$$\sqrt{g} = \begin{vmatrix} \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_1} \rangle & & \\ & \ddots & \\ & & \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_1} \rangle \end{vmatrix}^{\frac{1}{2}} > 0$$

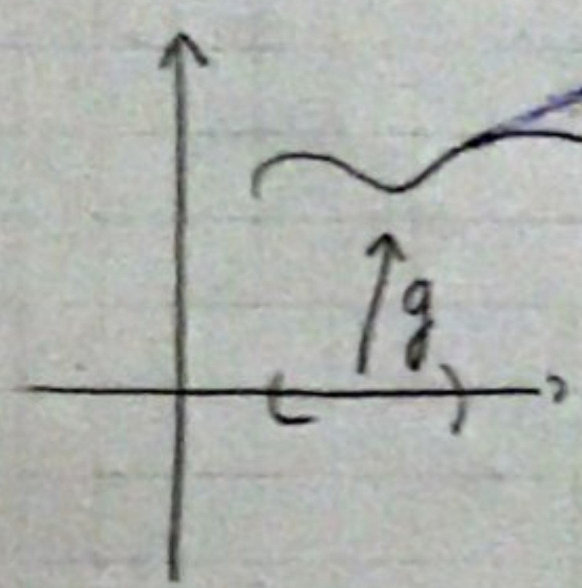
Integral po mnogostrukosti

$A \subseteq M^n$

(u, h) - karta

$$\int_A f = \int_{h(A)} f \circ h \sqrt{h}$$

Ako je $n = 1$

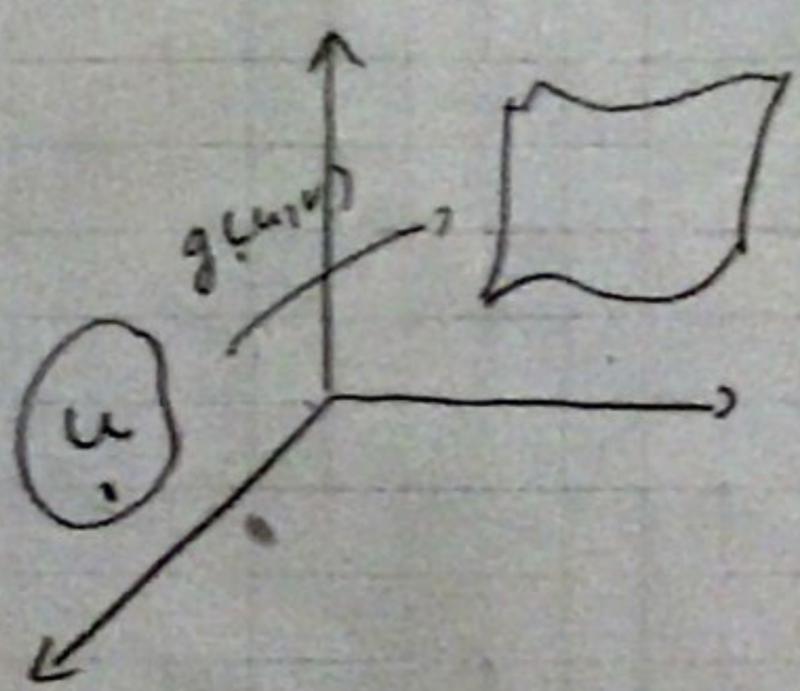


Dg:

$$\left| \langle \frac{\partial g}{\partial t}, \frac{\partial g}{\partial t} \rangle \right|^{\frac{1}{2}} = \left\| \frac{\partial g}{\partial t} \right\|$$

→ ako pravimo param, parametar će biti 1 (mnog. 1)

Ako je $n = 2$



Dg:

$$\begin{vmatrix} \langle g_u, g_u \rangle & \langle g_u, g_v \rangle \\ \langle g_v, g_u \rangle & \langle g_v, g_v \rangle \end{vmatrix}^{\frac{1}{2}} = \sqrt{EG - F^2}$$

$E = \langle g_u, g_u \rangle$

$F = \langle g_u, g_v \rangle$

$G = \langle g_v, g_v \rangle$

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V 27.04.2018.

(1.) Integral $\int_C (x^2+y^2+z^2)\,dl$ gdje je c: $x = R\cos t$

$y = R\sin t$

$z = bt$, $t \in [0, 2\pi)$

R//

$g(t) = (R\cos t, R\sin t, bt)$ $t \in [0, 2\pi)$

Г

$$\int_A f = \int_{g^{-1}(A)} f\circ g \sqrt{g}$$ u opštem sl

$c^* \subseteq \mathbb{R}^n$ $\quad g(t) = (g_1(t), \dots, g_n(t))$

$$\|Dg(t)\| = \sqrt{(g_1'(t))^2 + \dots + (g_n'(t))^2}$$ ┘

$$\int_C (x^2+y^2+z^2)\,dl = \int_0^{2\pi} (R^2\cos^2 t + R^2\sin^2 t + b^2t^2)\sqrt{(-R\sin t)^2 + (R\cos t)^2 + b^2}\,dt$$

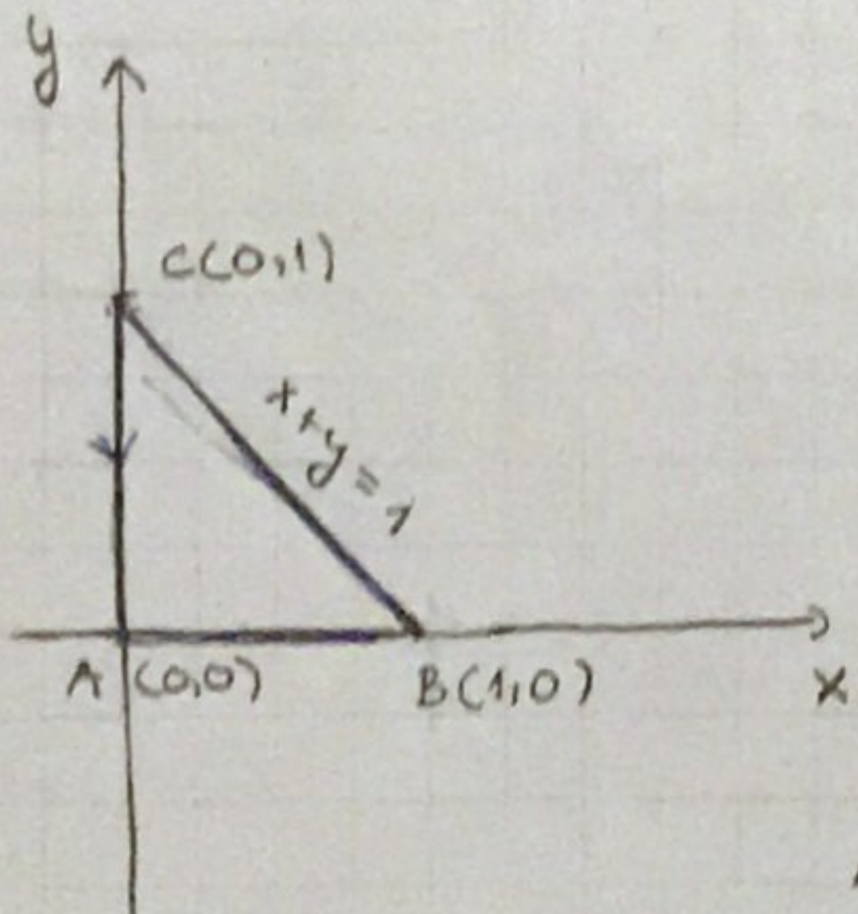
$$= \sqrt{R^2+b^2}\int_0^{2\pi}(R^2+b^2t^2)\,dt = \sqrt{R^2+b^2}\left(2\pi R^2 + b^2\frac{(2\pi)^3}{3}\right) ✓$$

(2.) $\int_C (x+y)\,dl$ gdje je $c = \partial(\triangle ABC)$, $A \to (0,0)$,

$B \to (1,0)$

$C \to (0,1)$

R//



$C = \overset{\frown}{AB} \cup \overset{\frown}{BC} \cup \overset{\frown}{CA}$

$\overset{\frown}{AB}$: $g_1(t) = (t, 0)$, $t \in (0,1)$

$g_1'(t) = (1, 0)$

$$\int_{\overset{\frown}{AB}} (x+y)\,dl = \int_0^1 t\sqrt{1^2+0^2}\,dt = \frac{1}{2}$$

$\overset{\frown}{BC}$: $g_2(t) = (1-t, t)$, $t \in (0,1)$ $\qquad \tilde{g}_2'(t) = (t, 1-t)$, $t \in (0,1)$

$g_2'(t) = (-1, 1)$

$$\int_{BC} (x+y)\,dl = \int_0^1 (1-\cancel{t}+\cancel{t})\sqrt{2}\,dt = \sqrt{2}$$

Г

nije nam bitna orijentacija ┘

$\widehat{CA}: g_3(t) = (0, 1-t)$, $t \in (0,1)$ $\quad \tilde{g}_3(t) = (0,t)$ $t \in (0,1)$

$g_3'(t) = (0,-1)$ $\quad \vdots$

$$\int_{\widehat{CA}} (x+y)\,dl = \int_0^1 (1-t)\sqrt{1}\,dt = \frac{1}{2}$$

$$\Rightarrow \int_C (x+y)\,dl = \int_{\widehat{AB}} (x+y)\,dl + \int_{\widehat{BC}} (x+y)\,dl + \int_{\widehat{CA}} (x+y)\,dl = 1+\sqrt{2}$$

③ $\int_{\mathcal{R}} |y|\,dl$, gdje je $\mathcal{R}: (x^2+y^2)^2 = a^2(x^2-y^2)$

