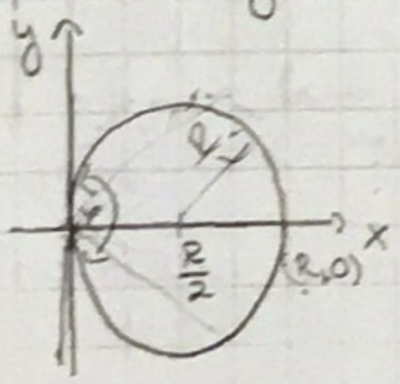
$z = \pm\sqrt{R^2-x^2-y^2}$

$-\sqrt{-11-} < z < \sqrt{-11-}$

$-\sqrt{-11-} \le z \le \sqrt{R^2-(\frac{R}{2}+\cos\varphi)^2+r^2\sin^2\varphi}$

$\Rightarrow V = \int_0^{2\pi} d\varphi \int_0^{\frac{R}{2}} dr \int_{-\sqrt{\dots}}^{\sqrt{\dots}} r\,dz = \dots$

II način:



$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

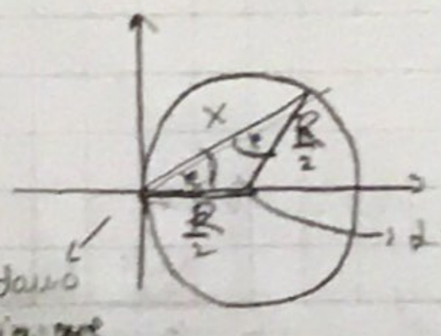
$\Rightarrow \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$x^2+y^2 = Rx$

$r^2 = R\cdot r\cos\varphi$

za $r \neq 0 \Rightarrow r = R\cos\varphi \quad \Rightarrow r \in [0, R\cos\varphi]$

③



Kosinusna teorema:

$x^2 = (\frac{R}{2})^2+(\frac{R}{2})^2-2\frac{R}{2}\cdot\frac{R}{2}\cos(180^\circ-2\varphi)$

$x^2 = \frac{R^2}{2}+\frac{R^2}{2}(\cos 2\varphi) = R^2\cos^2\varphi \Rightarrow$

$x = R\cos\varphi$

Scanned by TapScanner

$$= -\sqrt{R^2 - r^2} \quad dz = dr\sqrt{R^2 - r^2}$$

$$\Rightarrow V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{0}^{R\cos\varphi} dr \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\, dz$$

② $\int\limits_{T^n} \cdots \int \sqrt{x_1 + \ldots + x_n}\, dx_1 \ldots dx_n$, gdje je $T_n = \{(x_1, \ldots, x_n) \mid x_i \geq 0 \sum_{i=1}^{n} x_i \leq 1\}$

supstitucija: $y_1 = x_1$

$y_2 = x_1 + x_2$

$y_3 = x_1 + x_2 + x_3$

$\vdots$

$y_n = x_1 + \ldots + x_n$

$$J = \frac{D(x_1, \ldots, x_n)}{D(y_1, \ldots, y_n)} \text{ - Jakobijan} = \frac{1}{\frac{D(y_1, \ldots, y_n)}{D(x_1, \ldots, x_n)}} = \frac{1}{1} = 1$$

$$\frac{D(y_1, \ldots, y_n)}{D(x_1, \ldots, x_n)} = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 0 \\ \vdots & & & \\ 1 & \ldots & & 1 \end{vmatrix} = 1$$

$y_1 = x_1 \geq 0$

$y_2 - y_1 = x_2 \geq 0 \Rightarrow y_1 \leq y_2$

$y_3 - y_2 = x_3 \geq 0 \Rightarrow y_2 \leq y_3$

$\vdots$

$y_n - y_{n-1} = x_n \geq 0 \Rightarrow y_{n-1} \leq y_n$

$y_n \leq 1$

$$\Rightarrow \int\limits_{T^n} \cdots \int \sqrt{x_1 + \ldots + x_n}\, dx_1 \ldots dx_n = \int\limits_{T^n} \cdots \int \sqrt{y_n} \cdot 1 \cdot dy_1 \ldots dy_n =$$

