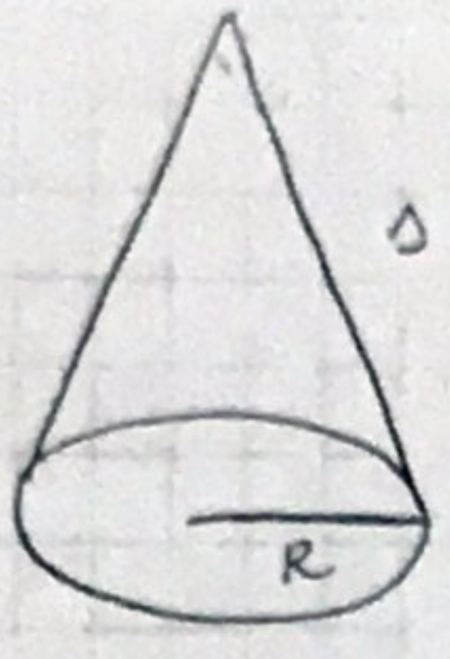
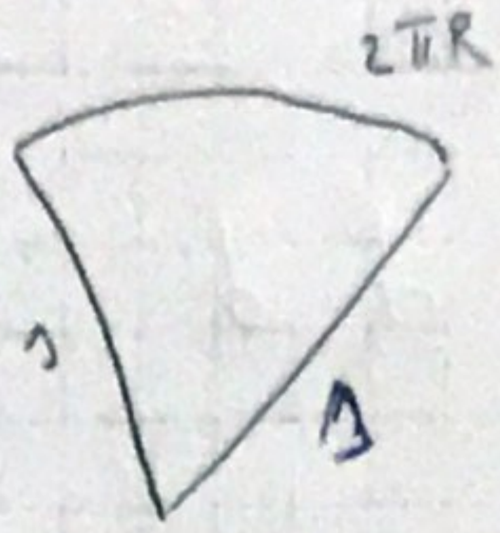
13.

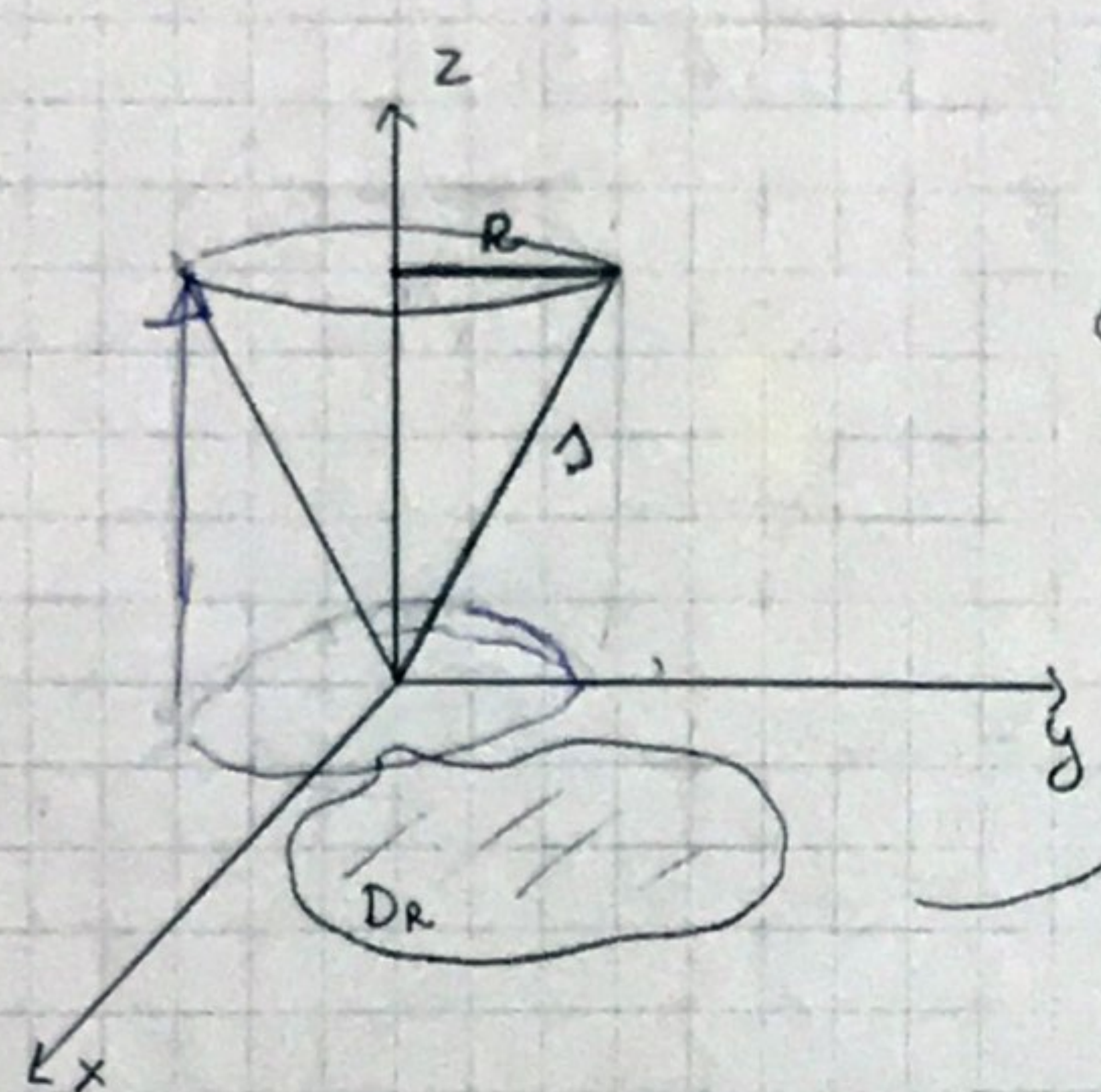




$$\frac{\Pi}{\pi s^2} = \frac{2\pi R}{2\pi \cancel{s}}$$

$$\Rightarrow \boxed{\Pi = \pi R s}$$

Preko integrala:



$k^2(x^2+y^2) = z^2$ - j-na konusa

$(x,y) \longrightarrow (x, y, k\sqrt{x^2+y^2})$

$$\iint_K dS = \iint_{D_R} \sqrt{1 + \left(\frac{kx}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{ky}{\sqrt{x^2+y^2}}\right)^2} = \iint_{D_R} \sqrt{\frac{(k^2+1)\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}} =$$

$$= \sqrt{k^2+1}\, \pi R^2$$

$k = ?$ ako imamo s i R

slika!

ako je visina $H = \sqrt{s^2 - R^2}$ $\Rightarrow$ $x^2+y^2 = R^2$ (kreću x i y po krugu R)