$x = r\cos\varphi$

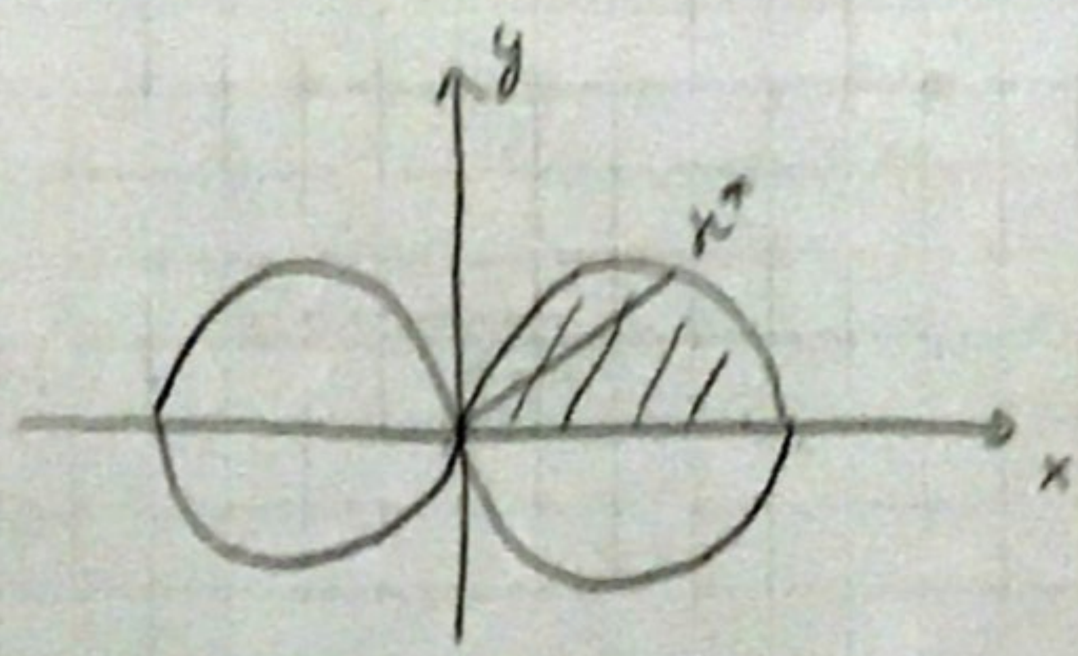
$y = r\sin\varphi$

$x^2+y^2 = r^2$

$r^4 = a^2 r^2 \cos 2\varphi \quad / : r^2$

$r^2 = a^2 \cos 2\varphi$

$r = a\sqrt{\cos 2\varphi}$



$$\int_{\mathcal{R}} |y|\,dl = 4 \int_{\mathcal{R}^*} |y|\,dl \quad , \varphi \in (0, \tfrac{\pi}{4})$$

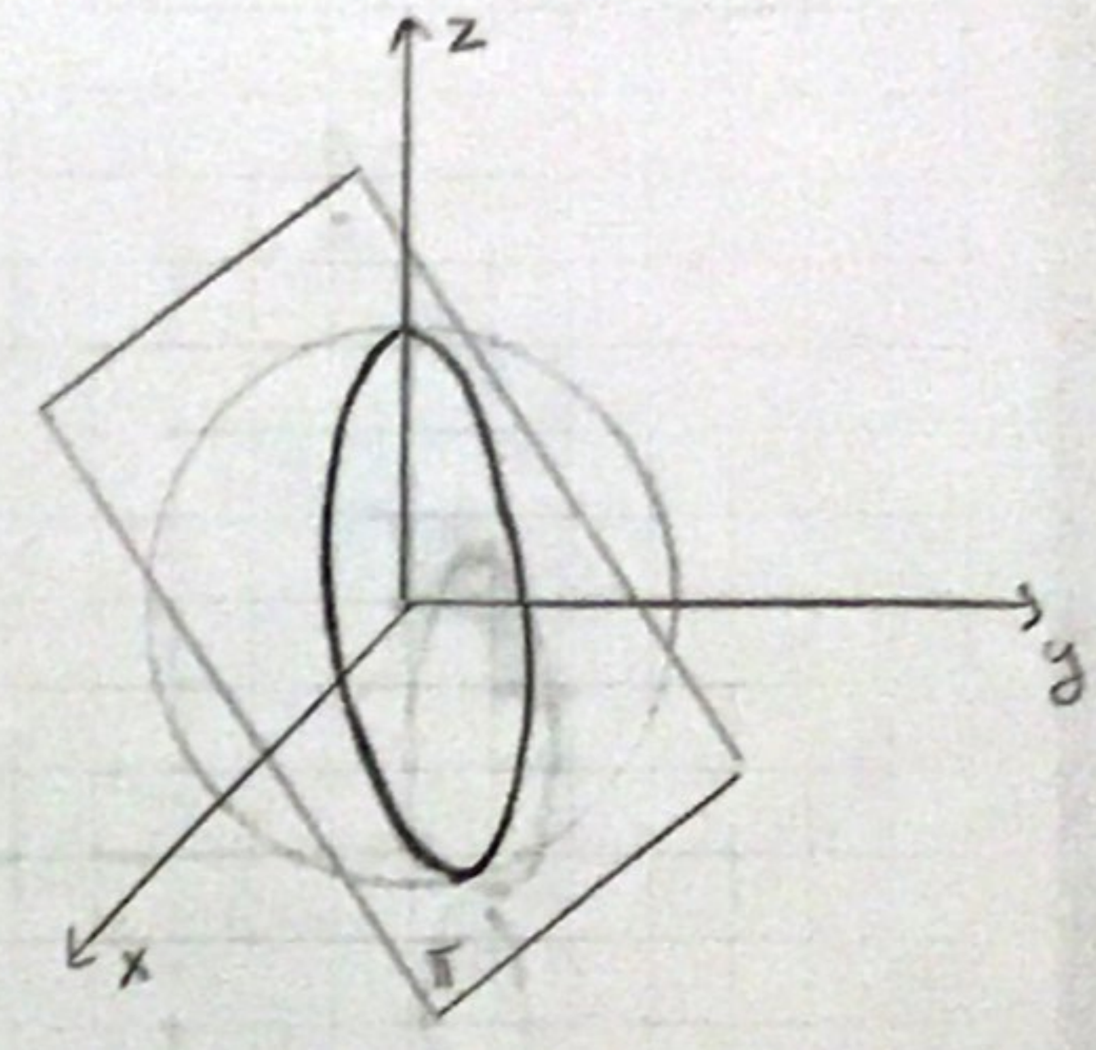
zbog simetričnosti

$\varphi \to (\underbrace{a\sqrt{\cos 2\varphi}\cdot\cos\varphi}_{g_1}, \underbrace{a\sqrt{\cos 2\varphi}\,\sin\varphi}_{g_2})$

$\vdots$

$$= 4 \cdot \int_0^{\frac{\pi}{4}} a\sqrt{\cos 2\varphi}\,\sin\varphi \sqrt{g_1'^2 + g_2'^2}\,d\varphi$$

4) $\int_C x^2 \, dl$, c - kriva dobijena u presjeku $\begin{cases} x^2+y^2+z^2=a^2 & (1) \\ x+y+z=0 & (2) \end{cases}$



(2) $z = -x-y$

(1) $x^2+y^2+(-x-y)^2=a^2$

$x^2+y^2+x^2+2xy+y^2=a^2 \quad /:2$

$x^2+y^2+xy = \frac{a^2}{2}$ - elipsa

$(x+\frac{1}{2}y)^2 + \frac{3}{4}y^2 = \frac{a^2}{2}$

$(x+\frac{1}{2}y)^2 + (\frac{\sqrt{3}}{2}y)^2 = \frac{a^2}{2}$

$x+\frac{1}{2}y = \frac{a}{\sqrt{2}}\cos t \Rightarrow x = \frac{a}{\sqrt{2}}\cos t - \frac{a\sin t}{\sqrt{6}}$

$\frac{\sqrt{3}}{2}y = \frac{a}{\sqrt{2}}\sin t \Rightarrow y = \frac{2a\sin t}{\sqrt{6}}$

$\Rightarrow z = -x-y = -\frac{a}{\sqrt{2}}\cos t + \frac{a\sin t}{\sqrt{6}} - \frac{2a\sin t}{\sqrt{6}}$

$t \xrightarrow{g} (x(t), y(t), z(t))$, $t \in (0, 2\pi)$

Može se iz ovoga odmah postaviti integral ali će biti težak za izračunati. Primijetimo: $\int_C x^2 dl = \int_C y^2 dl = \int_C z^2 dl$