$$= \int_0^1 dy_n \int_0^{y_n} dy_{n-1} \int_0^{y_{n-1}} dy_{n-2} \int_0^{y_3} \int_0^{y_2} \sqrt{y_n}\, dy_1 =$$

$$= \int_0^1 \sqrt{y_n}\, dy_n \int_0^{y_n} dy_{n-1} \cdots \int_0^{y_3} y_2\, dy_2 = \int_0^1 \sqrt{y_n}\, dy_n \int_0^{y_n} \cdots \int_0^{y_4} \frac{y_3^2}{2}\, dy_3 = \cdots \frac{2}{(n-1)!(2n+1)}$$

Scanned by TapScanner

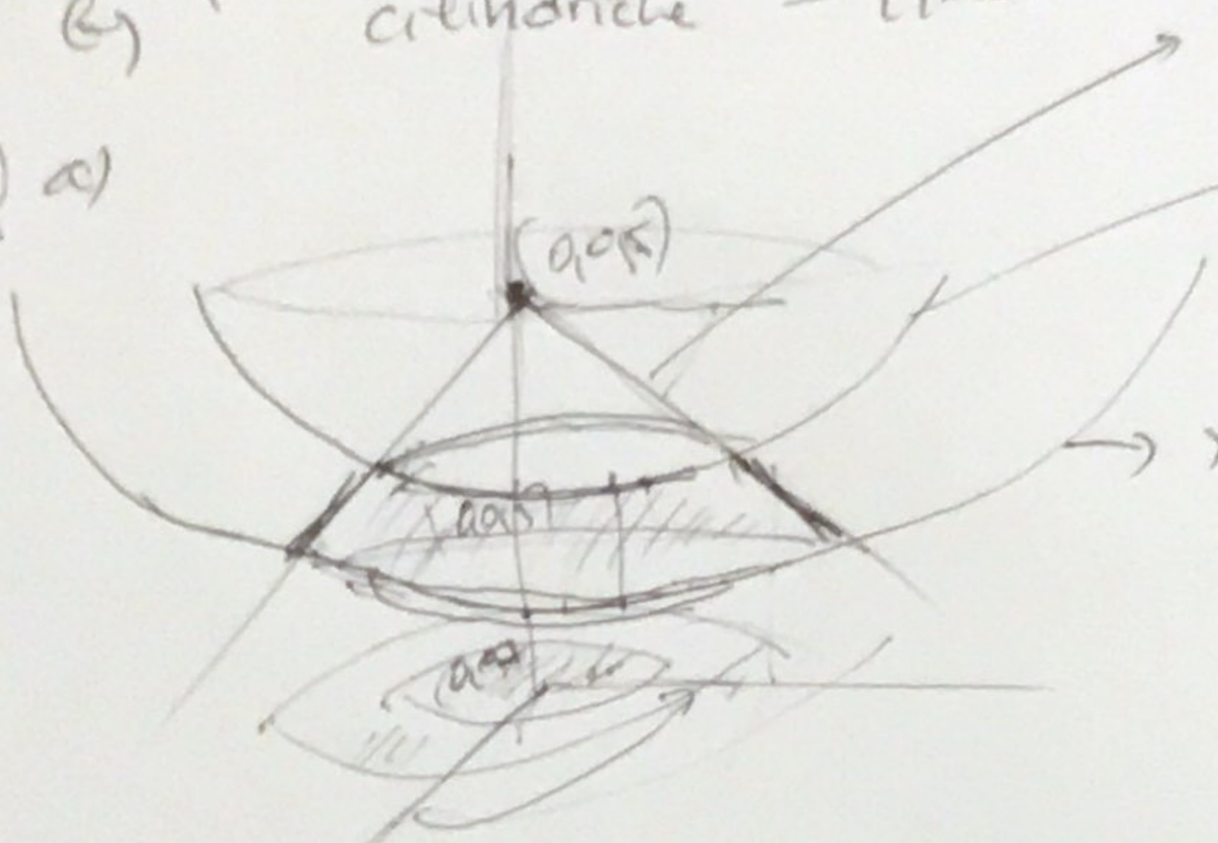
(1) u prvom oktantu $z = 5 - \sqrt{x^2+y^2}$