$k^2(x^2+y^2) = z^2$ $\quad s^2 - R^2 = R^2 k^2$

$k^2 R^2 = z^2$

$$\Rightarrow k^2 = \frac{s^2 - R^2}{R^2} \quad \left(\begin{matrix} \text{za nas konus} \\ R = 1 \\ s = \sqrt{2} \end{matrix} \right)$$

$$k^2 + 1 = \frac{s^2 - R^2 + R^2}{R^2}$$

$$\sqrt{k^2+1} = \frac{s}{R}$$

Naš rez: $\frac{s}{R}\pi R^2 = \pi R s$

(14.) $\displaystyle\int_{M^2} \frac{ds}{d^3}$, pri čemu je $M^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

a $d = d(O, T(x,y,z))$

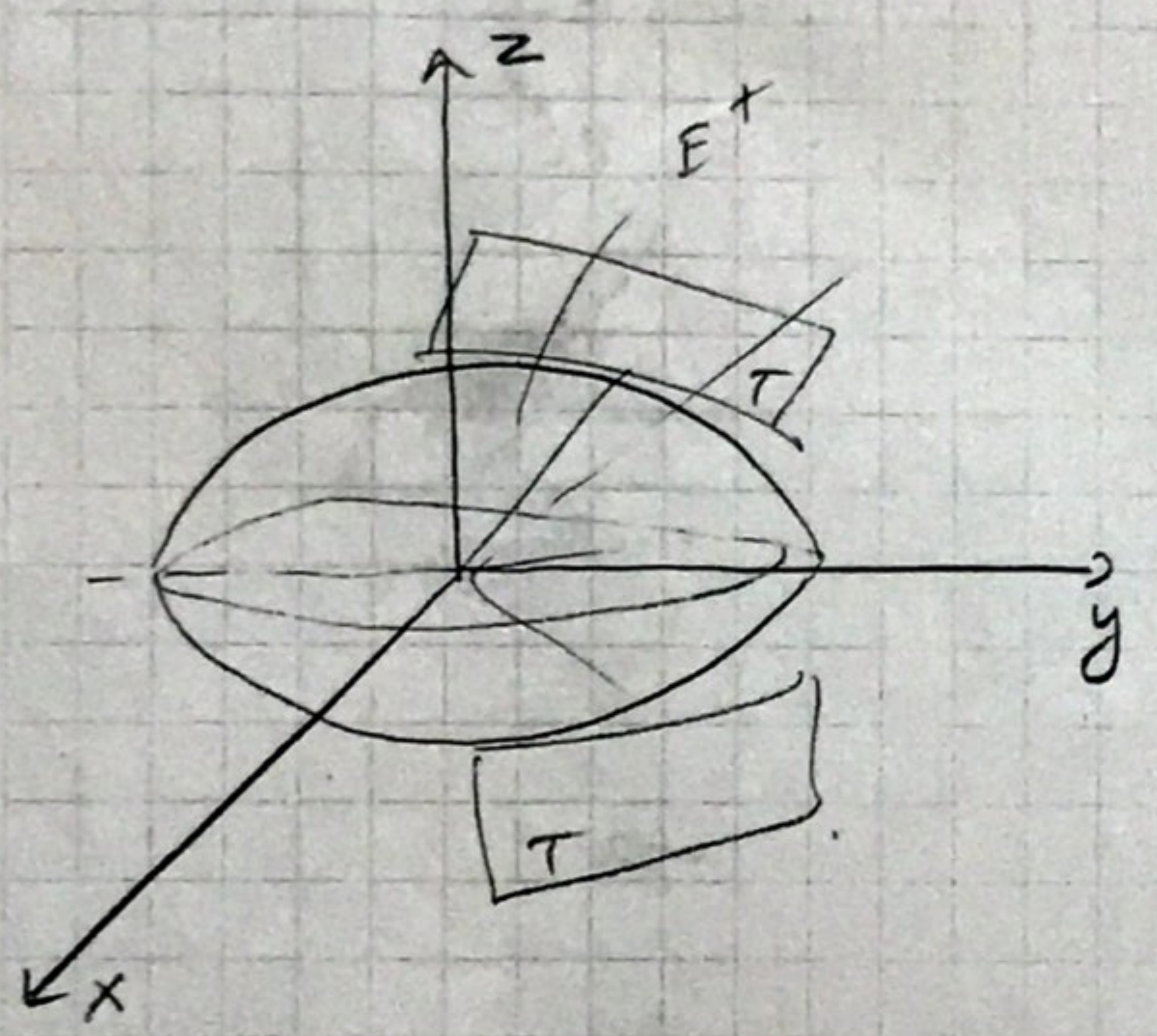
↳ tangentna ravan u tački (x,y,z)

R

tang. ravan: $\frac{Xx_0}{a^2} + \frac{Yy_0}{b^2} + \frac{Zz_0}{c^2} = 1$

$$d((0,0,0), T) = (\text{formula}) = \frac{1}{\sqrt{\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4}}}$$

$$\frac{1}{d^3} = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{3}{2}}$$

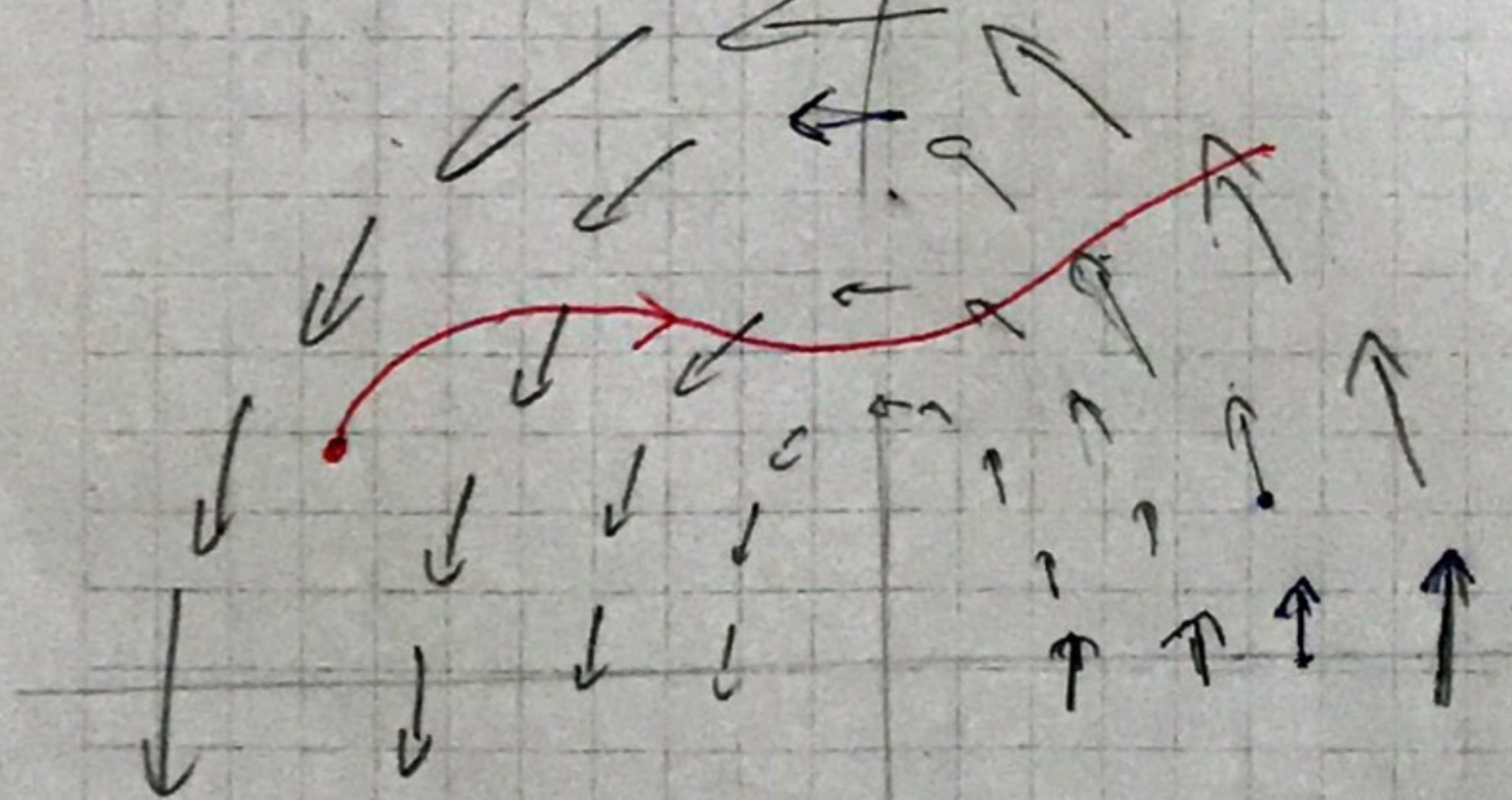


$$I = 2\iint_{E^+} \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{3}{2}} ds =$$

$$(x,y) \to \left(x, y, +c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\right)$$

↳ za gornju str.