$\int_C x^2 dl = \frac{1}{3}\int_C (x^2+y^2+z^2)\,dl =$

$= \frac{1}{3}\int_C a^2 \, dl =$

$= \frac{a^2}{3}\int_C \underbrace{dl}_{\text{dužina krive}} =$

$= \frac{a^2}{3} \cdot \underbrace{2\pi a}_{\text{obim kružnice}} =$

$= \frac{a^3}{3} 2\pi$

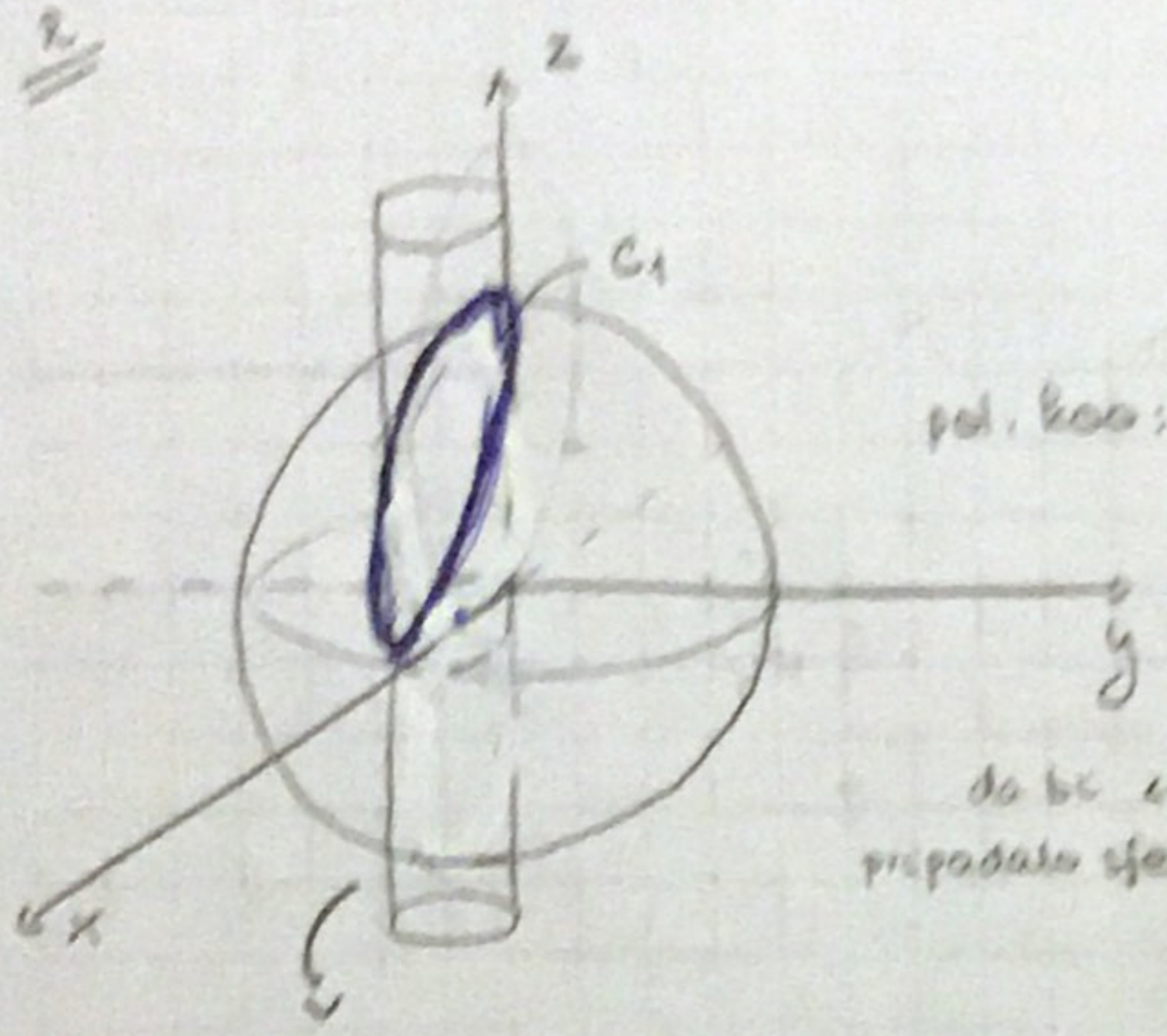
(5) Izračunati dužinu krive koja nastaje u presjeku

$$c: \begin{cases} x^2+y^2+z^2=a^2 \\ x^2+y^2=ax \ , \ a>0 \quad \text{- cilindar} \end{cases}$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2 \quad \text{- projekcija na } Oxy \text{ ravan}$$

$$\ell(c)=2\ell(c_1)$$

$$c_1, \ z>0$$



Vivijanijeva kriva

pol. koo: $x-\frac{a}{2}=\frac{a}{2}\cos t \Rightarrow x=\frac{a}{2}(1+\cos t)$

$$y=\frac{a}{2}\sin t$$

da bi pripadalo sferi $\leftarrow z^2=a^2-x^2-y^2=$

$$=a^2-\left(\frac{a}{2}(1+\cos t)\right)^2-\left(\frac{a}{2}\sin t\right)^2=$$

$$=a^2-\frac{a^2}{4}(1+2\cos t+\cos^2 t)-\frac{a^2}{4}\sin^2 t=$$

$$=a^2-\frac{a^2}{4}-\frac{a^2}{4}-\frac{a^2}{2}\cos t=$$

$$=\frac{a^2}{2}(1-\cos t)=a^2\sin^2\frac{t}{2}$$

$$\Rightarrow z=(\text{radimo sa } z>0)=$$

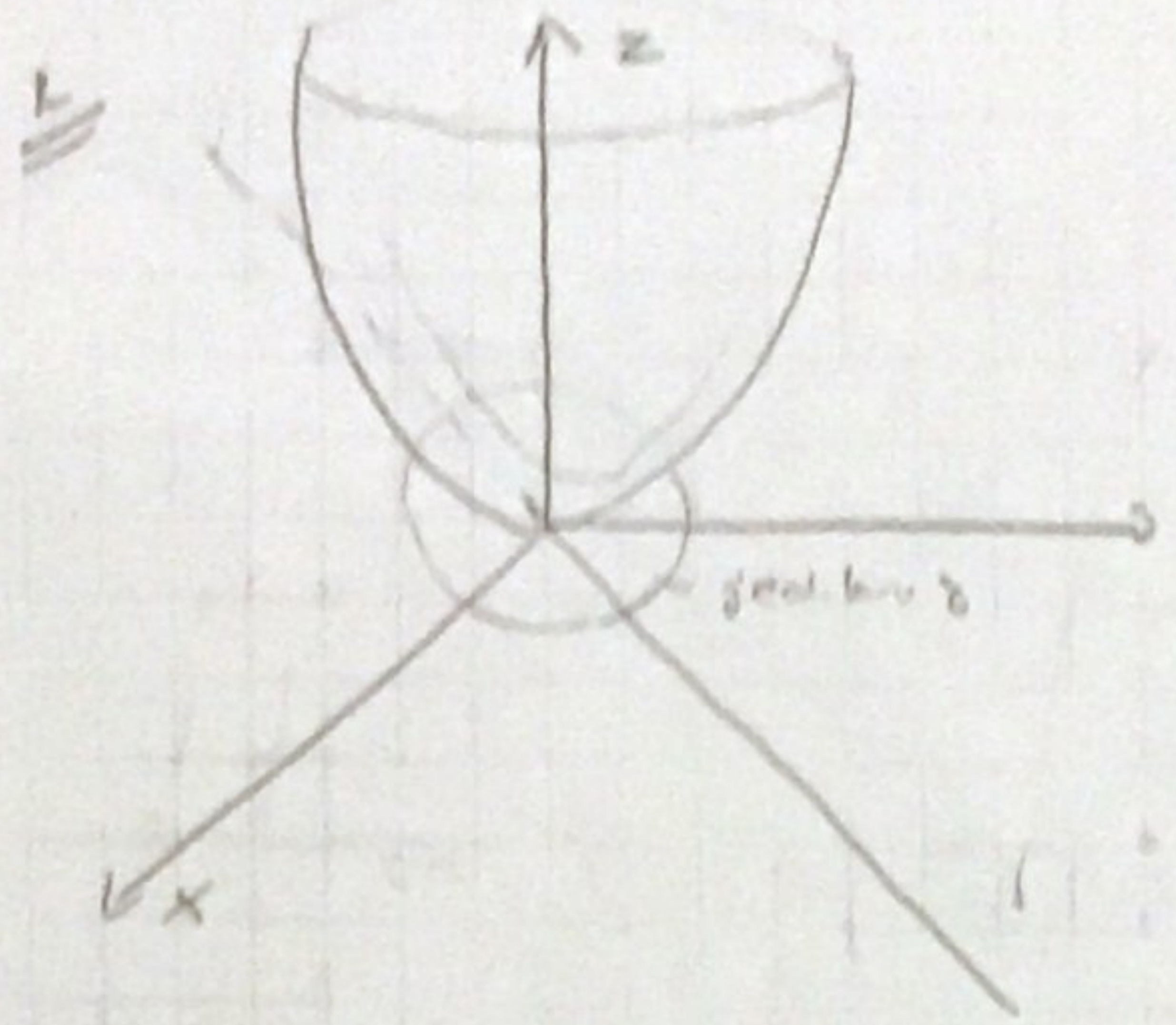
$$=a\sin\frac{t}{2}$$

$$\Rightarrow t \xrightarrow{g} \Big(\underbrace{\frac{a}{2}(1+\cos t)}_{g_1}, \underbrace{\frac{a}{2}\sin t}_{g_2}, \underbrace{a\sin\frac{t}{2}}_{g_3}\Big)$$