$x^2+y^2+(z-1)^2 = 4$ $\quad x^2+y^2+(z-5)^2 = 16$ $\quad V = ?$

a) ~~pola~~ sferne koordinate

b) cilindrične — II —

(2) a)

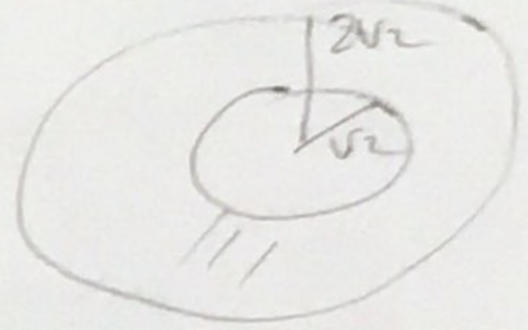


$x = r\cos\varphi\sin\theta$

$y = r\sin\varphi\sin\theta$ $\quad |J| = r^2\sin\theta$

$z - 5 = r\cos\theta$.

$$\int_0^{\frac{\pi}{2}} \int_2^4 \int_{\frac{3\pi}{4}}^{\pi} r^2\sin\theta \, d\theta dr d\varphi = \dots$$



b) $\begin{cases} z = 5 - \sqrt{x^2+y^2} \\ x^2+y^2+(z-0)^2 = 4 \end{cases}$ $2(x^2+y^2) = 4$ $\quad x^2+y^2 = 2 = (\sqrt{2})^2$

$\begin{cases} z = 5 - \sqrt{x^2+y^2} \\ x^2+y^2+(z-5)^2 = 16 \end{cases}$ $2(x^2+y^2) = 16$ $\quad x^2+y^2 = 8 = (2\sqrt{2})^2$.

$$\int_0^{\sqrt{2}} \int_0^{\frac{\pi}{2}} \int_{5-\sqrt{16-r^2}}^{5-\sqrt{4-r^2}} r \, dz d\varphi dr + \int_{\sqrt{2}}^{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \int_{5-\sqrt{16-r^2}}^{5-r} r \, dz d\varphi dr.$$

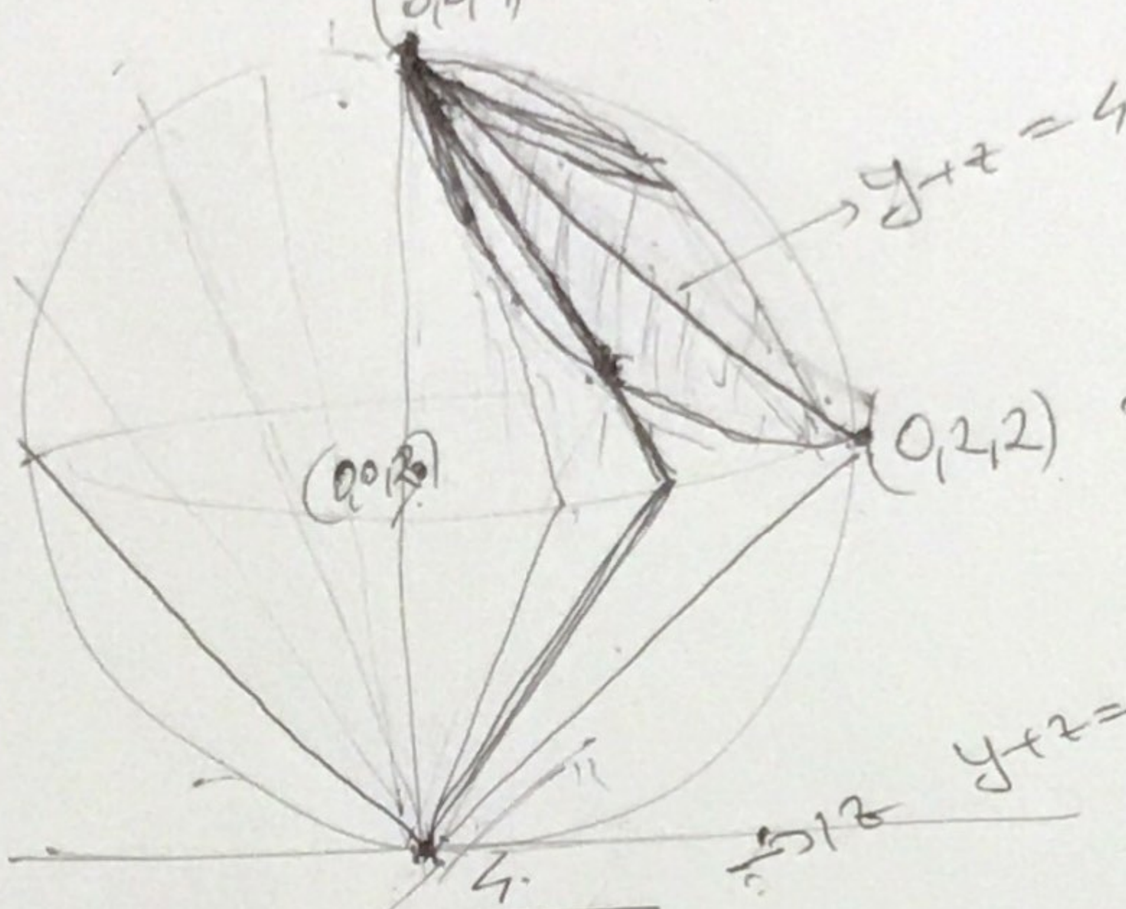
$= \dots$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$