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}$$



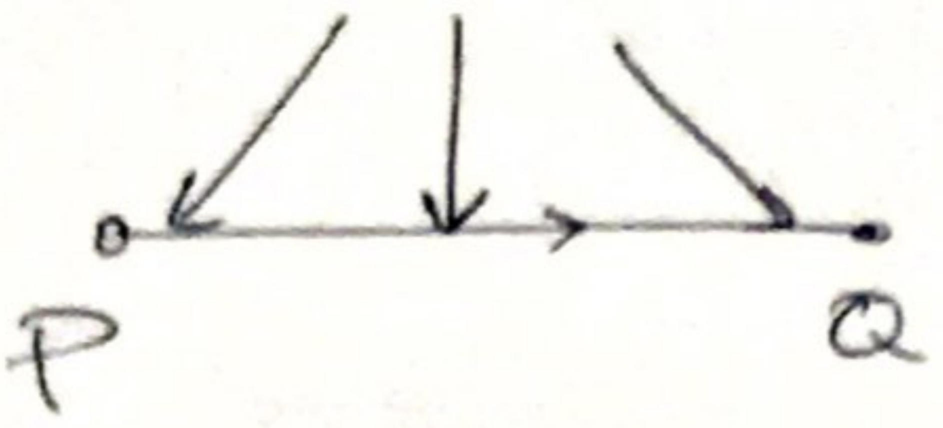
$F(x,y) = (-y, x)$

Integral polja po orjentisanoj krivoj

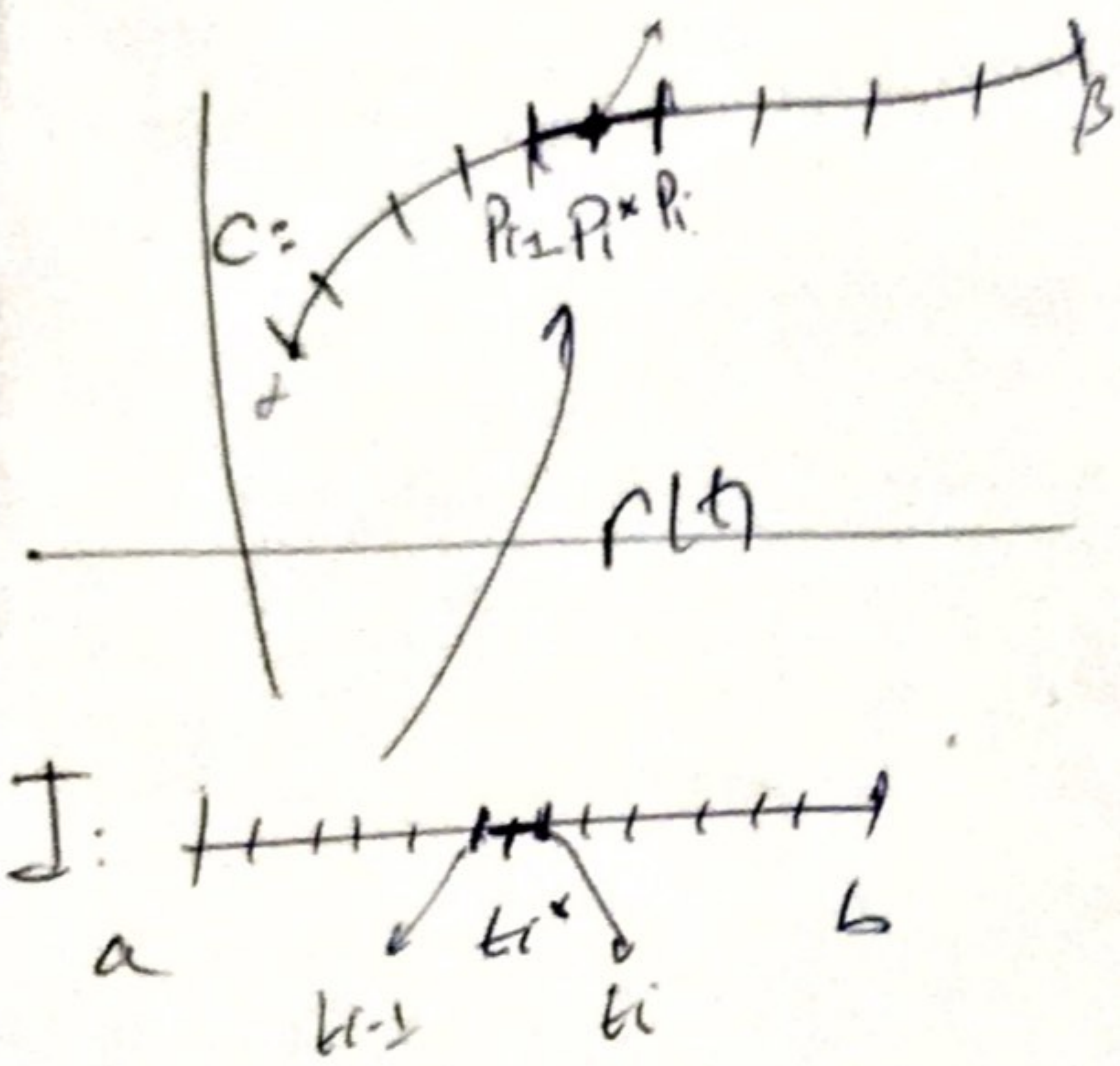
$\vec{F}$ djeluje na pravolinijskom putu od P do Q

Njen rad po tom putu se računa kao

$$\langle \vec{F}, \vec{PQ} \rangle = \|\vec{F}\| \, \|\vec{PQ}\| \cos\angle(\vec{F}, \vec{PQ})$$



Šta se dešava ako put nije pravolinijski?