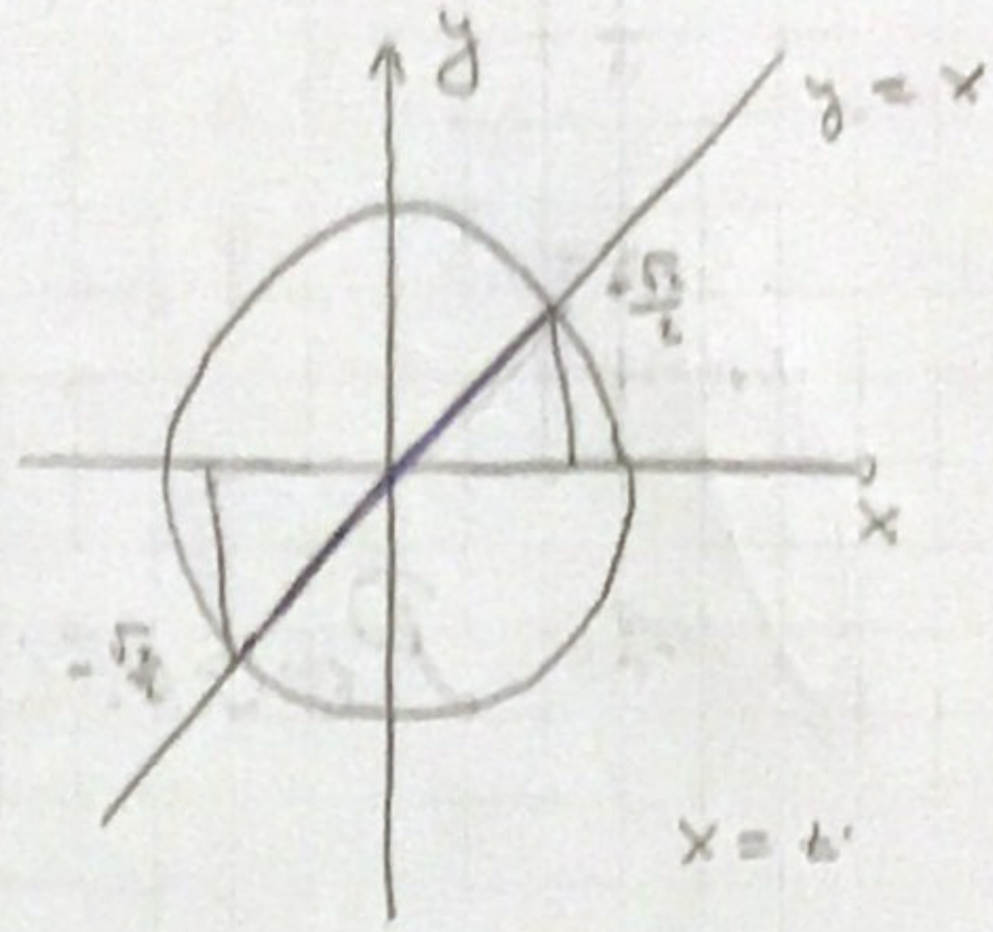
$$\int_{c_1} dl=\int_0^{2\pi}\sqrt{g_1'^2+g_2'^2+g_3'^2}\,dt=\dots$$

⑥ Izračunati dužinu dijela parabole $p: \begin{cases} x^2+y^2=z \\ y=x \end{cases}$ koji leži iznad jediničnog kruga.

R//



projekcija na Oxy:



$x = t$

$y = t$

$z = 2t^2$

treba da zad: $x^2+y^2=1$

$$2t^2 = 1 \Rightarrow t^2 = \frac{1}{2}$$

$$t = \pm \frac{\sqrt{2}}{2}$$

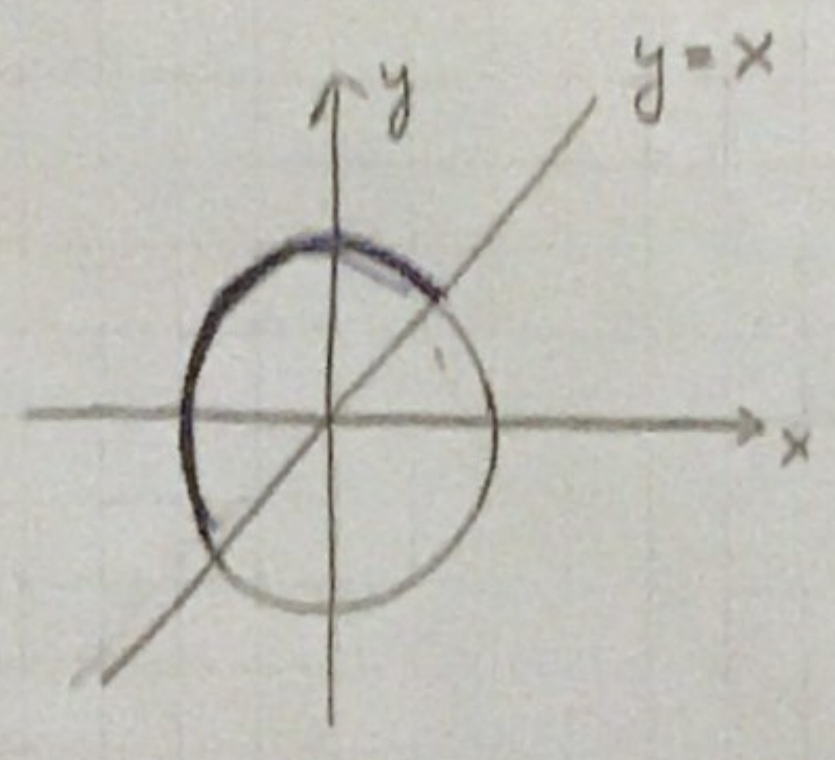
$$t \xrightarrow{\gamma} (t, t, 2t^2), \quad t \in \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

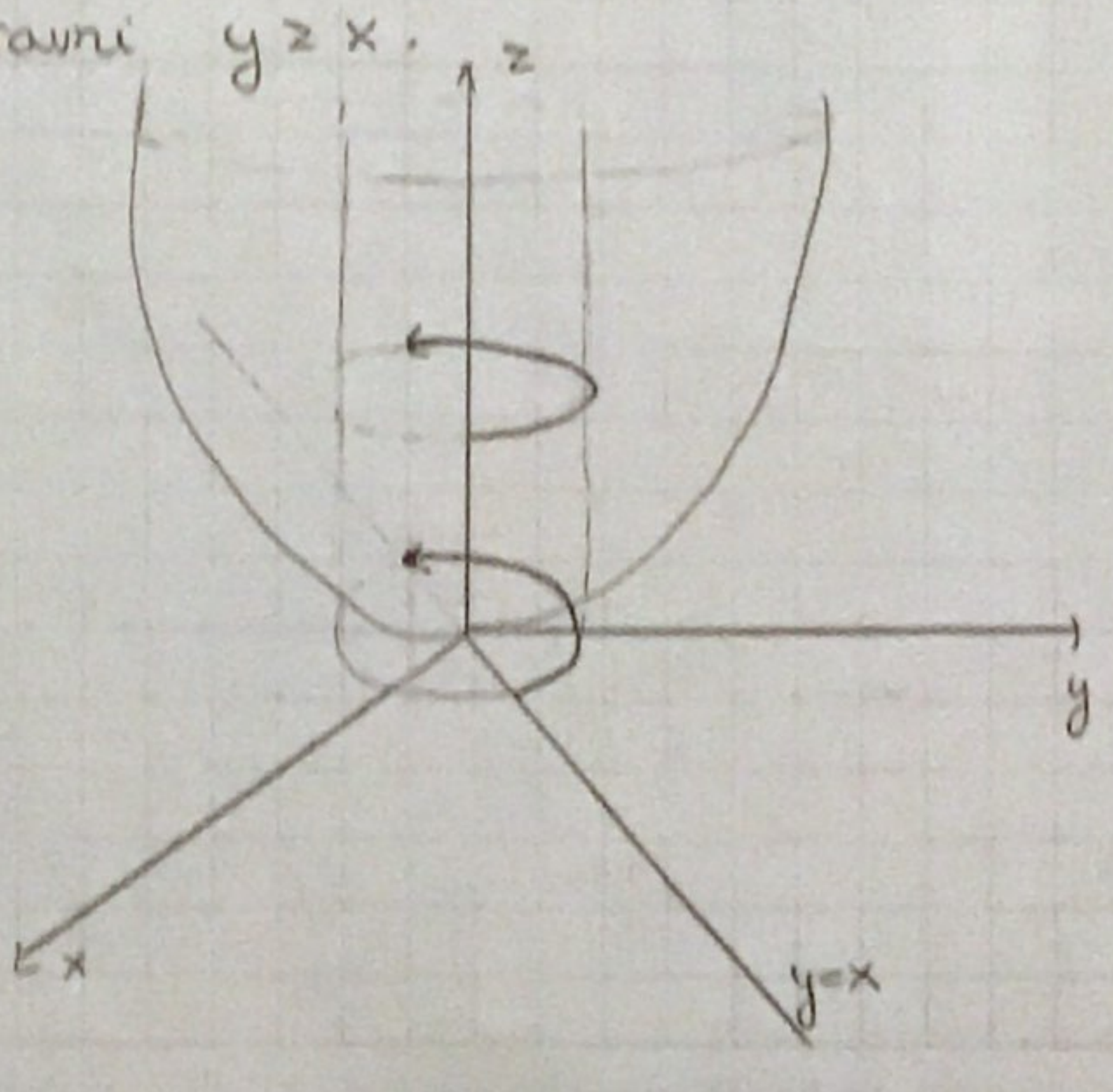
$$\int_c dl = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \sqrt{\underbrace{1+1+16t^2}_{2+16t^2}}\, dt = \dots$$

s = tgt ili parcijalno.

⑦ $\begin{cases} x^2+y^2=z \\ x^2+y^2=1 \end{cases}$ dužinu dijela kruga koji leži iznad poluravni $y \geq x$.