Scanned by TapScanner

Zad Izračunati sljemim koordinatama zapremnu V

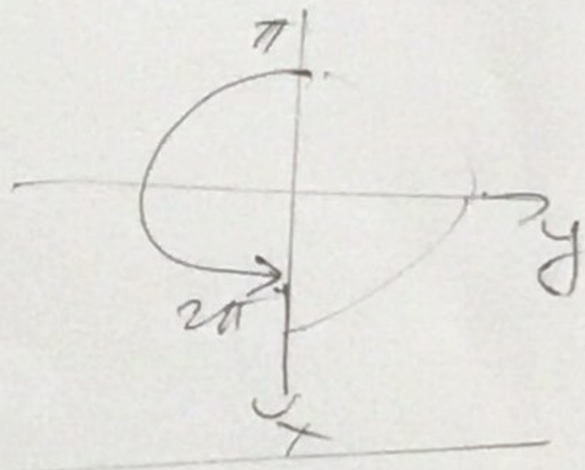
$x^2+y^2+z^2 \le 4z$, $z \ge \sqrt{x^2+y^2}$, $y+z \le 4$.



$z = \sqrt{x^2+y^2}$ i $x^2+y^2+z^2=4z$ $\Rightarrow$ $2z^2 = 4z$ $\Rightarrow$ $z=0$ ili $z=2$

$$\int_0^{\pi} \int_0^{\arctg(\sin\varphi)} \int_0^{\frac{4}{\sin\varphi\sin\theta+\cos\theta}} r^2\sin\theta \, dr\,d\theta\,d\varphi + \int_{\pi}^{2\pi} \int_{\frac{\pi}{4}}^{\arctg(\sin\varphi)} \int_0^{4\cos\theta} r^2\sin\theta \, dr\,d\theta\,d\varphi +$$

$$+ \int_{\pi}^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{4\cos\theta} r^2\sin\theta \, dr\,d\theta\,d\varphi = \dots$$



$x = r\cos\varphi\sin\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\theta$

$|J| = r^2\sin\theta$

$x^2+y^2+z^2 = 4z$

$r^2 = 4r\cos\theta$

$r = 4\cos\theta$

$y+z=4$

$x^2+y^2+z^2 = 4z$

$r = 4\cos\theta$

$r\sin\varphi\sin\theta + r\cos\theta = 4$

$4\cos\theta\sin\varphi\sin\theta + 4\cos^2\theta = 4$

$\cos\theta\sin\varphi\sin\theta = \sin^2\theta$

$\cos\theta\sin\varphi = \sin\theta$

$\sin\varphi = \operatorname{tg}\theta$

$\theta = \operatorname{arctg}(\sin\varphi)$

Scanned by TapScanner