$\vec{F} = (P, Q, R)(x, y, z)$

$P_i^* \in \widehat{P_{i-1}P_i}$

$t_i^* \in (t_{i-1}, t_i)$

$F_i^* = \vec{F}(r(t_i^*))$, $r(t_i^*) = P_i^*$

W_i rad sile po luku $\widehat{P_{i-1}P_i}$

$$W_i \approx \langle F_i^*, \vec{T} \rangle \cdot \Delta l_i$$

jedinični tangentni vektor

dužina luka $\widehat{P_{i-1}P_i}$

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} \langle F_i^*, T_i \rangle \cdot \Delta l_i =$$

$$= \sum_{i=1}^{n} \left\langle F \circ r(t_i^*), \frac{r'}{\|r'\|}(t_i^*) \right\rangle \cdot \Delta l_i \xrightarrow[n \to \infty]{d(P) \to 0} \int_a^b \langle F \circ r, r' \rangle \, dt$$

$\Delta l_i \approx \|r'(t_i^*)\| \cdot \Delta t_i$

$$= \int_C \langle F, T \rangle \, ds$$

$ds = dl$

$\odot \begin{cases} \text{For – nas asocira da uvodimo param.} \\ dx, dy \text{ – asocira nas na parc. izv.} \end{cases}$

① $\int\limits_C \underset{P}{(x+y)}\,dx + \underset{Q}{(x-y)}\,dy$, gdje je C pozitivno orijentisana elipsa: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

R

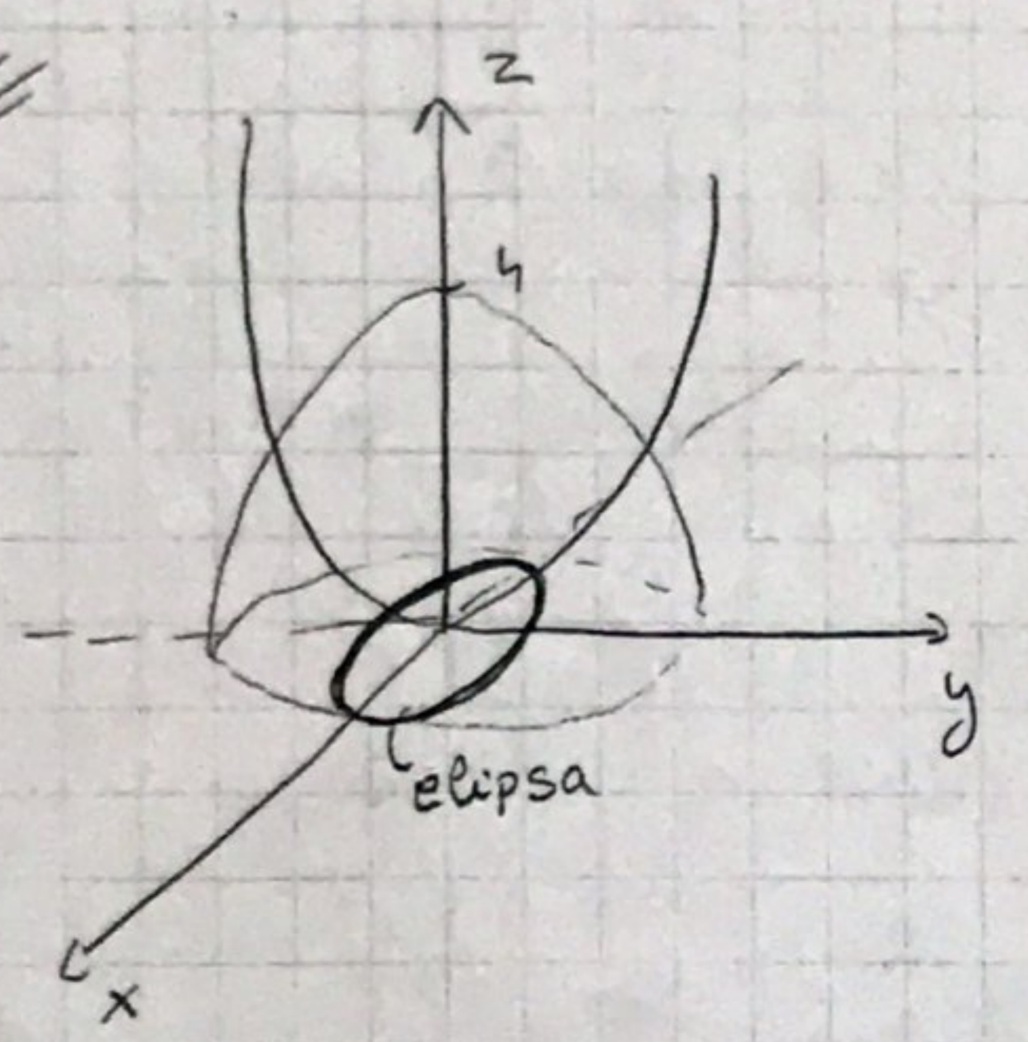
$g(t) = (a\cos t, b\sin t)$, $t \in (0, 2\pi)$ ← uvodimo parametrizaciju

$$= \int_0^{2\pi} (a\cos t + b\sin t)(-a\sin t) + (a\cos t - b\sin t)\cdot(b\cos t)\,dt =$$

$$= \dots = 0$$

② $\int\limits_C (4y^2 + 2x^2)\,dx + (z+x)\,dy + y\,dz$, gdje je $C: \begin{cases} z = 4 - x^2 - y^2 \\ z = y^2 \end{cases}$

R



– pravimo projekciju presjeka:

$y^2 = 4 - x^2 - y^2$

$\Rightarrow x^2 + 2y^2 = 4$ – elipsa

$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{(\sqrt{2})^2} = 1$

$x = 2\cos t$, $y = \sqrt{2}\sin t$

$\Rightarrow z = 2\sin^2 t$ (iz j-ne)

$t \in (0, 2\pi)$

$$I = \int_0^{2\pi} \Big[(4 \cdot 2\sin^2 t + 2 \cdot 4\cos^2 t)(-2\sin t) + (2\sin^2 t + 2\cos t)(\sqrt{2}\cos t)$$

$$+ \sqrt{2}\sin t (4\sin t\cos t) \Big]\,dt = \dots$$

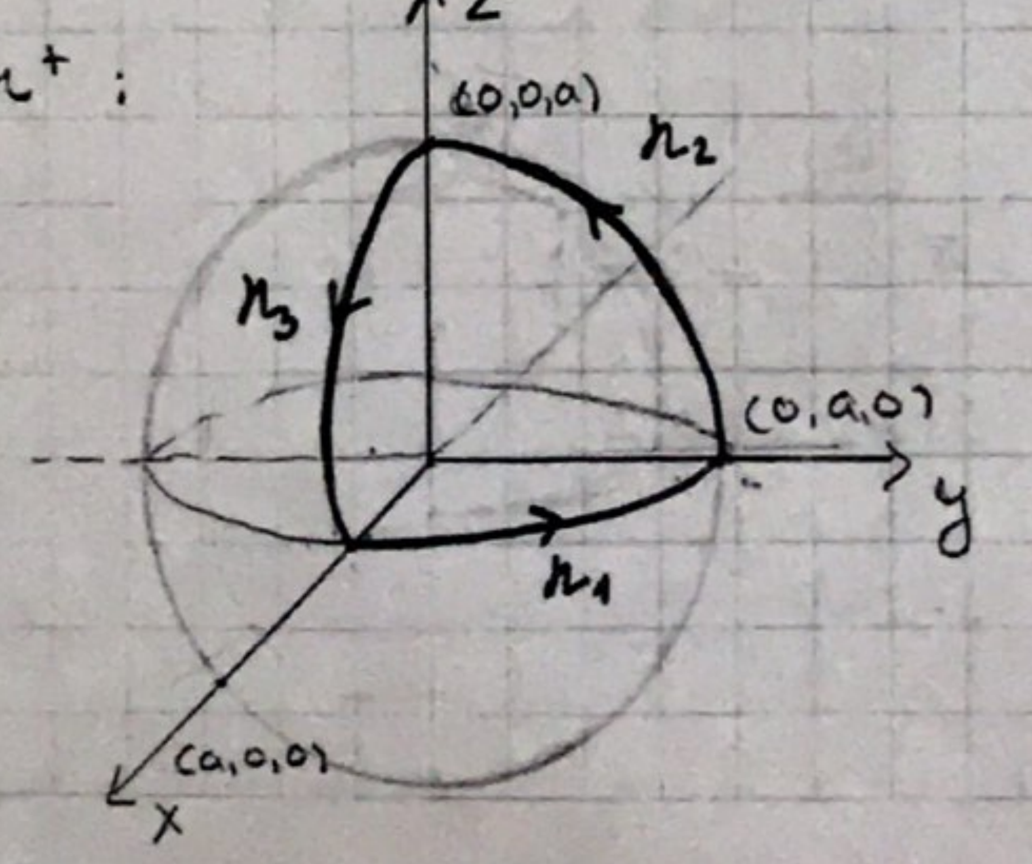
③ $\int\limits_{r^+} (y^2 - z^2)\,dx + (z^2 - x^2)\,dy + (x^2 - y^2)\,dz$