R//





$x = \cos t$

$y = \sin t$

$z = 1.$

$t \in \left(\frac{\pi}{4}, \frac{5\pi}{4} \right)$

$t \to (\cos t, \sin t, 1)$

$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1 \cdot \sqrt{1+0} = \frac{5\pi}{4} - \frac{\pi}{4} = \pi$

~ Površinski integrali ~

Primjer: a) Izračunati površinu sfere poluprečnika R

$\int_{S^2} 1 \, ds$

$\sqrt{EG - F^2}$

$(u,v) \xrightarrow{g} (x(u,v), y(u,v), z(u,v))$

$$D_g = \begin{vmatrix} \langle g_u, g_u \rangle & \langle g_u, g_v \rangle \\ \langle g_v, g_u \rangle & \langle g_v, g_v \rangle \end{vmatrix}^{\frac{1}{2}}$$

$u \to t, \ v \to \theta$

$x = R \cos t \sin \theta$

$y = R \sin t \sin \theta$

$z = R \cos \theta$

$(t, \theta) \xrightarrow{g} (x, y, z)$

$E = \langle g_t, g_t \rangle = \langle (-R \sin t \sin\theta, R\cos t \sin\theta, 0), (-//-) \rangle =$

$= R^2 \sin^2 t \sin^2\theta + R^2 \cos^2 t \sin^2 \theta + 0 =$

$= R^2 \sin^2 \theta$

$G = \langle g_\theta, g_\theta \rangle = R^2$

$g_\theta = (R\cos\varphi\cos\theta,\ R\sin\varphi\cos\theta,\ -R\sin\theta)$

$\sqrt{EG - F^2} = \sqrt{R^2\sin^2\theta \cdot R^2} = R^2 \sin\theta$

$F = \langle g_\varphi, g_\theta \rangle = 0$

$$\int_0^{2\pi}\int_0^{\pi} \sqrt{EG - F^2}\, d\theta\, d\varphi = \int_0^{2\pi}\int_0^{\pi} \sqrt{R^4 \sin^2\theta}\, d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta = R^2 \cdot 2\pi \cdot 2 = 4\pi R^2$$

b) Površina torusa

$g(\varphi, \theta) = ((R + r\cos\theta)\cos\varphi,\ (R + r\cos\theta)\sin\varphi,\ r\sin\theta) \quad \varphi, \theta \in [0, 2\pi]$

$$\int_{T^2} ds = \dots$$

$\sqrt{EG - F^2} = \dots$

$g_\varphi = (-(R + r\cos\theta)\sin\varphi,\ (R + r\cos\theta)\cos\varphi,\ 0)$

$g_\theta = (-r\sin\theta\cos\varphi,\ -r\sin\theta\sin\varphi,\ r\cos\theta)$

$E = (R + r\cos\theta)^2$

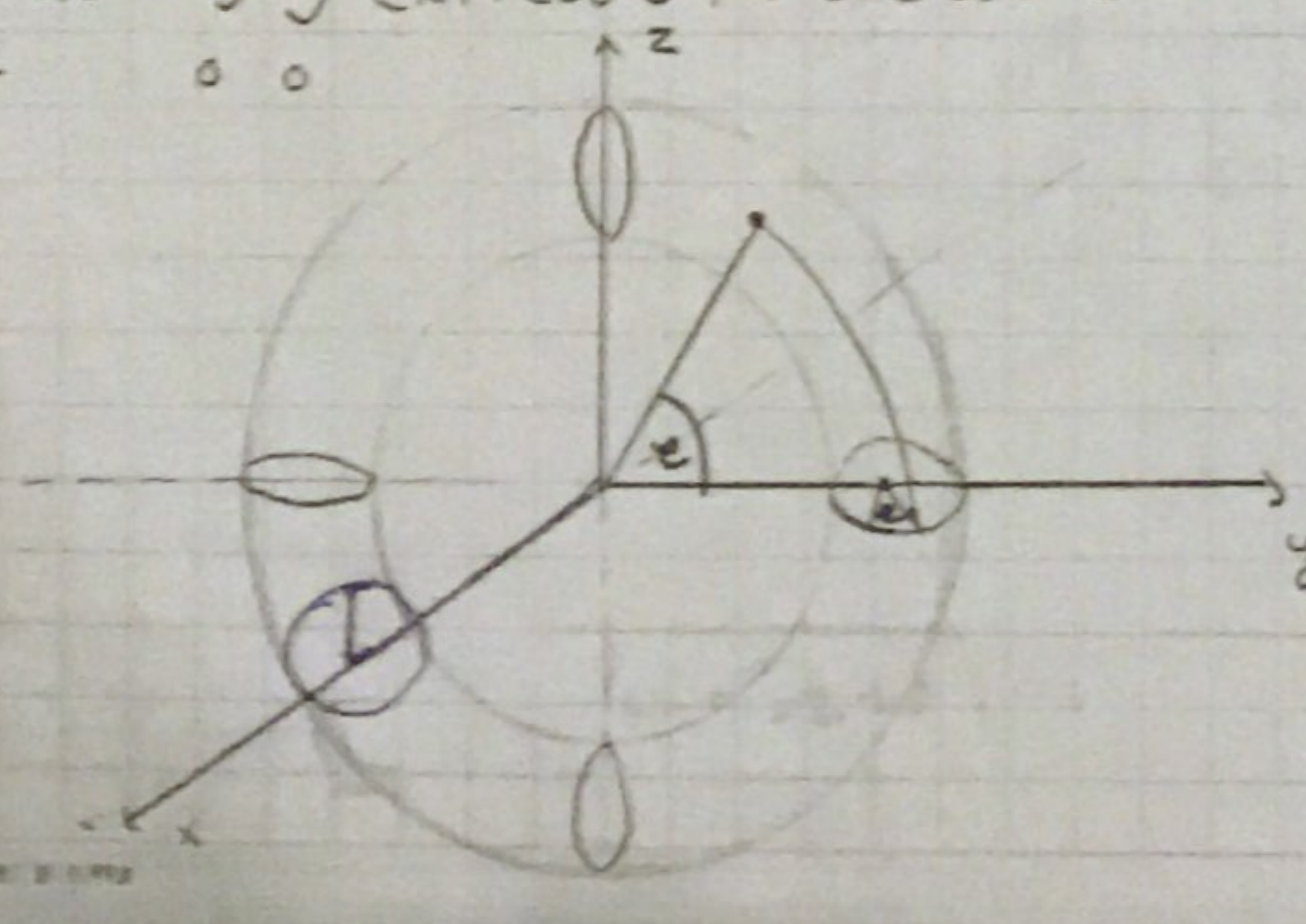
$G = r^2$

$F = 0$

$\Rightarrow \sqrt{EG - F^2} = (R + r\cos\theta) \cdot r$

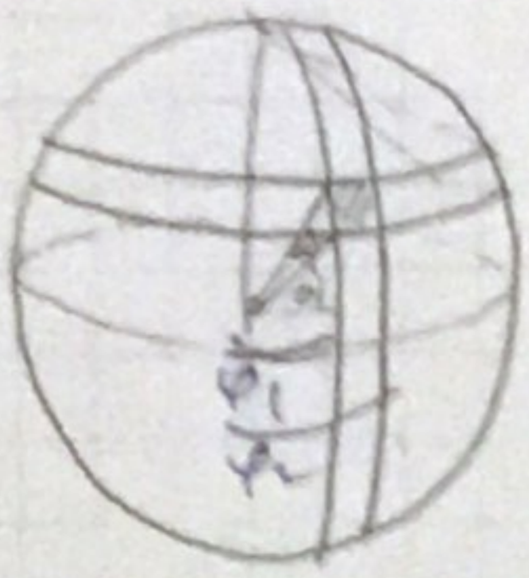
$$\int_{T^2} ds = \int_0^{2\pi}\int_0^{2\pi} (R + r\cos\theta)\, r\, d\varphi\, d\theta = R \cdot r \cdot (2\pi)^2 + 0 = 4\pi^2 R \cdot r$$

$R, r \uparrow \Rightarrow P \uparrow$



⑨ Izračunati površinu na S^2, T^2 koja leži između dvije paralele i dva meridijana:

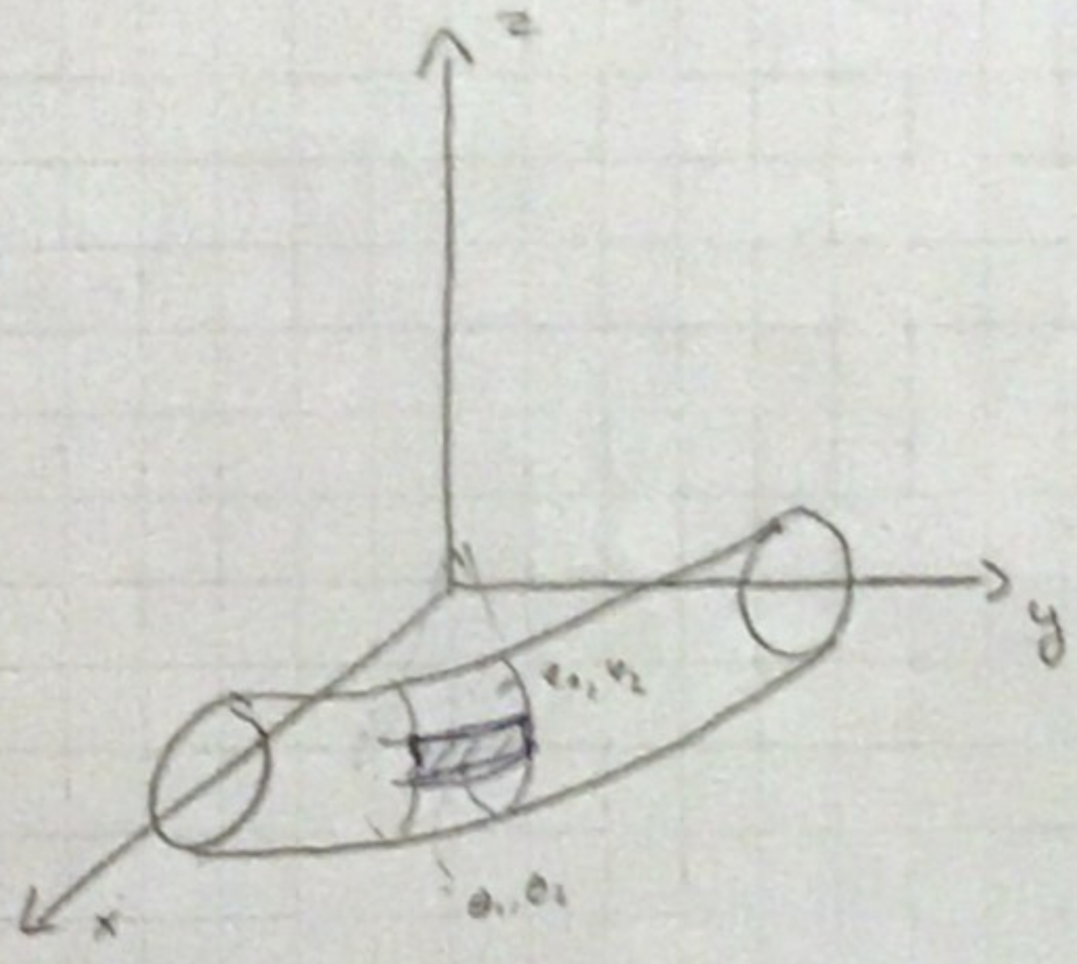
$\underline{\underline{S^2}}$



$$[\varphi_1, \varphi_2] \times [\theta_1, \theta_2]$$

$$P = \int_{\varphi_1}^{\varphi_2}\int_{\theta_1}^{\theta_2} \underbrace{\sqrt{EG - F^2}}_{R^2 \sin\theta}\, d\theta\, d\varphi = \dots$$

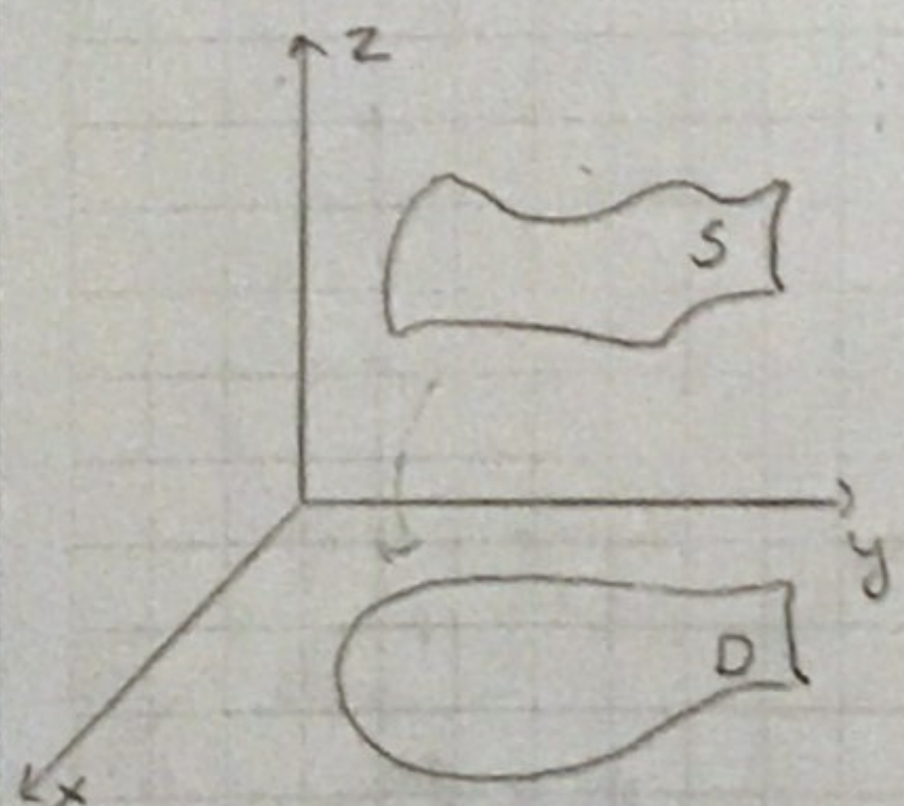
$\underline{\underline{T^2}}$



$$[\varphi_1, \varphi_2] \times [\theta_1, \theta_2]$$

⑩ Izračunati površinu tijela ograničenog sa
$$\begin{cases} x^2 + z^2 = a^2 & \text{- cilindar duž } y\text{-ose} \\ y^2 + z^2 = a^2 & \text{- cilindar duž } x\text{-ose} \end{cases}$$

Pomoćni zadatak: $(x, y) \xrightarrow{g} \underbrace{(x, y, f(x,y))}$ ~ graf neke f-je