r^+ - kontura koja ograničava dio sfere u I oktantu $(x^2 + y^2 + z^2 = a^2)$

R

II zapis: $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$

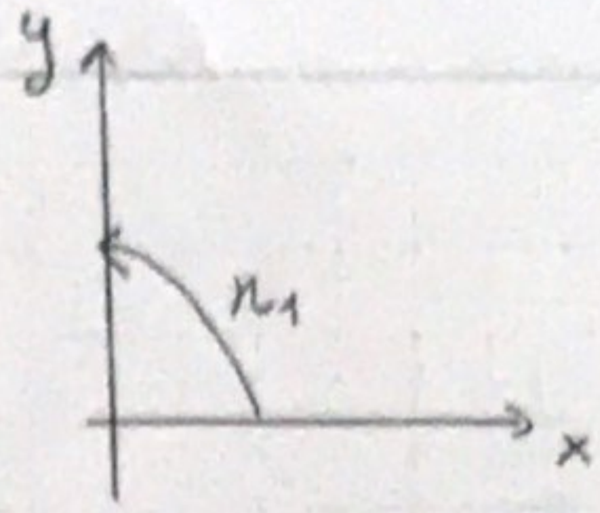
gdje je r^+:



r_1: $x = a\cos t$ $t \in (0, \frac{\pi}{2})$

$y = a\sin t$ t raste od 0 do $\frac{\pi}{2}$

$z = 0$



$$I_1 = \int_0^{\frac{\pi}{2}} (a^2\sin^2 t)(-a\sin t) + (-a^2\cos^2 t)(a\cos t) =$$

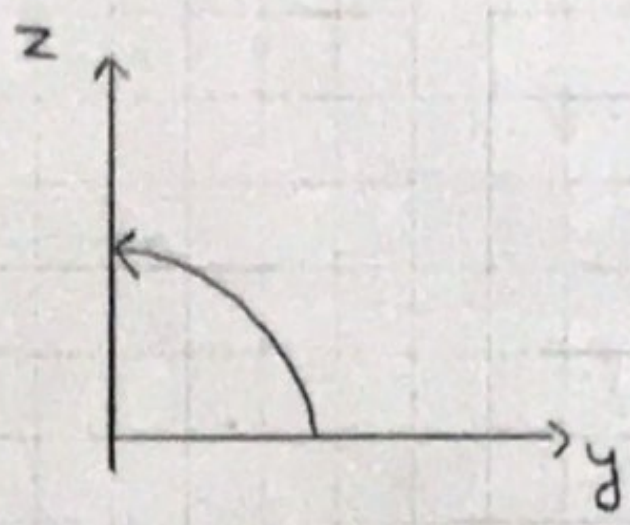
$$= \int_0^{\frac{\pi}{2}} (-a^3\sin^3 t - a^3\cos^3 t)\, dt = \dots =$$

$\sin^3 t = \sin t\,(1 - \cos^2 t)$

$u = \cos t$

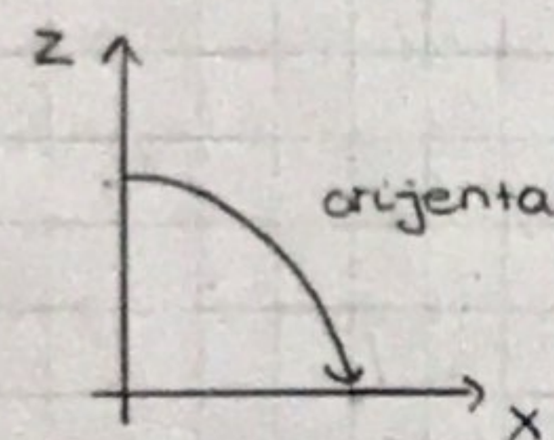
$du = \sin t$

r_2: $t \to (0, a\cos t, a\sin t)$



$t \in (0, \frac{\pi}{2})$

r_3:



t opada od $\frac{\pi}{2}$ do 0

$t \to (a\cos t,\ 0,\ a\sin t)$

$\varepsilon = -1$ jer je suprotna orijentacija

$t \in (0, \frac{\pi}{2})$

Teorema: $U \subseteq \mathbb{R}^3$

$A = (P, Q, R)$ vekt. polje (svakoj tački pridružuje vekt.)

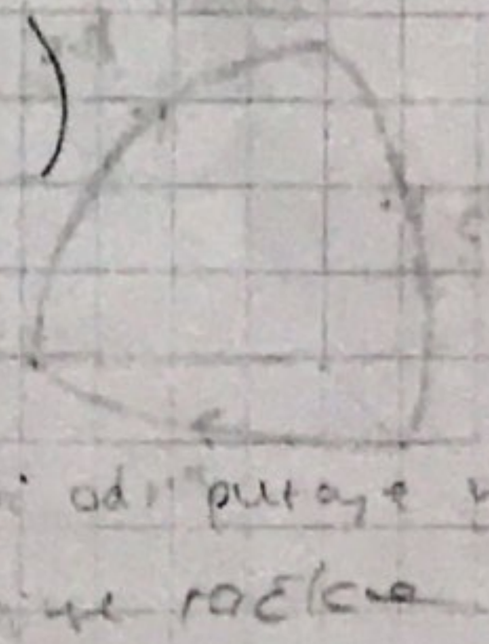
1) $\exists u : U \to \mathbb{R}$ grad $u = A$

2) $\int_{\widehat{ab}} A$ ne zavisi od puta $= u(b) - u(a)$

3) integral po konturi $c \subseteq U$ od vekt polja A jednak je nuli

primjer: Njutnovo polje

$$A(x, y, z) = \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{\dots}, \frac{z}{\dots} \right)$$



$$\int_c \vec{A} = \frac{1}{\|a\|} - \frac{1}{\|b\|}$$ → dakle, ne zavisi od putanje već samo od početne i krajnje tačke

$= u(b) - u(a)$

$u = \frac{-1}{\sqrt{x^2+y^2+z^2}}$

$u_x = \left(-(x^2+y^2+z^2)^{-\frac{1}{2}}\right)'_x = \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

$u_y = \dots$

$u_z = \dots$

$\Rightarrow \boxed{\text{grad } u = A}$

! Teorema: $U \subseteq \mathbb{R}^3$ prosto-povezana oblast

$A = (P, Q, R)$

A je totalni dif. $\Leftrightarrow \text{rot } A = 0$

$$\text{rot } A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0$$

$$= (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y) = (0, 0, 0)$$

A je tot. dif. $\Leftrightarrow \boxed{R_y = Q_z,\ P_z = R_x,\ Q_x = P_y}$

1. $P = yz + 2x\cos x^2$

$Q = xz - z\sin(yz)$

$R = xy - y\sin(yz)$

$A = (P, Q, R)$

$\int\limits_{c(a,b)} = ?$

R:

$R_y = x - \sin(yz) - y\cos(yz) \cdot z$

$Q_z = x - \sin(yz) - z\cos(yz) \cdot z$

$R_y = Q_z$

$P_z = y, \quad R_x = y \quad \Rightarrow P_z = R_x$

$Q_x = z$

$\Rightarrow Q_x = P_y$

$P_y = z$

Dalle, jeste j-na sa tot. dif. $\overset{T}{\Rightarrow} \exists u : U \to \mathbb{R}$ t.d. $\text{grad}(u) = A$

$u = xyz + \sin(x^2) + \cos(yz)$ - kad izr. ovo se dob.. $u = \int P\,dx + \mathcal{C}(y,z)$..

$$\frac{\partial u}{\partial x} = P, \quad \frac{\partial u}{\partial y} = Q, \quad \frac{\partial u}{\partial z} = R$$

$a = (\sqrt{\pi}, 0, 1)$

$b = (\sqrt{\frac{\pi}{2}}, 1, 0)$

$I = u(b) - u(a) = (0 + 1 + 1) - (0 + 0 + 1) = 1$

$u_x = P$

$u_y = Q$

$u_z = R$

$u = \int P\,dx + \mathcal{C}(y,z) = xyz + \sin(x^2) + \mathcal{C}(y,z)$

$u_y = Q \Rightarrow \cancel{xz} + \mathcal{C}_y = \cancel{xz} - z\sin(yz) \Rightarrow \mathcal{C}_y = -z\sin(yz)$

$u_z = R \Rightarrow x\cancel{y} + \mathcal{C}_z = x\cancel{y} - y\sin(yz) \Rightarrow \mathcal{C}_z = -y\sin(yz)$

$\mathcal{C}_y = \int(-z\sin(yz)) + \Psi(z)$

$= \ldots$

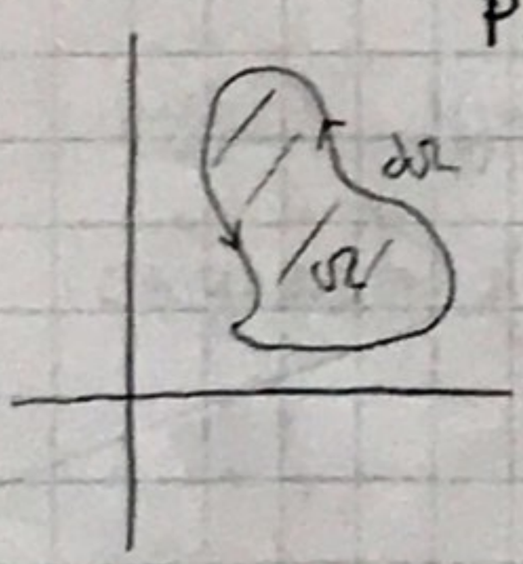
pa nađemo $\mathcal{C}(z)$

Grinova formula:

Teorema: $\Omega \subseteq \mathbb{R}^2$ oblast

ograničena dio po dio glatkom krivom

P, Q definisane na $\bar{\Omega}$, $P, Q \in C^1$



$\Rightarrow$ tada je $$\int_{\partial\Omega^+} P\,dx + Q\,dy = \iint_{\Omega} (Q_x - P_y)\,dx\,dy$$

veza između krivolinijskog i dvojnog integrala

(Zad) $\int\limits_C (\ln(z)+y^2)\,dx + (axy + ze^{zy})\,dy + \left(\frac{x}{z} + ye^{zy}\right)dz$

$z>0.$ $\quad A(1,1,e) \quad B(3,3,1)$

Odrediti a t.d. ne zavisi od krive c koji leži u oblasti $z>0$ i spaja A, B.

(Rj)

$P = \ln(z) + y^2$

$Q = axy + ze^{zy}$

$R = \frac{x}{z} + ye^{zy}$

$R_y = e^{zy} + yze^{zy}$ $\quad P_z = \frac{1}{z}$ $\quad Q_x = ay$

$\quad\ \ \|$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \Big\}\ a=2$

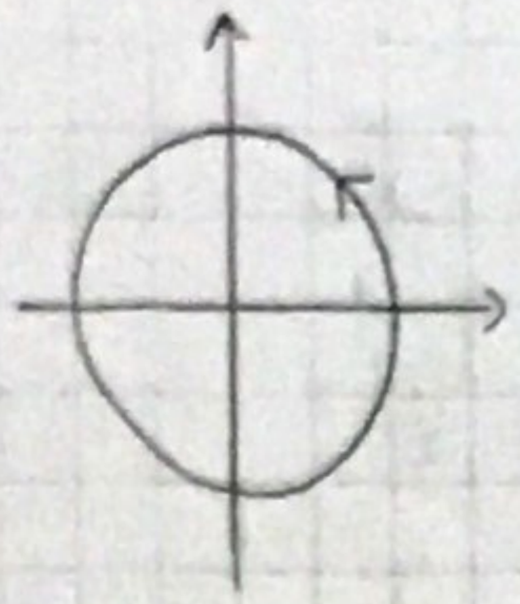
$Q_z = e^{zy} + z\cdot y\cdot e^{zy}$ $\quad R_x = \frac{1}{z}$ $\quad P_y = 2y$

$$\left.\begin{array}{l} u_x = P \\ u_y = Q \\ u_z = R \end{array}\right\} u = \ln(z)x + e^{zy} + xy^2$$