D = projekcija površi na Oxy ravan

$$\sqrt{EG - F^2}$$

$$g_x = (1, 0, f'_x)$$

$$g_y = (0, 1, f'_y)$$

$$E = 1 + f'^2_x$$

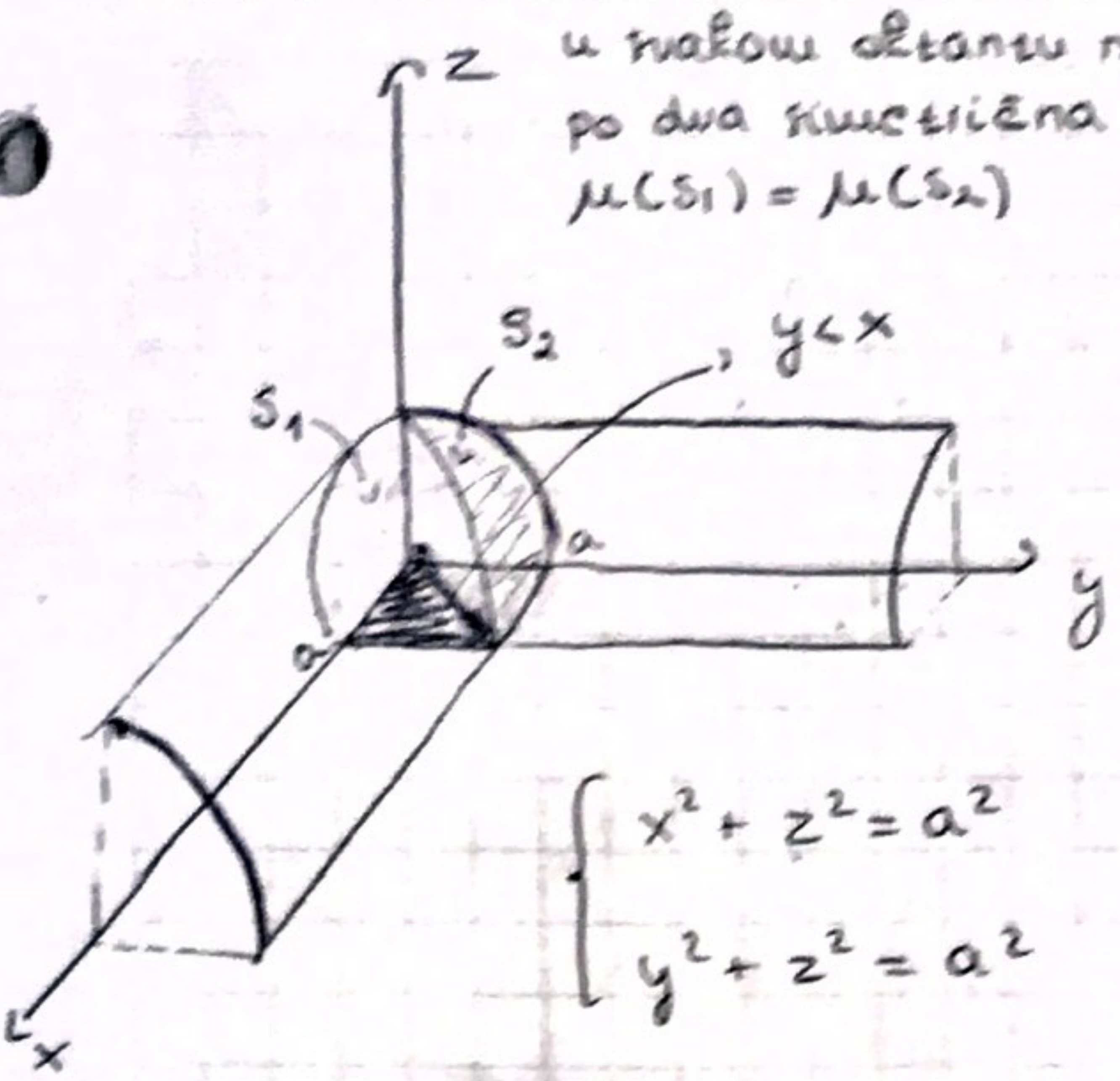
$$G = 1 + f'^2_y$$

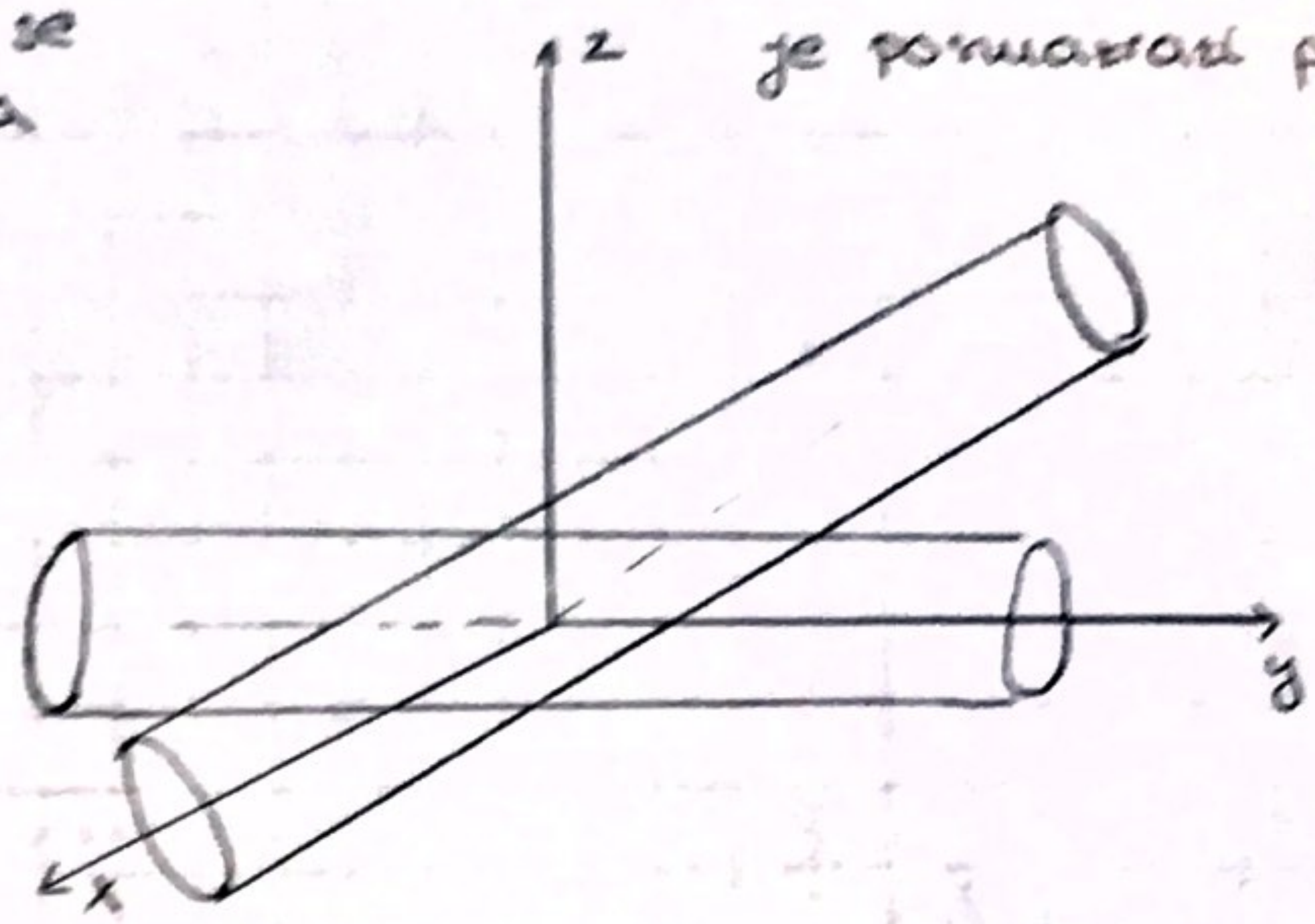
$$F = f'_x f'_y$$

$$\sqrt{EG - F^2} = \sqrt{(1 + f'^2_x)(1 + f'^2_y) - f'^2_x f'^2_y} = \sqrt{1 + f'^2_x + f'^2_y}$$

dijelo je simetrično u svih 8 oktanata, dovoljno je promatrati prvi oktant

u svakom oktantu nalaze se po dva simetrična dijela
$\mu(S_1) = \mu(S_2)$





$$\begin{cases} x^2 + z^2 = a^2 \\ y^2 + z^2 = a^2 \end{cases} \Rightarrow y = x \quad (\text{ako gledamo } z \geq 0)$$

$$\begin{cases} z^2 = a^2 - x^2 \\ z^2 = a^2 - y^2 \end{cases}$$

$$a^2 - x^2 > a^2 - y^2 \Leftrightarrow y^2 > x^2$$
$$\Leftrightarrow |y| > |x|$$
$$\overset{\text{I kv.}}{\Leftrightarrow} y > x$$

$$S = 8 \cdot 2 S_1 = 16 S_1$$

$$\underset{z^2}{\underset{\parallel}{a^2 - x^2}} < \underset{z^2}{\underset{\parallel}{a^2 - y^2}} \Leftrightarrow y < x$$

$$z^2 = a^2 - x^2$$

$$z = \sqrt{a^2 - x^2}$$

$$(x, y) \longrightarrow (x, y, \sqrt{a^2 - x^2})$$

$$\sqrt{EG - F^2} = \sqrt{1 + \left(\frac{-2x}{2\sqrt{a^2 - x^2}}\right)^2 + 0^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}$$

$$S_1 = \int_0^a \int_0^x \frac{a}{\sqrt{a^2 - x^2}} \, dy\, dx = \int_0^a \frac{ax}{\sqrt{a^2 - x^2}} \, dx = \begin{pmatrix} a^2 - x^2 = t \\ -2x\,dx = dt \end{pmatrix} =$$

$$= -\frac{a}{2} \int_{a^2}^{0} \frac{dt}{\sqrt{t}} = \frac{a}{2} \int_0^{a^2} t^{-\frac{1}{2}} \, dt = \frac{a}{\cancel{2}} \left. \frac{t^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} \right|_0^{a^2} = a^2$$

$$S = 16 \cdot a^2$$

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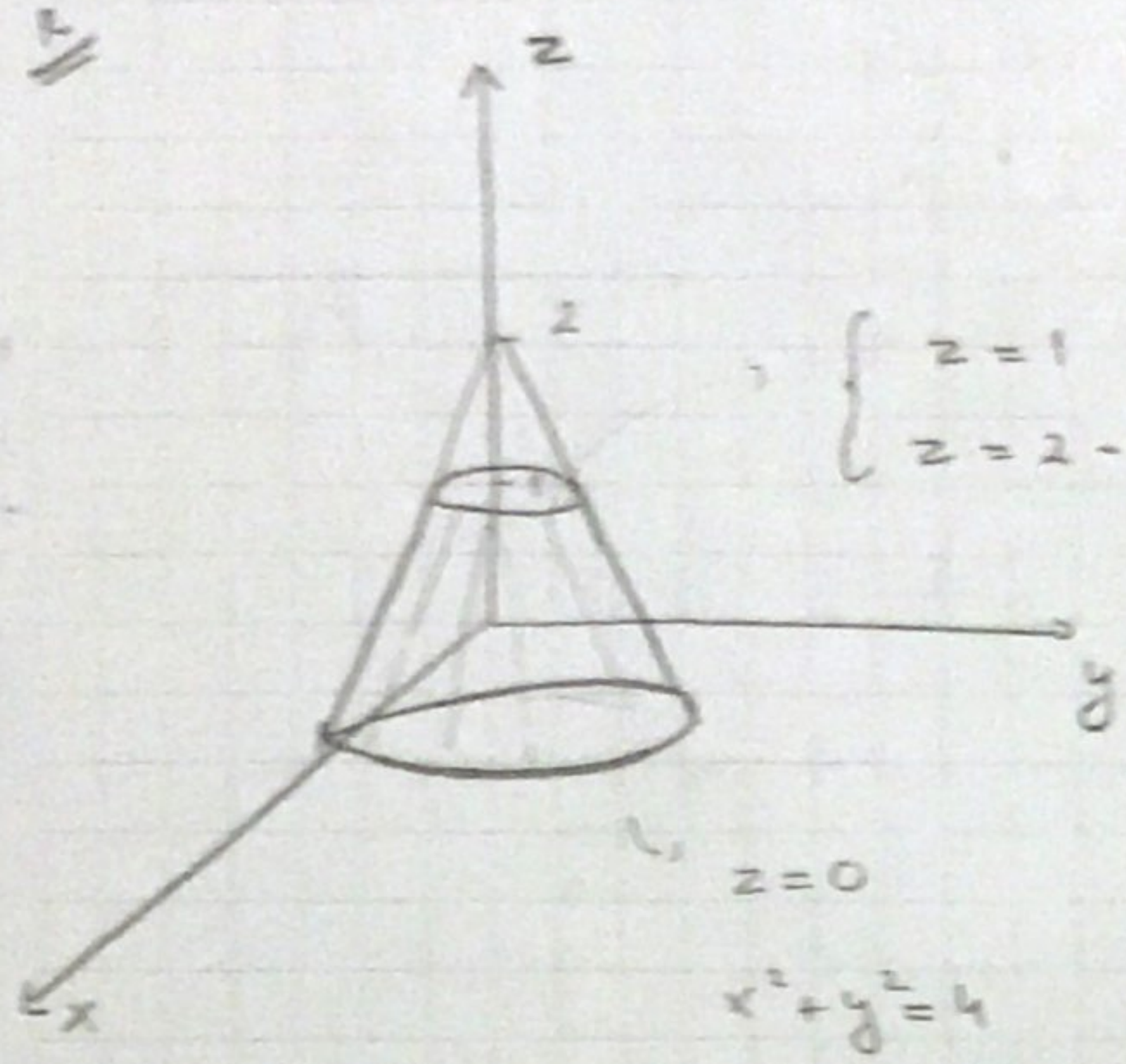
⑧ $\iint\limits_S (x+z-2)\,dS$, gdje je S dio površi $z = 2-\sqrt{x^2+y^2}$, $0 \le z \le 1$.