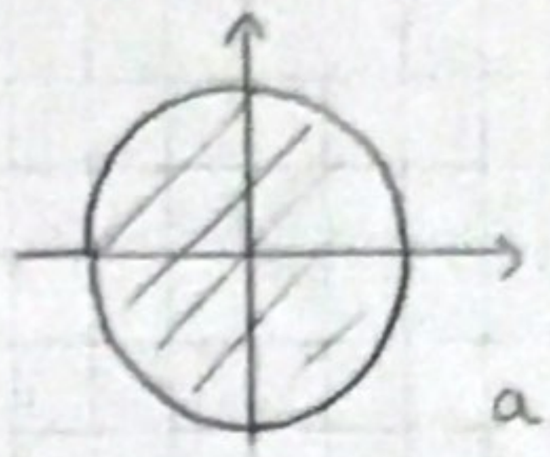
$\int = u(B) - u(A) = \ldots$

primjer: $\int_c \underbrace{(x+y)}_{P}dx + \underbrace{(x-y)}_{Q}dy$

$Q_x - P_y = 1-1 = 0$



možemo da svedemo na



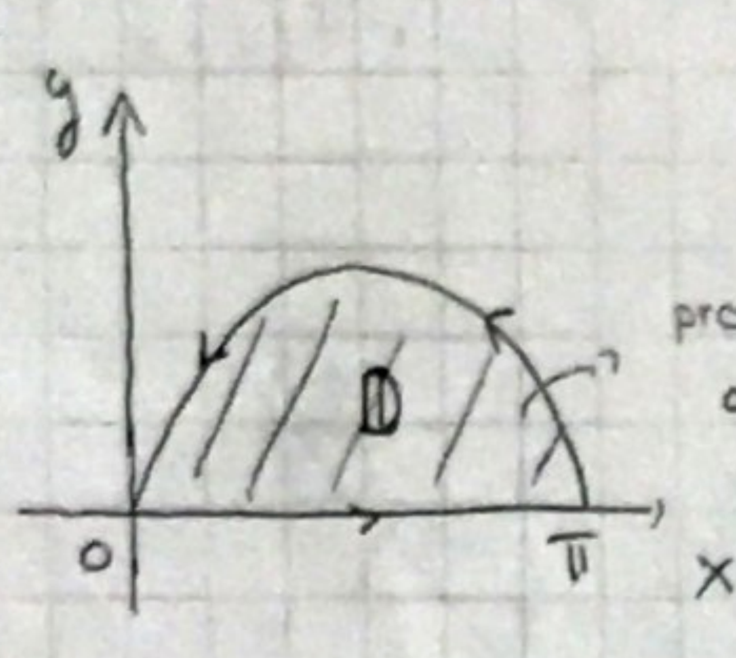
a to je $\underline{\underline{0}}$

① $\int_{k^+} e^x(1-\cos y)dx - e^x(y-\sin y)dy$, gdje je k kriva koja ograničava oblast gdje je

$0 \le x \le \pi$

$0 \le y \le \sin x$

R.



$P = e^x(1-\cos y)$

$Q = -e^x(y-\sin y)$

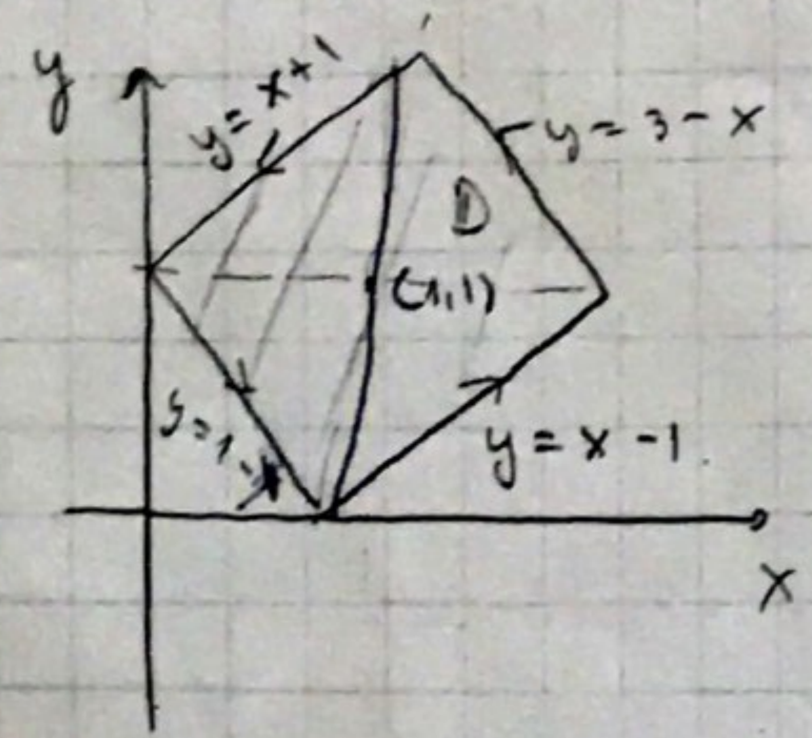
$Q_x - P_y = -e^x(y-\sin y) - e^x\sin y = -e^x y$

Grinova formula:

$\Rightarrow I = \iint_D -e^x y\,dxdy = \int_0^\pi \int_0^{\sin x} -e^x y\,dydx = \dots = -\frac{1}{5}(e^\pi - 1)$

② (sami) $\int_{k^+} (x^2+y^2)dx + (x^2-y^2)dy$, gdje je $k: |x-1|+|y-1| = 1$

R.



I način: preko 4 krive

II način: Grinova

$P = x^2+y^2$

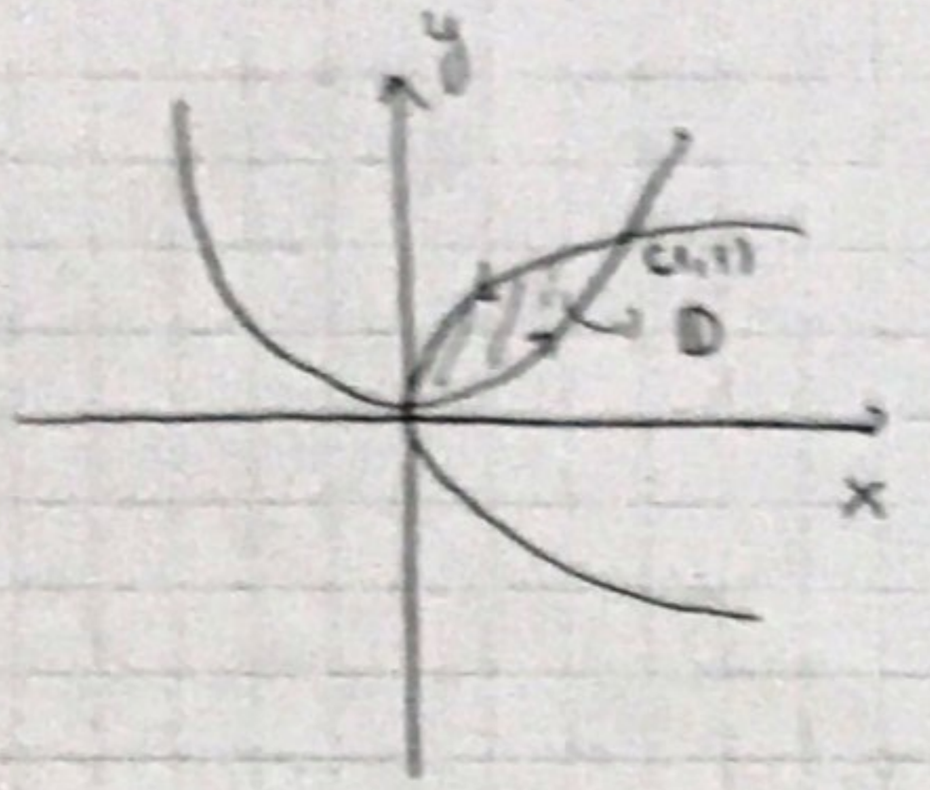
$Q = x^2-y^2$

$Q_x - P_y = 2x-2y$

$\underline{I} = \iint_D (2x-2y)dxdy = \int_0^1 dx \int_{1-x}^{x+1} (2x-2y)dy + \int_1^2 dx \int_{x-1}^{3-x} (2x-2y)dy$