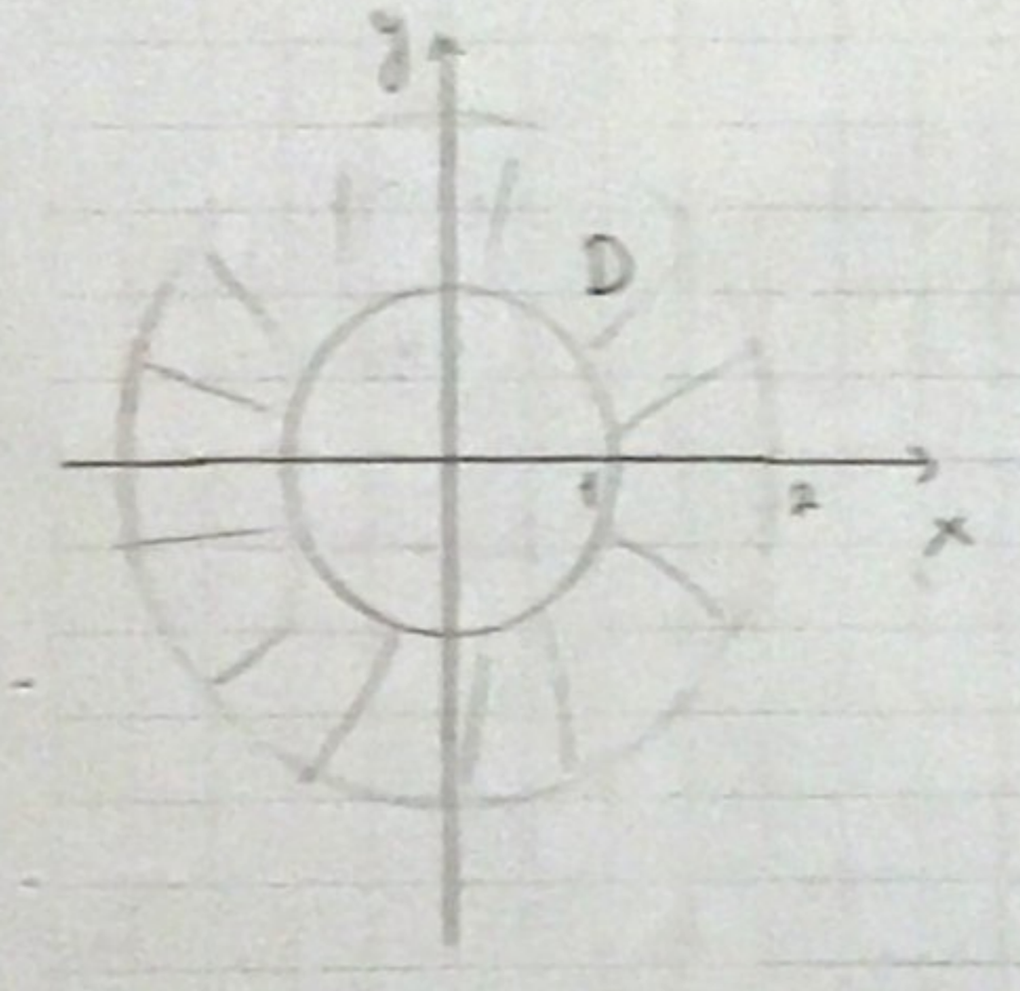
R/



$\begin{cases} z = 1 \\ z = 2-\sqrt{x^2+y} \end{cases} \Rightarrow x^2+y^2 = 1$

$z = 0$

$x^2+y^2 = 4$



$(x,y) \longrightarrow (x, y, 2-\sqrt{x^2+y^2})$

$(x,y) \in D$

$$I = \iint\limits_D (x+\cancel{2}-\sqrt{x^2+y^2}-\cancel{2})\sqrt{1+\left(\frac{-\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\right)^2+\left(\frac{-\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\right)^2}\,dx\,dy =$$

$$= \iint\limits_D (x-\sqrt{x^2+y^2})\sqrt{\frac{2\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}}\,dx\,dy =$$

$$= \sqrt{2}\iint\limits_D (x-\sqrt{x^2+y^2})\,dx\,dy = \begin{pmatrix} \text{uv. pol. koo:} & \\ x = r\cos\varphi & r \in (1,2) \\ y = r\sin\varphi & \varphi \in (0,2\pi) \end{pmatrix} =$$

$$= -\frac{14\sqrt{2}\pi}{3}$$

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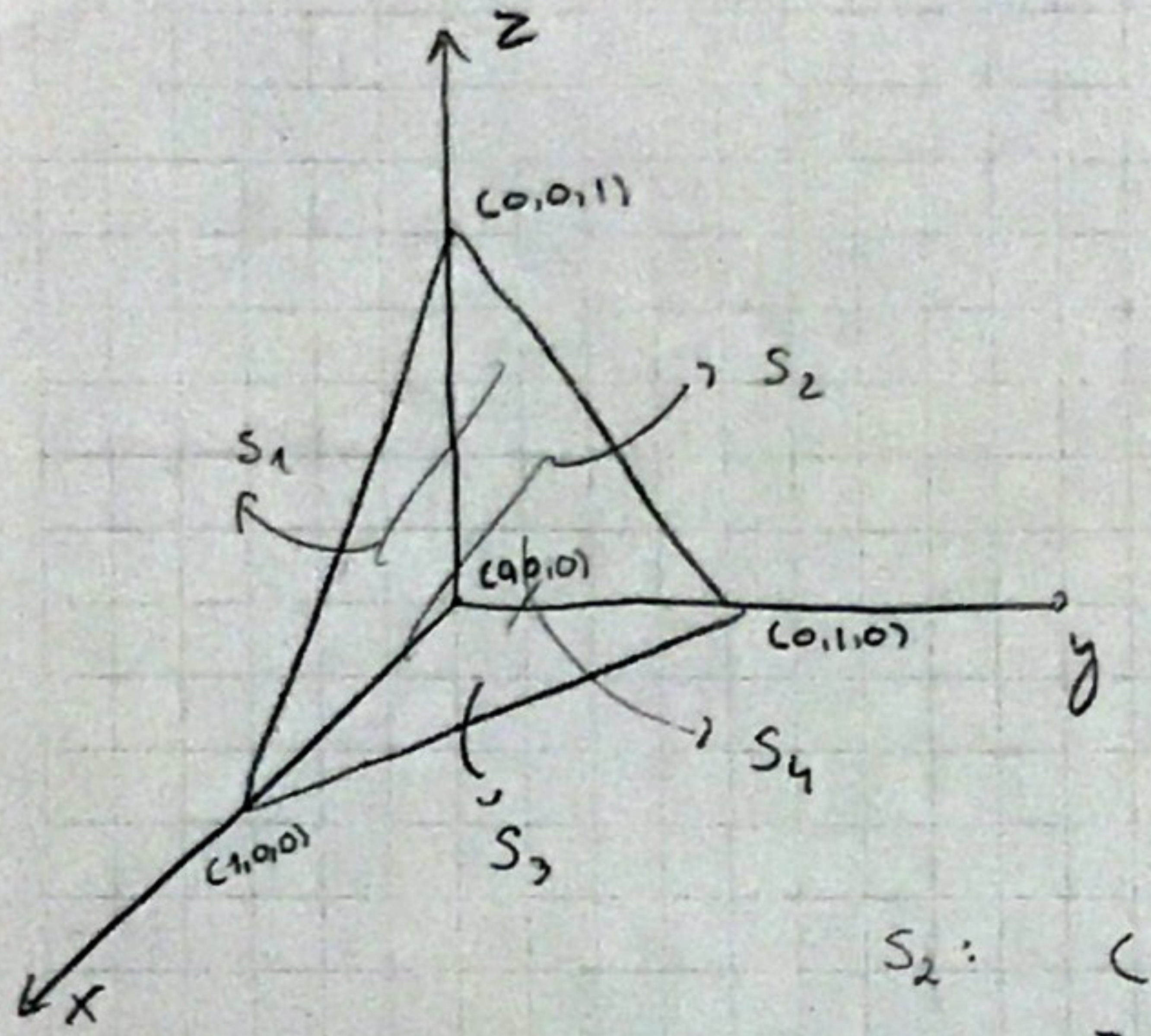
(12) $\displaystyle\iint_S \frac{1}{(1+x+y)^2}$, $S = \partial \Delta_T$, tetraedar, gdje je:

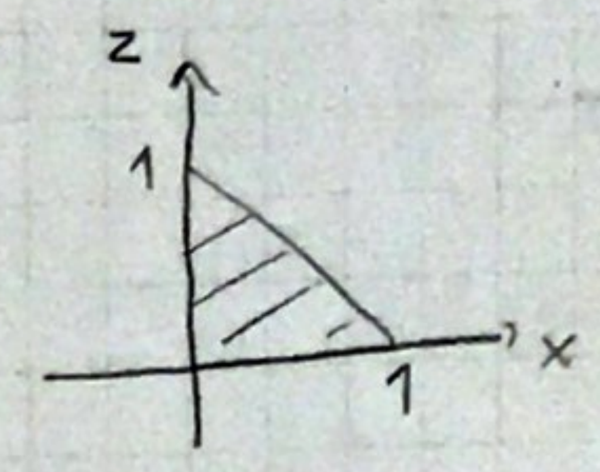
$$\Delta_T = \begin{cases} x = 0 \\ y = 0 \\ z = 0 \\ x+y+z = 1 \end{cases}.$$

R

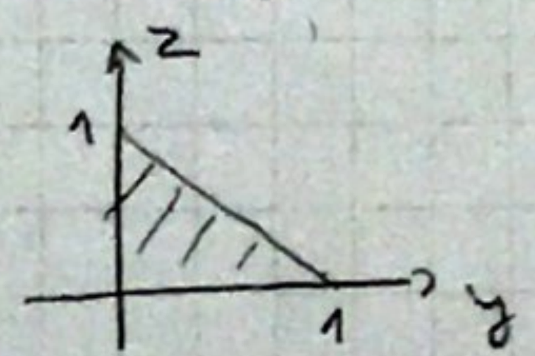


$S_1: (x,z) \to (x,0,z) \quad (y=0)$

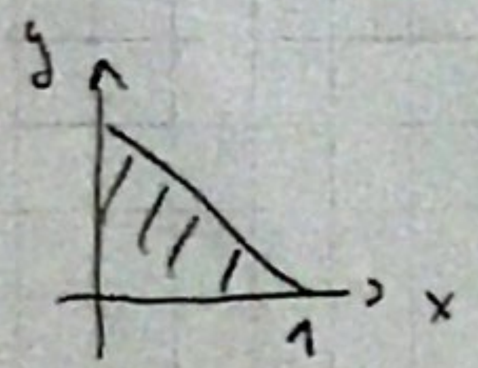
$(x,z) \in \Delta$



$S_2: (y,z) \to (0,y,z)$



$S_3: (x,y) \to (x,y,0)$



$S_4: (x,y) \to (x,y,1-x-y)$

$(x,y) \in \Delta$