③ $\iint\limits_{\partial G} (y+e^{\sqrt{x}})dx + (2x+\cos(y^2))dy$, G oblast u ravni xOy

ograničena sa $y=x^2$, $x=y^2$



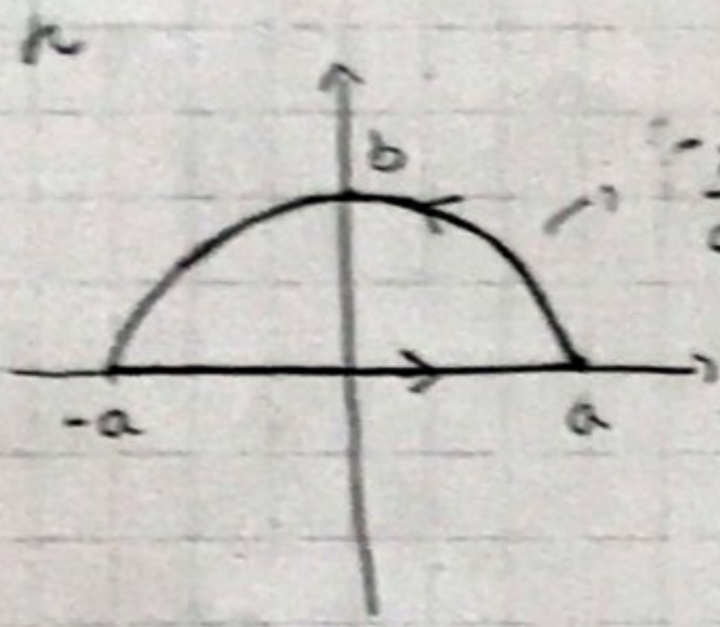
Grinova formula:

$P = y + e^{\sqrt{x}} \Rightarrow P_y = 1$

$Q = 2x + \cos(y^2) \Rightarrow Q_x = 2$

$\Rightarrow I = \iint\limits_D 1\,dx\,dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} 1\,dy$

Za vježbu: $\int\limits_K (x^2+y^2)dx + y\,dy$ (na 2 načina)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

T

$\int\limits_{\partial D^+} P\,dx + Q\,dy = \iint\limits_D (Q_x - P_y)\,dx\,dy$ ← Posljedica Grinove formule

1° ako dob. $Q_x - P_y = 1$, onda računamo mjeru

$Q = x$, $P = 0$

$\int\limits_{\partial D^+} x\,dy = \mu(D)$

2° $Q = 0$, $P = -y$

$\int\limits_{\partial D^+} -y\,dx = \mu(D)$

3° $Q = \frac{x}{2}$ $P = -\frac{y}{2}$

$\int\limits_{\partial D^+} -\frac{y}{2}dx + \frac{x}{2}dy = \mu(D)$

(4.) $\mu(E) = ?$ $E(a,b)$ - elipsa sa osama a i b

R

$$\int_{\partial E^+} x\,dy = \int_0^{2\pi} a\cos t\,(b\cos t)\,dt = ab\int_0^{2\pi} \cos^2 t\,dt =$$

$$= ab\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = \pi ab$$

param: $x = a\cos t$

$, t \in (0, 2\pi)$

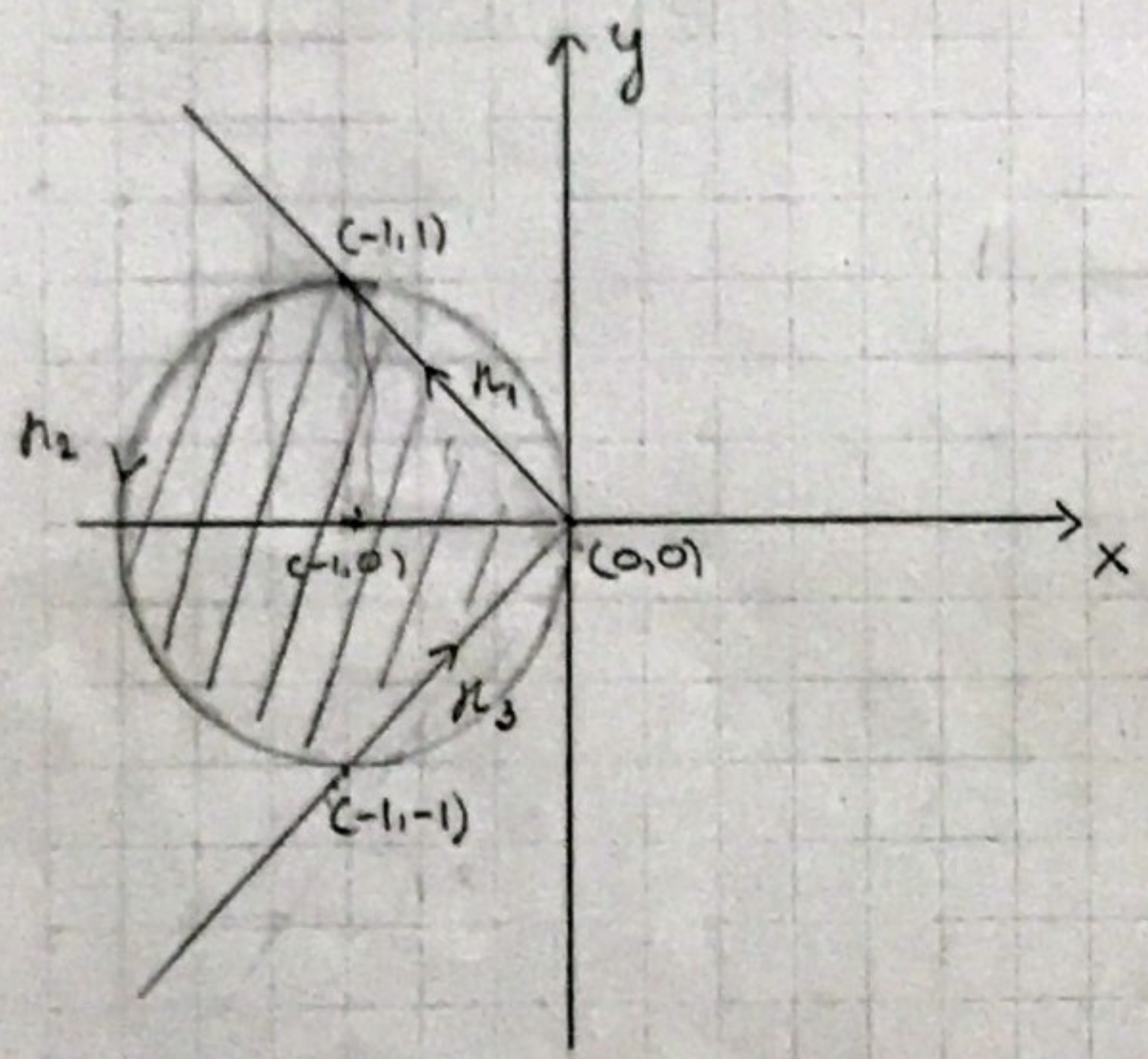
$y = b\sin t$

(5.) $D = \{ (x,y) \mid x^2+y^2 \leq -2x \ , \ x+|y| \leq 0 \}$

R

$x^2+2x+1+y^2 \leq 1$ $\qquad$ $x \leq -|y|$

$(x+1)^2+y^2 \leq 1$



I način: krive

II način: Grin

κ_1: $t \to (t, -t)$

$t \in (-1, 0)$

$\varepsilon = -1$

κ_2: $x = -1+\cos t$

$y = \sin t$ $\qquad \varepsilon = 1$

$t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

κ_3: $t \to (t, t)$

$t \in (-1, 0)$ $\qquad \varepsilon = 1$

$$\int_{\partial D^+} x\,dy = \int_{\kappa_1^+} x\,dy + \int_{\kappa_2^+} x\,dy + \int_{\kappa_3^+} x\,dy$$

$$= \frac{\pi \cdot 1^2}{2} + 2 \cdot \frac{1 \cdot 1}{2} = \frac{\pi}{2} + 1$$